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Mathematical analysis of extended Fisher-Kolmogorov equation with Neumann boundary conditions

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ABSTRACT: We investigate the extended Fisher-Kolmogorov equation (EFK) under Neumann boundary conditions in a spatial dimension of up to three ($d \leq 3$). This is done on a bounded convex domain with a boundary that possesses C^2 smoothness. The EFK equation, which is a fourth-order PDE, is reduced into a system of two second-order PDEs. We prove the global existence, uniqueness, and continuous dependence on initial conditions for both strong and weak solutions using Lions' classic Faedo-Galerkin approach and compactness arguments. Furthermore, we provide results on the regularity of weak forms.

Key Words: existence, Faedo-Galerkin, extended Fisher-Kolmogorov equation, weak formulation, strong solution.

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1. Introduction

The analysis of the worldwide existence of solutions to partial differential equations (PDEs) is a basic problem in mathematics, with applications in many scientific and engineering areas. The Faedo-Galerkin method was used in this work because it provides a systematic approach to determining the presence of solutions for a wide variety of PDEs. PDEs appear in a variety of applications, showing how functions of several variables change through space and time. These equations play a crucial role in modeling a wide range of scientific phenomena, including fluid dynamics, heat conduction, electromagnetic fields, and quantum physics, among others. Solving these equations is essential for gaining insights into the behavior of complex systems, which is vital for making predictions and informed decisions. The Faedo-Galerkin technique, a form of functional analysis, is employed to analyze the existence and behavior of solutions to partial differential equations (PDEs). This method involves approximating the original PDE with a finite-dimensional subspace of functions, typically represented by a collection of basis functions. By doing so, the problem is transformed into a set of ordinary differential equations, which are often more manageable for analysis. The accuracy of these approximations is crucial to ensure that as the dimension of the subspace increases, the solutions to the finite-dimensional problems converge to the correct solution of the original PDE. One of the strengths of the Faedo-Galerkin technique lies in its versatility, as it can be applied to various types of PDEs, different boundary conditions, and diverse domains. However, its effectiveness often hinges on carefully selecting the approximation space and basis functions, and it may require a deep understanding of the underlying problem's structure for optimal

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results. For further information on the Faedo-Galerkin approach and how to utilize it to show the global existence of the solution to PDEs, see [36,47,48,42,11]. Given the importance of this topic, recent studies have also focused on the existence and uniqueness of solutions for differential equations, as indicated in [13,3,34,22,37,8,23,44,26,1,25,38,24,46,43,7,27,16,9,29,10,14,15,4,5,6].

The Fisher-Kolmogorov (FK) equation, which is attributed to Fisher and Kolmogorov, falls into the class of nonlinear PDEs. It serves as a means to depict the diffusion of organisms and interactions during adaptation processes. A modification to the FK equation involves the inclusion of a fourth-order derivative term, leading to a novel model referred to as the extended Fisher-Kolmogorov (EFK) equation. This extended version was introduced in [18], and it bears significant relevance within the realm of physics. The EFK equation possesses notable significance in various physical phenomena, encompassing fields such as hydrodynamics, plasma physics, thermonuclear reactions, population growth, and the propagation of infectious diseases [2]. However, these equations often exhibit nonlinearity or intricate computational domains, particularly when the source term lacks predefined patterns. Although theoretical analytical solutions may exist [19], practical engineering applications frequently encounter challenges in determining exact solutions. Consequently, resorting to numerical solutions emerges as a viable approach. The nonlinear extended EFK is formulated as proposed in [30,31,32] with Neumann boundary conditions:
(P) Find $\{u\}$ such that

$$\partial_t u + k_1 \Delta^2 u - k_2 \Delta u = F(u), \qquad \text{in} \quad \mathcal{R} \times (0,T), \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \qquad \text{on} \quad \partial\mathcal{R} \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \qquad \text{in} \quad \mathcal{R}, \tag{1.3}$$

where k_1 and k_2 are positive constants, $F(u) = u - u^3$, and $\mathcal{R} \subset \mathcal{R}^d$ is a bounded domain with a Lipschitz continuous boundary $\partial\mathcal{R}$.

The EFK equation, originally introduced in references [51,52,21], serves as a fundamental model for the analysis of pattern formation in a wide range of physical, chemical, and biological systems. Additionally, it finds relevance in the field of phase transitions near Lifshitz points. Over the past few decades, extensive research has been dedicated to the study of the EFK equation. In [40,39], spatial patterns and various types of stationary solutions of the EFK equation have been thoroughly examined. Furthermore, in [50,33], the uniqueness of solutions for the EFK equation has been rigorously established. Moreover, periodic solutions of the EFK Equation have been explored in studies such as those conducted in [49,45,35].

In this paper, we assume that $w = -k_1\Delta u + k_2 u$, then System (1.1)-(1.3) can be formulated as follows

$$\partial_t u - \Delta w = F(u), \tag{1.4}$$

$$w = k_2 u - k_1 \Delta u, \tag{1.5}$$

$$F(u) = u - u^3, \tag{1.6}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \qquad \text{on} \quad \partial\mathcal{R} \times (0,T), \tag{1.7}$$

$$u(x,0) = u_0(x), \qquad \text{in} \quad \mathcal{R}. \tag{1.8}$$

The following excerpt presents an overview of the paper's structure. In Section 2, we provide an introduction to the fundamental notation used in Sobolev spaces, along with time-dependent Sobolev spaces. Moving on to Section 3, we will present the weak formulation of the problem, discuss Galerkin approximations, and present the main theorem asserting the existence of a weak solution for System (1.4)-(1.8). Section 3.1 delves into the global existence of weak solutions, while the more theoretical aspect of this paper, passage to the limit of Galerkin approximations, is covered in Section 3.2. Following this, Section 3.3 includes a proof of uniqueness. In Section 4, we derive additional regularity outcomes for the weak form through supplementary estimates, leading to conclusions about strong solutions. Furthermore, we demonstrate the continuous dependency of strong solutions on the initial data in $H^1(\mathcal{R})$).

2. Notation and auxiliary results

We represent the inner product in the $L^2(\mathcal{R})$ space, equipped with the norm denoted as $\|\cdot\|_0$, as $(\cdot,\cdot)$. Additionally, we use $\langle\cdot,\cdot\rangle$ to signify the duality pairing between the dual space $(H^1(\mathcal{R}))'$ and the space $H^1(\mathcal{R})$. It is worth noting that $(H^1(\mathcal{R}))'$ itself can have its own norm, which is defined as:

$$\|\phi\|_{(H^1(\mathcal{R}))'} := \sup_{\eta\neq 0}\frac{|\langle\phi,\eta\rangle|}{\|\eta\|_1} \equiv \sup_{\|\eta\|_1=1}|\langle\phi,\eta\rangle|. \tag{2.1}$$

Moreover, we denote to the function spaces which are depending on time and space as $L^\alpha(0,T;Y)$ $(1\leq\alpha\leq\infty)$; where Y is a Banach space. These spaces consist of all functions ϕ so that for a.e. $s\in(0,T)$ $\phi\in Y$ and the following norm is finite:

$$\|\phi(s)\|_{L^\alpha(0,T;Y)} = \left(\int_0^T \|\phi(s)\|_Y^\alpha ds\right)^{\frac{1}{\alpha}},$$

$$\|\phi(s)\|_{L^\infty(0,T;Y)} = ess \sup_{s\in(0,T)}\|\phi(s)\|_Y.$$

Additionally, we define the spaces $L^\alpha(\mathcal{R}_T)=L^\alpha(0,T;L^\alpha(\mathcal{R}))$, $\alpha\in[1,\infty]$. Also, we define $C([0,T];Y)$, the space of continuous functions from $[0,T]$ into Y, which consists $\phi(s):[0,T]\to Y$ so that $\phi(s)\to\phi(s_0)$ in Y as $s\to s_0$. The norm which is associated to the space $C([0,T];Y)$ can be defined as [48]:

$$\|\phi(s)\|_{C([0,T];Y)} = \sup_{s\in[0,T]}\|\phi(s)\|_Y.$$

We should also bring to mind the following commonly acknowledged results in the field of Sobolev theory, as found in well-established references such as [12] and [17]:

$$H^1(\mathcal{R}) \overset{c}{\hookrightarrow} L^\rho(\mathcal{R}) \hookrightarrow (H^1(\mathcal{R}))' \text{ holds for } \rho\in\begin{cases}[1,\infty] & \text{if } d=1,\\ [1,\infty) & \text{if } d=2,\\ [1,6] & \text{if } d=3,\end{cases} \tag{2.2}$$

where $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ are the continuous embedding and compact embedding, respectively.

For time-dependent problems defined as [41], if $u\in L^2(0,T;Y)$ then we have that

$$-\Delta u\in L^2(0,T;Y'). \tag{2.3}$$

3. Weak solutions

Let V be the trial space such that

$$V=\left\{v:\int_{\mathcal{R}}(|\nabla(v)|^2+v^2)dx<\infty\right\},$$

then by multiplying (1.4) and (1.5) by a test function $v\in V$, integrating over $\mathcal{R}$ and rearranging the terms, we find that

$$(\partial_t u,v)=(\Delta w,v)+(F(u),v), \tag{3.1}$$

$$(w,v)=-k_1(\Delta u,v)+k_2(u,v). \tag{3.2}$$

By applying Green's formula

$$\int_{\mathcal{R}}\Delta uv dx=\int_\Gamma v\frac{\partial u}{\partial v}d\sigma-\int_{\mathcal{R}}\nabla u\nabla v dx, \tag{3.3}$$

we then find the weak form as follows:
(P) Find $\{u,w\}\in H^1(\mathcal{R})\times H^1(\mathcal{R}), t\in[0,T]$ such that $\forall\eta\in H^1(\mathcal{R})$,

$$(\partial_t u,\eta)=-(\nabla w,\nabla\eta)+(F(u),\eta), \tag{3.4}$$

$$(w,\eta)=k_1(\nabla u,\nabla\eta)+k_2(u,\eta), \tag{3.5}$$

$$u(.,0)=u^0. \tag{3.6}$$

Theorem 3.1 *Given $u^0 \in H^1(\mathcal{R})$, then there exists a solution $\{u, w\}$ to the problem (P) such that*

$$u(x,t) \in L^\infty(0,T;(H^1(\mathcal{R})))\bigcap H^1(0,T;(H^1(\mathcal{R}))')\bigcap L^2(0,T;H^1(\mathcal{R}))$$

$$\bigcap C([0,T];L^2(\mathcal{R}))\bigcap L^2(\mathcal{R}_T), \tag{3.7}$$

$$w \in L^2(0,T;H^1(\mathcal{R})), \tag{3.8}$$

$$\frac{\partial u}{\partial t} \in L^2(0,T;(H^1(\mathcal{R}))'). \tag{3.9}$$

Proof: To prove the theorem, we apply the Faedo-Galerkin method [36]. We separate the proof into three parts.

To establish the existence, we employ the Faedo-Galerkin method as outlined in [36]. Consider an orthogonal basis denoted as $y_i{}_{i=1}^{\infty}$ for the space $H^1(\mathcal{R})$, which also serves as an orthonormal basis for $L^2(\mathcal{R})$. This basis is composed of eigenfunctions for

$$-\Delta y_i + y_i = \lambda_i y_i, \text{ in } \mathcal{R}, \quad \frac{\partial y_i}{\partial \nu} = 0 \text{ on } \partial\mathcal{R}, \tag{3.10}$$

where

$$1 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \ldots \quad \text{with} \lim_{i\to\infty} \lambda_i = \infty, \tag{3.11}$$

is an infinite set of corresponding eigenvalues. Note $(y_i, y_j)_{H^1(\mathcal{R})} = \lambda_i \delta_{ij}$ and $(y_i, y_j)_{L^2(\mathcal{R})} = \delta_{ij}$. Now set $V^k := \text{span}\{y_i\}_{i=0}^k \subset H^1(\mathcal{R})$, and seek a finite-dimensional weak form corresponding to (P):
Find $\{u^k(x,t), w^k(x,t)\} \in V^k \times V^k$, in the form

$$u^k(x,t) = \sum_{j=1}^{k} c_j(t) y_j, \tag{3.12}$$

$$w^k(x,t) = \sum_{j=1}^{k} d_j(t) y_j, \tag{3.13}$$

such that

$$(\partial_t u^k, \eta^k) = -(\nabla w^k, \nabla \eta^k) + (F(u^k), \eta^k), \tag{3.14}$$

$$(w^k, \eta^k) = k_1(\nabla u^k, \nabla \eta^k) + k_2(u^k, \eta^k), \tag{3.15}$$

$$u^k(.,0) = P^k u^0, \tag{3.16}$$

where P^k is a projection from $H^1(\mathcal{R})$ in to V^k define by

$$P^k\chi = \sum_{j=1}^{k} (\chi, y_i) y_i, \tag{3.17}$$

which satisfies that

$$(P^k\chi - \chi, \eta^k) = (\nabla(P^k\chi - \chi), \nabla\eta^k) = 0, \quad \forall, \ \eta^k \in V^k, \tag{3.18}$$

$$\|P^k\|_{\mathcal{L}(H^1,V^k)} = \|P^k\|_{\mathcal{L}(L^2,V^k)} = 1. \tag{3.19}$$

Simple calculations indicate that this projection operator meets, for $i = 0, 1$, the following properties:

$$\|P^k\chi - \chi\|_i \le \|\zeta^k - \chi\|_i, \quad \forall \zeta^k \in V^k, \tag{3.20}$$

$$\|P^k\chi\|_i \le \|\chi\|_i, \quad \forall \chi \in H^1(\mathcal{R}). \tag{3.21}$$

Since V^k is a dense in $H^1(\mathcal{R})$ and the injection of $H^1(\mathcal{R})$ in to $L^2(\mathcal{R})$ is compact (see [20] page 140) it follows that

$$P^k\chi \to \chi \ \ in \ \ L^2(\mathcal{R}). \tag{3.22}$$

By taking $\eta^k = y_j$ in (3.14) we have, for $j = 1, 2, ..., k$, and that

$$(\partial_t c_j^k(t) y_j, y_j) = -\Big(\nabla\big(\sum_{j=1}^{k} d_j^k(t) y_j\big), \nabla y_j\Big) + \Big(F\big(\sum_{j=1}^{k} c_j^k(t) y_j\big), y_j\Big),$$

then by using (3.10), (3.11), (3.12), and (3.13) we can rewrite the above equations as a coupled system of first order differential equations for $j = 1, 2, ..., k$, in the following form:

$$\partial_t c_j^k(t) = -\sum_{j=1}^{k} d_j^k(t) \int_{\mathcal{R}} \nabla y_j \nabla y_j dx + \Big(F\big(\sum_{j=1}^{k} c_j^k(t) y_j\big), y_j\Big). \tag{3.23}$$

Furthermore, by selecting $\eta^k = y_j$ in (3.15), then we have, for $j = 1, 2, ..., k$, that:

$$(w^k, \eta^k) = k_1(\nabla(\sum_{j=1}^{k} y_j), \nabla y_j) + (\sum_{j=1}^{k} y_j, y_j),$$

then, using (3.10), (3.11), (3.12), and (3.13), it follows that

$$d_j^k(t) = k_1 \lambda_j c_j^k(t) + \delta_j c_j^k(t). \tag{3.24}$$

Since $u^k(.,0) = P^k u^0$, then we have, from (3.12) and (3.17), that

$$c_j^k(t)(0) = (u_0, y_j)_{L^2(\mathcal{R})}. \tag{3.25}$$

Lemma 3.1 *For any $\psi \in H^1(\mathcal{D})$, it holds that:*

$$(\nabla(P^k\psi), \nabla\eta^k) = (\nabla\psi, \nabla\eta^k) \quad \forall \eta^k \in V^k. \tag{3.26}$$

A straightforward calculation demonstrates that this projection operator fulfills the following characteristics:

$$\|\nabla P^k \psi\|_0 \leq \|\nabla\psi\|_0 \quad \forall \psi \in H^1(\mathcal{R}). \tag{3.27}$$

The initial approximations are structured as follows:

$$u^k(\cdot, 0) := P^k u_0^k, \quad w^k(\cdot, 0) := P^k w_0^k, \tag{3.28}$$

where the following property is satisfied:

$$\{u_0^k, w_0^k\} \longmapsto \{u_0, w_0\} \quad \text{in} \quad L^2(\mathcal{R}) \quad \text{as} \quad k \longmapsto \infty. \tag{3.29}$$

We can express equations (3.14) and (3.15) as a set of ordinary differential equations involving the functions $c_{ik}(t)$ and $d_{ik}(t)$. This system of ordinary differential equations can be represented in the following composite form:

$$\frac{du^k}{dt} = \Delta u^k + P^k f(u^k), \qquad u^k(\cdot, 0) := P^k u_0^k, \tag{3.30}$$

$$w^k = k_1 \Delta u^k + k_2 u^k. \tag{3.31}$$

Our next step is to demonstrate the local Lipschitz continuity of the nonlinearity within the system of ordinary differential equations. We will focus on the function F as follows:

$$|F(u_1) - F(u_2)| \leq L_1(u_1, u_2)\Big[|u_1 - u_2| + |u_1 - u_2|\Big], \tag{3.32}$$

where $L_1(u_1, u_2) = \frac{3}{2}(|u_1|^2 + |u_2|^2) + 1$. As a result, we can establish that F exhibits local Lipschitz continuity. Leveraging the local existence theorem, often referred to as Picard's Theorem (for details, refer to [28], p. 9), we can conclude that the system of ordinary differential equations possesses a unique solution denoted as u^k, w^k, defined over a finite time interval $(0, t_k)$ with $t_k > 0$.

3.1. Global existence

Now. we will show the existence of a global solution, and to achieve that, all we need is some a priori estimations of u^k, w^k, which are independent of k.

Firstly, we choosing $\eta^k = k_1 u^k$ and $\eta^k = w^k$ in (3.14) and (3.15) respectively, we have that

$$k_1(\partial_t u^k, u^k) = -k_1(\nabla w^k, \nabla u^k) + k_1(F(u^k), u^k) = -k_1(\nabla w^k, \nabla u^k) + k_1(u^k, u^k) - k_1((u^k)^3, u^k), \quad (3.33)$$

and

$$(w^k, w^k) = k_1(\nabla u^k, \nabla w^k) + k_2(u^k, w^k). \quad (3.34)$$

From (3.33) and (3.34), it follows, on noting Young's inequality, that

$$\frac{k_1}{2}\frac{d}{dt}\|u^k\|_0^2 + k_1\|u^k\|_{0,4}^4 + \frac{1}{2}\|w^k\|_0^2 \le \frac{3k_1}{2}\|u^k\|_0^2. \quad (3.35)$$

By integrating the above equation over $t \in (0,T)$, we can find that

$$\frac{k_1}{2}\|u^k\|_0^2 + k_1\int_0^T \|u^k\|_{0,4}^4 dt + \frac{1}{2}\int_0^T \|w^k\|_0^2 dt \le C. \quad (3.36)$$

From the above inequality, we deduce that u^k and w^k are uniformly bounded in $L^4(\mathcal{R}_T)\bigcap L^\infty(0,T;L^2(\mathcal{R}))$ and $L^2(\mathcal{R}_T)$, respectively.

By selecting $\eta^k = u^k$ in (3.15), and using Young's inequality, we can find

$$k_1\|\nabla u^k\|_0^2 + \frac{k_2}{2}\|u^k\|_0^2 \le \frac{1}{2}\|w^k\|_0^2. \quad (3.37)$$

By integrating the both hand side of (3.37) over $t \in (0,T)$, and utilising last bound in (3.36), we have that

$$\min\{k_1, \frac{k_2}{2}\}\int_0^T [\|\nabla u^k\|_0^2 + \|u^k\|_0^2]dt \le \frac{1}{2}\int_0^T \|w^k\|_0^2 dt \le C. \quad (3.38)$$

Therefore, we have that u^k is uniformly bounded in $L^2(0,T;H^1(\mathcal{R}))$.

Now, we choosing $\eta^k = w^k$ in (3.14) and $\eta^k = F(u^k), \partial_t u^k$ in (3.15), to find that

$$(\partial_t u^k, w^k) = -(\nabla w^k, \nabla w^k) + (F(u^k), w^k), \quad (3.39)$$

and

$$(\partial_t u^k, w^k) = k_1(\nabla u^k, \nabla \partial_t u^k) + k_2(u^k, \partial_t u^k). \quad (3.40)$$

Then, by combining (3.39) and (3.40), it follows that

$$\frac{k_1}{2}\frac{d}{dt}\|\nabla u^k\|_0^2 + \frac{k_2}{2}\frac{d}{dt}\|u^k\|_0^2 + \|\nabla w^k\|_0^2 + k_1\|u^k\nabla u^k\|_0^2 + k_2\|u^k\|_{0,4}^4 = k_1\|\nabla u^k\|_0^2 + k_2\|u^k\|_0^2, \quad (3.41)$$

by integrating (3.41) over $t \in (0,T)$ and using (3.38), we have that

$$\frac{k_1}{2}\|\nabla u^k\|_0^2 + \frac{k_2}{2}\|u^k\|_0^2 + \int_0^T \|\nabla w^k\|_0^2 dt + k_1\int_0^T \|u^k\nabla u^k\|_0^2 dt + k_2\int_0^T \|u^k\|_{0,4}^4 dt$$
$$\le C + \frac{k_1}{2}\|\nabla u^k(0)\|_0^2 + \frac{k_2}{2}\|u^k(0)\|_0^2 \le C. \quad (3.42)$$

From first and second bounds in (3.1), it follows that

$$\min\{\frac{k_1}{2}, \frac{k_2}{2}\}\big(\|\nabla u^k\|_0^2 + \|u^k\|_0^2\big) \le C. \quad (3.43)$$

Also, from third bound of (3.1) together with the last bound of (3.36), we obtain that w^k is uniformly bounded in $L^2(0,T;H^1(\mathcal{R}))$,

$$\|w^k\|_{L^2(0,T;H^1(\mathcal{R}))} \le C. \quad (3.44)$$

3.2. Passage to the limit

Recall that $L^1(0,T;H^1(\mathcal{R}))$ is a separable Banach space, but not reflexive, while the Banach spaces $L^2(0,T;H^1(\mathcal{R})), L^2(\mathcal{R}_T)$ and $L^4(\mathcal{R}_T)$ are reflexive. Thus from classical compactness arguments (see [20] Theorems 4, 5), from the uniformly bounded sequences of functions $\{u^k\}_{k=1}^{\infty}$ and $\{w^k\}_{k=1}^{\infty}$ we can extract convergent subsequences, still denoted $\{u^k\}$, $\{w^k\}$, such that

$$\{u^k\} \rightharpoonup \{u\} \quad \text{in} \quad L^2(0,T;H^1(\mathcal{R})) \cap L^2(\mathcal{R}_T) \cap L^4(\mathcal{R}_T) \quad \text{as} \quad k \to \infty, \tag{3.45}$$

$$\{w^k\} \rightharpoonup \{w\} \quad \text{in} \quad L^2(0,T;H^1(\mathcal{R})) \cap L^2(\mathcal{R}_T) \quad \text{as} \quad k \to \infty, \tag{3.46}$$

and

$$\{u^k\} \rightharpoonup^* \{u\} \quad \text{in} \quad L^\infty(0,T;H^1(\mathcal{R})) \quad \text{as} \quad k \to \infty, \tag{3.47}$$

where '$\rightharpoonup$' and '$\rightharpoonup^*$' represent weak and weak-star convergence, respectively. We demonstrate the process of taking the limit for the terms in the initial composite Galerkin approximation (3.30). Let us focus on the term $P^kF(u^k)$, and it can be readily proven that

$$|F(u^k)| \leq C\Big(|u^k|^3 + |u^k|\Big), \tag{3.48}$$

then we find that

$$\int_0^T \int_{\mathcal{R}} \big|F(u^k)\big|^{\frac{4}{3}} dxdt \leq C \int_0^T \int_{\mathcal{R}} \Big(|u^k|^4 + |u^k|^{\frac{4}{3}}\Big) dxdt. \tag{3.49}$$

On noting the bounds (3.45), and the injections, $L^4(\mathcal{R}_T) \hookrightarrow L^{\frac{4}{3}}(\mathcal{R}_T)$, we have that $F(u^k)$ is uniformly bounded in $L^{\frac{4}{3}}(\mathcal{R}_T)$ and so from weak compactness arguments there exists some $\zeta \in L^{\frac{4}{3}}(\mathcal{R}_T)$ such that

$$F(u^k) \rightharpoonup \zeta \text{ in } L^{\frac{4}{3}}(\mathcal{R}_T) \quad \text{as } k \to \infty. \tag{3.50}$$

We will demonstrate that $P^kF(u^k)$ also converges weakly to ζ in the $L^{\frac{4}{3}}(\mathcal{R}_T)$ space. To do this, we define Q^k as $I - P^k$, which represents the orthogonal projection with respect to P^k. Now, let's recall that $(P^k\psi, \vartheta^k)_V = (\psi, \zeta^k)_V$ holds for all ζ^k in V^k and ψ in $H^1(\mathcal{R})$. This implies that $|P^k\psi - \psi|1 \leq |\psi - \zeta^k|_1$ for all ζ^k in V^k and ψ in $H^1(\mathcal{R})$. As V^k is dense in $H^1(\mathcal{R})$, we conclude that P^ku converges to u in $H^1(\mathcal{R})$ for all u in $H^1(\mathcal{R})$. In other words, Q^ku tends to 0 in $H^1(\mathcal{R})$ as k approaches infinity. Furthermore, we can assert that Q^ku converges to 0 in $L^4(\mathcal{R})$ for all u in $L^4(\mathcal{R})$, given that H^1 is embedded in $L^4(\mathcal{R}_T)$. Now, let us consider an arbitrary ξ in $L^4(\mathcal{R}_T)$. By utilizing Hölder's inequality and leveraging the orthogonality of Q^k, we can establish the following relationship:

$$\begin{aligned} \Big|\int_0^T (P^kF(u^k) - \zeta, \xi)dt\Big| &= \Big|\int_0^T [(F(u^k) - \zeta, \xi) - (F(u^k), Q^k\xi)]dt\Big| \\ \leq \Big|\int_0^T (F(u^k) - \zeta, \xi)dt\Big| &+ \int_0^T \|F(u^k)\|_{0,\frac{4}{3}} \|Q^k\xi\|_{0,4} dt \quad \longrightarrow \quad 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Upon observing the strong convergence of Q^k to 0 in the $L^4(\mathcal{R}_T)$, along with (3.50), we can conclude that

$$P^kf(u^k) \rightharpoonup \zeta \text{ in } L^{\frac{4}{3}}(\mathcal{R}_T) \quad \text{as } k \to \infty. \tag{3.51}$$

Noting (2.3) we have that $\Delta w^k \in L^2(0,T;(H^1(\mathcal{R})')$ and $P^kF(u^k) \in L^{\frac{4}{3}}(\mathcal{R}_T)$ it follows from (3.30) that $\frac{du^k}{dt}$ is uniformly bounded in $L^2(0,T;(H^1(\mathcal{R}))') + L^{\frac{4}{3}}(\mathcal{R}_T)$ and from weak compactness arguments $\frac{du^k}{dt}$ tends weakly to some $\dot{\eta}$ in $L^2(0,T;(H^1(\mathcal{R}))') + L^{\frac{4}{3}}(\mathcal{R}_T)$. From the uniqueness of the weak convergence, it follows that $\dot{\eta} = \frac{du}{dt}$, i.e.

$$\frac{du^k}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T;(H^1(\mathcal{R}))') + L^{\frac{4}{3}}(\mathcal{R}_T) \quad \text{as } k \to \infty. \tag{3.52}$$

First, recall from(3.45) that $\{u^k\} \rightharpoonup \{u\}$ in the space $L^2(0,T;H^1(\mathcal{R})) \cap L^4(\mathcal{R}_T)$ with dual space $L^2(0,T;(H^1(\mathcal{R}))') + L^{\frac{4}{3}}(\mathcal{R}_T)$. Moreover, from the Sobolev embedding theorem and the fact that $H^1(\mathcal{R})$ is dense in $L^2(\mathcal{R})$, we have the dense inclusion $H^1(\mathcal{R}) \hookrightarrow L^4(\mathcal{R})$, thus $L^{\frac{4}{3}}(\mathcal{R}) \hookrightarrow (H^1(\mathcal{R}))'$, and so $L^2(0,T;(H^1(\mathcal{R}))')+L^{\frac{4}{3}}(\mathcal{R}_T) \subset L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R}))')$. Now consider an arbitrary $\xi(t) \in C_0^\infty(0,T;H^1(\mathcal{R})) \subset L^{\frac{4}{3}}(0,T;H^1(\mathcal{R}))$. Integrating by parts and using the weak convergence of u^k to u in $L^2(0,T;(H^1(\mathcal{R}))')+L^{\frac{4}{3}}(\mathcal{R}_T)$ and hence $L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R}))')$ yields

$$\int_0^T (\frac{du^k}{dt}, \xi)dt = -\int_0^T (u^k, \frac{d\xi}{dt})dt \to -\int_0^T (u, \frac{d\xi}{dt})dt = \int_0^T (\frac{du}{dt}, \xi)dt,$$

after noting $\frac{d\xi}{dt} \in C_0^\infty(0,T;H^1(\mathcal{R}))$. From the weak convergence of $\frac{du^k}{dt}$ to $\dot{\eta}$ in $L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R}))')$, we have

$$\int_0^T (\frac{du^k}{dt}, \xi)dt \to \int_0^T (\dot{\eta}, \xi)dt,$$

and so by the uniqueness of weak limits we have $\frac{du}{dt} = \dot{\eta}$, i.e. $\frac{du^k}{dt} \rightharpoonup \frac{du}{dt}$ in $L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R}))')$ as $k \to \infty$.

Now we take $\phi \in L^2(0,T;H^1(\mathcal{R}))$ and consider

$$\int_0^T (\Delta w^k, \phi)dt = \int_0^T (\Delta\phi, w^k)dt \to \int_0^T (\Delta\phi, w)dt, \quad ask \to \infty. \tag{3.53}$$

Since $L^2(0,T;H^1(\mathcal{R}))$ and $L^2(0,T;(H^1(\mathcal{R}))')$ are dual space and due to the weak convergence of w^k to w in $L^2(0,T;H^1(\mathcal{R}))$, so, we have that $\Delta\phi \in L^2(0,T;(H^1(\mathcal{R}))')$. However,

$$\int_0^T (\Delta\phi, w)dt = \int_0^T (\Delta w, \phi)dt. \tag{3.54}$$

From (3.54) and (3.53) we have that

$$\int_0^T (\Delta w^k, \phi)dt = \int_0^T (\Delta\phi, w^k)dt \to \int_0^T (\Delta\phi, w)dt = \int_0^T (\Delta w, \phi)dt, \quad as \;\; k \to \infty, \tag{3.55}$$

then, we arrive at

$$\int_0^T (\Delta w^k, \phi)dt \to \int_0^T (\Delta w, \phi)dt, \quad as \;\; k \to \infty. \tag{3.56}$$

Hence, we have successfully established the necessary convergence of all terms in $L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R})')$. To prove that $\zeta = F(u)$ in equation (3.50), we utilize classical theorems. By applying the Lions-Aubin theorem [36] with the following:

$$\mathfrak{V} = \{\eta : \eta \in L^2(0,T;H^1(\mathcal{R})); \frac{d\eta}{dt} \in L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R})')\} \overset{c}{\hookrightarrow} L^2(\mathcal{R}_T).$$

As $w^k \in \mathfrak{V}$ belongs to the set v, we can identify a subsequential sequence, still denoted as w^k, such that w^k converges to w in the $L^2(\mathcal{R}_T)$ space. Consequently, w^k also converges to w 'pointwise' almost everywhere in $\mathcal{R}_T$. Since the function F is locally Lipschitz within $\mathcal{R}_T$, this implies, due to its continuity, that $F(u^k)$ converges to $F(u)$ 'pointwise' almost everywhere in $\mathcal{R}_T$. Applying Lemma 1.3 from Lions' work [36] leads to the following result:

$$F(u^k) \rightharpoonup F(u) \in L^{\frac{4}{3}}(\mathcal{R}_T). \tag{3.57}$$

In the same way of (3.56) we can prove $\Delta u^k \rightharpoonup \Delta u$ in the space $L^2(0,T;(H^1(\mathcal{R})') \subset L^{\frac{4}{3}}(0,T;(H^1(\mathcal{R}))')$. By taking $\phi \in L^2(0,T;H^1(\mathcal{R}))$ and thus $\Delta\phi \in L^2(0,T;(H^1(\mathcal{R}))')$, since $L^2(0,T;H^1(\mathcal{R}))$ and $L^2(0,T;(H^1(\mathcal{R}))')$ are dual space and due to the weak convergence of u^k to u in $L^2(0,T;H^1(\mathcal{R}))$, we have that

$$\int_0^T (\Delta u^k, \phi)dt = \int_0^T (\Delta\phi, u^k)dt \to \int_0^T (\Delta\phi, u)dt = \int_0^T (\Delta u, \phi)dt, \quad \text{as} \;\; k \to \infty. \tag{3.58}$$

Therefor, we have show that

$$\int_0^T (\Delta u^k, \phi)dt \to \int_0^T (\Delta u, \phi)dt, \quad \text{as} \quad k \to \infty. \tag{3.59}$$

Then, we have shown that

$$\frac{du^k}{dt} - \Delta w^k - P^k F(u^k) \rightharpoonup \frac{du}{dt} - \Delta w - F(u) \quad in \quad L^2(0,T;(H^1(\mathcal{R}))'),$$

and

$$w^k - \Delta u^k - u^k \rightharpoonup w - \Delta u - u, \quad \text{in} \;\; L^2(0,T;(H^1(\mathcal{R}))'),$$

which means for all $\phi \in L^2(0,T;H^1(\mathcal{R}))$

$$\int_0^T \Big(\frac{du^k}{dt} - \nabla w^k - P^k F(u^k), \phi\Big)dt \to \int_0^T \Big(\frac{du}{dt} - \nabla w - F(u), \phi\Big)dt, \tag{3.60}$$

and

$$\int_0^T \Big(w^k - \nabla u^k - u^k, \phi\Big)dt \to \int_0^T \Big(w - \nabla u - u, \phi\Big)dt. \tag{3.61}$$

To establish that $u \in C([0,T];L^2(\mathcal{R}))$, we rely on a modified version of another classical result from [41]. We have already demonstrated that $u \in L^2(0,T;H^1(\mathcal{R})) \cap L^4(\mathcal{R}_T)$ and $\frac{du}{dt} \in L^2(0,T;(H^1(\mathcal{R}))') + L^{\frac{4}{3}}(\mathcal{R}_T)$. Moreover, since $L^2(0,T;(H^1(\mathcal{R}))') + L^{\frac{4}{3}}(\mathcal{R}_T)$ be the dual spaces of $L^2(0,T;H^1(\mathcal{R})) \cap L^4(\mathcal{R}_T)$, therefore, we conclude that $u \in C([0,T];L^2(\mathcal{R}))$.

3.3. Uniqueness

To prove uniqueness we assume there are two solutions u_1, u_2, and w_1, w_2 of the weak form (3.1) and (3.2), with initial conditions $u_1(.,0) = u_{1,0}(.), u_2(.,0) = u_{2,0}(.)$, and $w_1(.,0) = w_{1,0}(.), w_2(.,0) = w_{2,0}(.)$, respectively. Setting $\omega_1 = u_1 - u_2$, and $\omega_2 = w_1 - w_2$, and setting $\eta = k_1\omega_1$, $\eta = \omega_2$ in (3.1) and (3.2), subtracting weak forms and using subtracting weak forms and Young's inequality, leads to,

$$\frac{k_1}{2}\frac{d}{dt}\|\omega_1\|_0^2 + \frac{1}{2}\|\omega_2\|_0^2 + k_1((u_1)^3 - (u_2)^3, \omega_1) \le \frac{3k_1}{2}\|\omega_1\|_0^2. \tag{3.62}$$

By using Young's inequality, we find that

$$\begin{aligned} ({u_1}^3 - {u_2}^3, \omega_1) &= ((u_1^2 + u_1u_2 + u_2^2)\omega_1, \omega_1) \\ = \int_{\mathcal{R}} (u_1^2 + u_1u_2 + u_2^2)\omega_1^2 dx &\ge \int_{\mathcal{R}} (u_1^2 - \frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_2^2)\omega_1^2 dx \\ &= \frac{1}{2}\int_{\mathcal{R}} (u_1^2 + u_2^2)\omega_1^2 dx. \end{aligned} \tag{3.63}$$

By substituting (3.3) in (3.62), we have

$$\frac{k_1}{2}\frac{d}{dt}\|\omega_1\|_0^2 + \frac{1}{2}\|\omega_2\|_0^2 + \frac{k_1}{2}\int_{\mathcal{R}} (u_1^2 + u_2^2)\omega_1^2 \le \frac{3k_1}{2}\|\omega_1\|_0^2. \tag{3.64}$$

By excluding the last term on the left-hand side of inequality (3.64) and subsequently multiplying the outcomes by 2, we reach

$$k_1\frac{d}{dt}\|\omega_1\|_0^2 + \|\omega_2\|_0^2 \le C\|\omega_1\|_0^2, \tag{3.65}$$

by Application of Grönwall lemma gives

$$\|\omega_1\|_0^2 + \int_0^T \|\omega_2\|_0^2 dt \le \exp(CT)\|\omega_1(0)\|_0^2. \tag{3.66}$$

Thus, if $(u_1(0), w_1(0)) = (u_2(0), w_2(0))$, we deduce uniqueness $u_1(t) = u_2(t)$ and $w_1(t) = w_2(t)$ for all t. However, if $(u_1(0), w_1(0)) \neq (u_2(0), w_2(0))$, then we have continuous dependence in $L^2(\mathcal{R})$. □

4. Regularity

Theorem 4.1 *For $\mathcal{R}$ sufficiently smooth, we have the following regularity results:*

$$\|\partial_t u^k\|_{L^2(\mathcal{R}_T)} \leq C, \tag{4.1}$$

$$u^k \in L^2(0,T;H^2(\mathcal{R}))\bigcap L^\infty(0,T,H^2(\mathcal{R}))\bigcap L^\infty(0,T;L^4(\mathcal{R})) \text{ and } \frac{\partial u}{\partial \nu}=0 \textit{ on } \partial\mathcal{R} \textit{ for a.e } t., \tag{4.2}$$

$$w^k \in L^2(0,T;H^2(\mathcal{R})). \tag{4.3}$$

Proof: Now, we choosing $\eta^k = \Delta u^k$ in (3.15) we get

$$k_1(\nabla u^k, \nabla\Delta u^k) + k_2(u^k, \Delta u^k) = (w^k, \Delta u^k). \tag{4.4}$$

We integrate by parts on (4.4) and apply Hölder and Young's inequality to the right hand of the result, we have that

$$k_1\|\Delta u^k\|_0^2 + \frac{k_2}{2}\|\nabla u^k\|_0^2 \leq \frac{1}{2}\|\nabla w^k\|_0^2. \tag{4.5}$$

By integrating (4.5) over $t \in (0,T)$, and noting the third bound of (3.1), we can show that

$$k_1\int_0^T \|\Delta u^k\|_0^2 dt + \frac{k_2}{2}\int_0^T \|\nabla u^k\|_0^2 dt \leq C. \tag{4.6}$$

From (3.38) and (4.6), it follows that

$$\|u^k\|_{L^2(0,T,H^2(\mathcal{R}))} \leq C. \tag{4.7}$$

Now, we choosing $\eta^k = k_1\Delta^2 u^k$ and $\eta^k = \Delta^2 w^k$ in (3.14) and (3.15) respectively, we get

$$k_1(\partial_t u^k, \Delta^2 u^k) = -k_1(\nabla w^k, \nabla\Delta^2 u^k) + k_1(F(u^k), \Delta^2 u^k), \tag{4.8}$$

and

$$(w^k, \Delta^2 w^k) = k_1(\nabla u^k, \nabla\Delta^2 w^k) + k_2(u^k, \Delta^2 w^k). \tag{4.9}$$

We also integrate by parts the terms of (1.6) to find that

$$k_1(\nabla\Delta w^k, \nabla\Delta u^k) = -k_1(\partial_t\Delta u^k, \Delta u^k) + k_1(\Delta u^k, \Delta u^k) - k_1((u^k)^3, \Delta^2 u^k), \tag{4.10}$$

and

$$k_1(\nabla\Delta u^k, \nabla\Delta w^k) = (\Delta w^k, \Delta w^k) - k_2(\Delta u^k, \Delta w^k). \tag{4.11}$$

From (4.10) and (4.12) and noting Hölder's and Young's inequality, we conclude that

$$\frac{k_1}{2}\frac{d}{dt}\|\Delta u^k\|_0^2 + \frac{1}{2}\|\Delta w^k\|_0^2 \leq \frac{3k_1}{2}\|\Delta u^k\|_0^2 - k_1((u^k)^3, \Delta^2 u^k). \tag{4.12}$$

By selecting $\eta^k = \Delta(u^k)^3$ in (3.15), we deduce that

$$\begin{aligned}
-k_1(\Delta^2 u^k, (u^k)^3) &= (\Delta w^k, (u^k)^3) + k_2\|u^k\nabla u^k\|_0^2 \\
&\leq \|\Delta w^k\|_0\|u^k\|_{0,6}^3 + k_2\|u^k\nabla u^k\|_0^2 \\
&\leq \|\Delta w^k\|_0\|u^k\|_1^3 + k_2\|u^k\nabla u^k\|_0^2 \\
&\leq \frac{1}{4}\|\Delta w^k\|_0^2 + \|u^k\|_1^6 + k_2\|u^k\nabla u^k\|_0^2.
\end{aligned} \tag{4.13}$$

Substitution (4) in (4.12), leads to

$$\frac{k_1}{2}\frac{d}{dt}\|\Delta u^k\|_0^2 + \frac{1}{4}\|\Delta w^k\|_0^2 \leq \frac{3k_1}{2}\|\Delta u^k\|_0^2 + \|u^k\|_1^6 + k_2\|u^k\nabla u^k\|_0^2. \tag{4.14}$$

By integrating (4.14) over $t \in (0,T)$, and using the first bound of (4.6), fourth bound of (3.1), (3.43) and the embedding $L^\infty(0,T,H^1(\mathcal{R})) \hookrightarrow L^6(0,T,H^1(\mathcal{R}))$, we can show that

$$\frac{k_1}{2}\|\Delta u^k\|_0^2 + \frac{1}{4}\int_0^T \|\Delta w^k\|_0^2 dt \leq \frac{3k_1}{2}\int_0^T \|\Delta u^k\|_0^2 dt + \int_0^T \|u^k\|_1^6 dt + k_2\int_0^T \|u^k\nabla u^k\|_0^2 dt \leq C. \quad (4.15)$$

From (3.43) and the first bound of (4.15), we have that

$$\|u^k\|_{L^\infty(0,T,H^2(\mathcal{R}))} \leq C. \quad (4.16)$$

From (3.44) and the second bound of (4.15), we find that

$$\|w^k\|_{L^2(0,T,H^2(\mathcal{R}))} \leq C. \quad (4.17)$$

Now, we choose $\eta^k = \partial_t u^k$ in (3.14), to obtain that

$$(\partial_t u^k, \partial_t u^k) = -(\nabla w^k, \nabla \partial_t u^k) + (F(u^k), \partial_t u^k). \quad (4.18)$$

By integrating by parts the terms of (4.18) and noting (1.6), we have that

$$\frac{1}{2}\|\partial_t u^k\|_0^2 + \frac{1}{4}\frac{d}{dt}\|u^k\|_{0,4}^4 = \frac{1}{2}\|\Delta w^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|u^k\|_0^2. \quad (4.19)$$

By integrating (4.19) over $t \in (0,T)$, and noting the second bound of (4.15) and the second bound of (3.1), we can show that

$$\frac{1}{2}\int_0^T \|\partial_t u^k\|_0^2 dt + \frac{1}{4}\|u^k\|_{0,4}^4 = \frac{1}{2}\int_0^T \|\Delta w^k\|_0^2 dt + \frac{1}{2}\|u^k\|_0^2 \leq C. \quad (4.20)$$

From first bound of (4.20) we have

$$\|\partial_t u^k\|_{L^2(\mathcal{R}_T)} \leq C. \quad (4.21)$$

It follows from the second bound of (4.20), that

$$\|u^k\|_{L^\infty(0,T;L^4(\mathcal{R}))} \leq C. \quad (4.22)$$

Thus, the result of (4.7), (4.16), (4.17), (4.21) and (4.22). This completes the proof of Theorem 4.1 □

4.1. Passage to the limit of the strong solution

Applying classical compactness arguments, as outlined in [20] Theorems 4 and 5, from the uniformly bounded sequences of functions ${u^k}_{k=1}^\infty$ and ${w^k}_{k=1}^\infty$, we can extract convergent subsequences denoted as u^k and w^k, such that

$$\{u^k\} \rightharpoonup \{u\} \quad \text{in} \quad L^2(0,T;H^2(\mathcal{R})) \quad \text{as} \quad k \to \infty, \quad (4.23)$$

$$\{w^k\} \rightharpoonup \{w\} \quad \text{in} \quad L^2(0,T;H^2(\mathcal{R})) \cap L^2(\mathcal{R}_T) \quad \text{as} \quad k \to \infty, \quad (4.24)$$

$$\{\partial_t u^k\} \rightharpoonup \{\partial_t u\} \quad \text{in} \quad L^2(\mathcal{R}_T) \quad \text{as} \quad k \to \infty, \quad (4.25)$$

and

$$\{u^k\} \rightharpoonup^* \{u\} \quad \text{in} \quad L^\infty(0,T;L^4(\mathcal{R})) \cap L^\infty(0,T,H^2(\mathcal{R})) \quad \text{as} \quad k \to \infty. \quad (4.26)$$

We demonstrate the process of taking the limit of the terms in the composite Galerkin approximation (3.30). It is easy to show that

$$|F(u^k)| \leq C\Big(|u^k|^3 + |u^k|\Big). \quad (4.27)$$

Subsequently, we determine that

$$\int_0^T \int_{\mathcal{R}} \left|F(u^k)\right|^2 dxdt \leq C \int_0^T \int_{\mathcal{R}} \Big(|u^k|^6 + |u^k|^2\Big) dxdt. \tag{4.28}$$

It follows from (4.23), and the injections, $L^2(0,T;H^1(\mathcal{R})) \hookrightarrow L^6(\mathcal{R}_T)$, that $F(u^k)$ is uniformly bounded in $L^2(\mathcal{R}_T)$ and so from weak compactness arguments there exists some $\zeta \in L^2(\mathcal{R}_T)$ such that

$$F(u^k) \rightharpoonup \zeta \text{ in } L^2(\mathcal{R}_T) \quad \text{as } k \to \infty. \tag{4.29}$$

We demonstrate that $P^k F(u^k)$ also converges weakly to ζ in $L^2(\mathcal{R}_T)$. To do this, let's define $Q^k := I - P^k$ as the orthogonal projection to P^k. Now, recall the equation $(P^k\psi, \vartheta^k)_V = (\psi, \zeta^k)_V$ for all $\zeta^k \in V^k$ and $\psi \in H^1(\mathcal{R})$. This implies that $\|P^k\psi - \psi\|_1 \leq \|\psi - \zeta^k\|_1$ for all $\zeta^k \in V^k$ and $\psi \in H^1(\mathcal{R})$. As V^k is dense in $H^1(\mathcal{R})$, we conclude that $P^k u \longrightarrow u$ in $H^1(\mathcal{R})$ for all $u \in H^1(\mathcal{R})$. In other words, $Q^k u \longrightarrow 0$ in $H^1(\mathcal{R})$ as $k \longrightarrow \infty$. We also have $H^1 \hookrightarrow L^2(\mathcal{R}_T)$ and so $Q^k u \longrightarrow 0$ in $L^2(\mathcal{R})$ for all $u \in L^2(\mathcal{R})$. When considering an arbitrary $\xi \in L^2(\mathcal{R}_T)$, we can utilize Hölder's inequality and take advantage of the orthogonality of Q^k, resulting in the following:

$$\begin{aligned} \Big|\int_0^T (P^k F(u^k) - \zeta, \xi) dt\Big| &= \Big|\int_0^T [(F(u^k) - \zeta, \xi) - (F(u^k), Q^k\xi)] dt\Big| \\ \leq \Big|\int_0^T (F(u^k) - \zeta, \xi) dt\Big| &+ \int_0^T \|F(u^k)\|_0 \|Q^k\xi\|_0 dt \longrightarrow 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

On noting the strong convergence of Q^k to 0 in $L^2(\mathcal{R})$ and (4.29), we deduce that

$$P^k f(u^k) \rightharpoonup \zeta \text{ in } L^2(\mathcal{R}_T) \quad \text{as } k \to \infty. \tag{4.30}$$

Now, since $u^k \in L^2(0,T;H^2(\mathcal{R}))$ then we get $u^k \in L^2(\mathcal{R}_T)$ and $w^k \in L^2(\mathcal{R}_T)$ and these space is dual to it self then, by using (2.3) we have that

$$\Delta u^k, \quad \Delta w^k \in L^2(\mathcal{R}_T). \tag{4.31}$$

Then, from (4.24), (4.25), (4.30) and (4.31), we have shown that

$$\frac{du^k}{dt} - \Delta w^k - P^k F(u^k) \rightharpoonup \frac{du}{dt} - \Delta w - F(u) \quad in \quad L^2(\mathcal{R}_T),$$

and

$$w^k - \Delta u^k - u^k \rightharpoonup w - \Delta u - u, \quad in\ L^2(\mathcal{R}_T),$$

which means for all $\phi \in L^2(\mathcal{R}_T)$

$$\int_0^T \Big(\frac{du^k}{dt} - \nabla w^k - P^k F(u^k), \phi\Big) dt \to \int_0^T \Big(\frac{du}{dt} - \nabla w - F(u), \phi\Big) dt, \tag{4.32}$$

and

$$\int_0^T \Big(w^k - \nabla u^k - u^k, \phi\Big) dt \to \int_0^T \Big(w - \nabla u - u, \phi\Big) dt. \tag{4.33}$$

4.2. Continuous dependence

Assume $\{u_1, w_1\}$ and $\{u_2, w_2\}$ satisfy the weak form (3.4) and (3.5), with initial conditions $u_1(.,0) = u_{1,0}(.), u_2(.,0) = u_{2,0}(.)$, and $v_1(.,0) = v_{1,0}(.), w_2(.,0) = w_{2,0}(.)$, respectively, such that $u_{1,0}(.) \neq u_{2,0}(.)$, and $w_{1,0}(.) \neq w_{2,0}(.)$. Setting $\omega_1 = u_1 - u_2$, and $\omega_2 = w_1 - w_2$, and setting $\eta = k_1(-\Delta\omega_1 + \omega_1)$ and $\eta = -\Delta\omega_2 + \omega_2$ in (3.4) and (3.5), subtracting weak forms leads after integrating by parts to,

$$k_1(\Delta\omega_1, \Delta\omega_2) + k_1(\nabla\omega_1, \nabla\omega_2)$$

$$= -k_1(\partial_t\nabla\omega_1,\nabla\omega_1)-k_1(\partial_t\omega_1,\omega_1)+k_1(\nabla\omega_1,\nabla\omega_1)+k_1(\omega_1,\omega_1)+k_1(u_1^3-u_2^3,\Delta\omega_1)-k_1(u_1^3-u_2^3,\omega_1), \quad (4.34)$$

and

$$(\nabla\omega_2,\nabla\omega_2)+(\omega_2,\omega_2)=k_1(\Delta\omega_1,\Delta\omega_2)+k_1(\nabla\omega_1,\nabla\omega_2)+k_2(\nabla\omega_1,\nabla\omega_2)+k_2(\omega_1,\omega_2). \quad (4.35)$$

From (4.2) and (4.35), we have that

$$\frac{k_1}{2}\frac{d}{dt}\|\nabla\omega_1\|_0^2+\frac{k_1}{2}\frac{d}{dt}\|\omega_1\|_0^2+\frac{3}{4}\|\nabla\omega_2\|_0^2+\frac{3}{4}\|\omega_2\|_0^2+k_1(u_1^3-u_2^3,\omega_1)$$
$$\le \|\nabla\omega_1\|_0^2+\|\omega_1\|_0^2+k_1(u_1^3-u_2^3,\Delta\omega_1)+\frac{k_2}{2}\|\nabla\omega_1\|_0^2+\frac{k_2}{4}\|\omega_1\|_0^2. \quad (4.36)$$

Now, we choose $\eta=-\Delta\omega_1+\omega_1$ in (3.5), then by using Young's inequality, we find that

$$k_1\|\Delta\omega_1\|_0^2+k_1\|\nabla\omega_1\|_0^2+k_2\|\nabla\omega_1\|_0^2+k_2\|\omega_1\|_0^2\le\frac{k_1}{2}\|\nabla\omega_1\|_0^2+\frac{1}{4}\|\nabla\omega_2\|_0^2+\frac{k_2}{4}\|\omega_1\|_0^2+\frac{1}{4}\|\omega_2\|_0^2. \quad (4.37)$$

Summing (4.2) and (4.37), yields

$$\frac{k_1}{2}\frac{d}{dt}\|\nabla\omega_1\|_0^2+\frac{k_1}{2}\frac{d}{dt}\|\omega_1\|_0^2+\frac{1}{2}\|\nabla\omega_2\|_0^2+\frac{1}{2}\|\omega_2\|_0^2+k_1\|\Delta\omega_1\|_0^2+\frac{k_1}{2}\|\nabla\omega_1\|_0^2$$
$$+\frac{k_2}{2}\|\nabla\omega_1\|_0^2+\frac{k_2}{2}\|\omega_1\|_0^2+k_1(u_1^3-u_2^3,\omega_1)\le\|\nabla\omega_1\|_0^2+\|\omega_1\|_0^2+k_1(u_1^3-u_2^3,\Delta\omega_1). \quad (4.38)$$

We apply Young's inequality (1.6), Hölder's inequality, and the continuous injections of $H^1(\mathcal{D})\hookrightarrow L^4(\mathcal{D})$ for $d=1,2,3$, and $H^1(\mathcal{D})\hookrightarrow L^8(\mathcal{D})$ for $d=1,2$, to second and first terms on the right hand side of (4.2), respectively, to give

$$\begin{aligned}
&k_1\int_{\mathcal{R}}(u_1^3-u_2^3)\Delta\omega_1 dx\\
\le\ &\frac{k_1^2}{k_1}\int_{\mathcal{R}}(u_1^3-u_2^3)^2dx+\frac{k_1}{4}\int_{\mathcal{R}}|\Delta\omega_1|^2dx\\
\le\ &C\int_{\mathcal{R}}\omega_1^2(u_1^2+u_2^2)^2dx+\frac{k_1}{4}\int_{\mathcal{R}}|\Delta\omega_1|^2dx\\
\le\ &C\int_{\mathcal{R}}\omega_1^2(u_1^4+u_2^4)dx+\frac{k_1}{4}\int_{\mathcal{R}}|\Delta\omega_1|^2dx\\
\le\ &C\Big[\|u_1\|_{0,8}^4+\|u_2\|_{0,8}^4\Big]\|\omega_1\|_{0,4}^2+\frac{k_1}{4}\int_{\mathcal{R}}|\Delta\omega_1|^2dx\\
\le\ &C\Big[\|u_1\|_1^4+\|u_2\|_1^4\Big]\|\omega_1\|_1^2+\frac{k_1}{4}\int_{\mathcal{R}}|\Delta\omega_1|^2dx. \quad (4.39)
\end{aligned}$$

Substituting (3.3) and (4.39) into (4.2), leads to,

$$\frac{k_1}{2}\frac{d}{dt}\|\nabla\omega_1\|_0^2+\frac{k_1}{2}\frac{d}{dt}\|\omega_1\|_0^2+\frac{1}{2}\|\nabla\omega_2\|_0^2+\frac{1}{2}\|\omega_2\|_0^2+\frac{3k_1}{4}\|\Delta\omega_1\|_0^2+\frac{k_1}{2}\|\nabla\omega_1\|_0^2+\frac{k_2}{2}\|\nabla\omega_1\|_0^2$$
$$+\frac{k_2}{2}\|\omega_1\|_0^2+\frac{k_1}{2}\int_{\mathcal{R}}(u_1^2+u_2^2)\omega_1^2dx\le\|\nabla\omega_1\|_0^2+\|\omega_1\|_0^2+C\Big[\|u_1\|_1^4+\|u_2\|_1^4\Big]\|\omega_1\|_1^2. \quad (4.40)$$

By removing some positive terms from the left-hand side and then multiplying both sides by 2, we obtain the following:

$$k_1\frac{d}{dt}\|\nabla\omega_1\|_0^2+k_1\frac{d}{dt}\|\omega_1\|_0^2+\|\nabla\omega_2\|_0^2+\|\omega_2\|_0^2+\frac{3k_1}{2}\|\Delta\omega_1\|_0^2+k_1\|\nabla\omega_1\|_0^2+k_2\|\nabla\omega_1\|_0^2+k_2\|\omega_1\|_0^2$$
$$\le 2\|\nabla\omega_1\|_0^2+2\|\omega_1\|_0^2+2C\Big[\|u_1\|_1^4+\|u_2\|_1^4\Big]\|\omega_1\|_1^2$$

$$\leq C[1 + \|u_1\|_1^4 + \|u_2\|_1^4][k_1\|\nabla\omega_1\|_0^2 + k_1\|\omega_1\|_0^2]. \tag{4.41}$$

Utilizing the Grönwall lemma results in the following:

$$k_1|\omega_1(t)|_1^2 + k_1\|\omega_1(t)\|_0^2 + \int_0^T \|\nabla\omega_2\|_0^2 dt + \int_0^T \|\omega_2\|_0^2 dt + \frac{3k_1}{2}\int_0^T \|\Delta\omega_1\|_0^2 dt + k_1\int_0^T \|\nabla\omega_1\|_0^2 dt$$

$$+k_2\int_0^T \|\nabla\omega_1\|_0^2 dt + k_2\int_0^T \|\omega_1\|_0^2 dt$$

$$\leq \Big(|\omega_1(0)|_1^2 + \|\omega_1(0)\|_0^2\Big)\exp\left(2CT + 2C\int_0^T \Big[\|u_1\|_1^4 + \|u_2\|_1^4\Big]dt\right). \tag{4.42}$$

From the uniform bounds (3.43) and the continuous injections $L^\infty(0,T;H^1(\mathcal{R})) \hookrightarrow L^4(0,T;H^1(\mathcal{R}))$ for $d = 1, 2$, we deduce that

$$|\omega_1(t)|_1^2 + \|\omega_1(t)\|_0^2 + \int_0^T \|\nabla\omega_2\|_0^2 dt + \int_0^T \|\omega_2\|_0^2 dt \leq C\Big(|\omega_1(0)|_1^2 + \|\omega_1(0)\|_0^2\Big). \tag{4.43}$$

Thus if $(u_1(0), w_1(0)) = (u_2(0), w_2(0))$ then $(\omega_2(0), \omega_2(0)) = (0, 0)$, therefore, it can be inferred from equation (4.43) that $(\omega_2(t), \omega_2(t)) = (0, 0)$, which implies that $u_1(t) =_2 (t)$ and $w_1(t) = w_2(t)$ for all values of t. However, if $(u_1(0), w_1(0)) \neq (u_2(0), w_2(0))$, then we have continuous dependence in $H^1(\mathcal{R})$.□

Competing Interests

We have no competing interests.

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