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On nonnil-finite conductor rings

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ABSTRACT: Let R be a commutative ring with nonzero identity and let $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime idealof $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. In this paper, we introduce and investigate new generalizations of nonnil-coherent rings: R is said to be a nonnil-finite conductor ring if $Ra \cap Rb$ and $(0 : c)$ are finitely generated ideals of R for all non-nilpotent elements $a, b, c \in R$; R is said to be a nonnil-quasi coherent ring if $a_1R \cap \cdots \cap a_nR$ and $(0 : c)$ are finitely generated ideals of R for any finite set of non-nilpotent elements c and $a_1, \ldots, a_n$ of R. Some basic properties of nonnil-finite conductor (resp., nonnil-quasi coherent) rings are studied. Further, we study the possible transfer to trivial ring extension and amalgamated algebra along an ideal. Examples illustrating the aims and scopes of our results are given.

Key Words: nonnil-finite conductor; nonnil-quasi cohrent; ϕ-finite conductor; nonnil-coherent; trivial ring extensions.

Contents

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring and M denote such an R-module. $Nil(R)$ denotes the set of all nilpotent elements of R; $Z_R(M)$ denotes the set of $r \in R$ such that $rm = 0$ for some nonzero $m \in M$ and $Z(R)$ denotes the the set of all zero-divisors of R. For an ideal I of R and an element $a \in R$, we denote by $(I : a) := \{x \in R \mid xa \subseteq I\}$ the conductor of Ra into I. Recall that a ring R is said to be a ZN-ring if $Z(R) = Nil(R)$. An ideal I of R is said to be a nonnil ideal if $I \nsubseteq Nil(R)$.

Recall from [9,15] that a prime ideal P of R is called a divided prime if it is comparable to every ideal of R. Set $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. A ϕ-ring R is called a strongly ϕ-ring if $Z(R) = Nil(R)$. The class of ϕ-rings and strongly ϕ-rings is a good extension of integral domains to commutative rings with zero-divisors. We recommend [2,3,10,16] for the study of the ring-theoretic characterizations on ϕ-rings. We start by recalling some background material. An R-module M is called a finitely presented module if there is an exact sequence of R-modules $F_1 \to F_0 \to M \to 0$ such that both F_0 and F_1 are finitely generated free modules. A ring R is said to be coherent if every finitely generated ideal of R is a finitely presented. An excellent summary of work done on coherence up to 1989 can be found in [19]. On the other hand, in [30], Zafrullah defined finite conductor domains as a new generalization of coherent domains. Moreover, Glaz extended the definition of finite conductor domains to rings with zero divisors [20]. A ring R is called a finite conductor ring if $Ra \cap Rb$ and $(0 : c)$ are finitely generated ideals of R for every $a, b, c \in R$. For more details on finite conductor rings, refer to [7].

Recently, in [8], Bacem and Benhissi investigated the nonnil-coherent (resp., ϕ-coherent) rings. A ϕ-ring R is called nonnil-coherent if each finitely generated nonnil ideal of R is a finitely presented ideal of R. A ϕ-ring R is said to be a ϕ-coherent ring if every finitely generated nonnil ideal of $\phi(R)$ is finitely presented nonnil ideal of $\phi(R)$. They showed that a ϕ-ring R is nonnil-coherent if and only if any direct product of ϕ-flat R-modules is ϕ-flat. See for instance [8,28].

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Some of our results use the $A \ltimes M$ construction. Let A be a ring and M be an A-module. Then $A \ltimes M$, the *trivial* (*ring*) *extension of* A *by* M, is the ring whose additive structure is that of the external direct sum $A \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in A$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [19,21]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [4,5,6,22,23].

In Section 2, we define nonnil-finite conductor (resp., nonnil-quasi coherent) rings and we investigate some proprieties of this kind of rings. In Theorem 2.1, we give some characterizations of nonnil-finite conductor (resp., nonnil-quasi coherent) rings. In addition, we show that a ϕ-ring R is a nonnil-finite conductor ring if and only if $R/Nil(R)$ is a finite conductor domain and $(0 : r)$ is a finitely generated ideal for every non-nilpotent element $r \in R$. Moreover, we prove that if ϕ-$w.gl.\dim R = 2$, then R is a nonnil-finite conductor ring if and only if R is a nonnil-quasi coherent ring. In Section 3, we study the possible transfer of the nonnil-finite conductor property to the trivial ring extension and the amalgamated algebra along an ideal.

2. On nonnil-finite conductor rings

Definition 2.1 *Let R be a ϕ-ring.*

(1) *R is called a nonnil-finite conductor ring if $aR \cap bR$ and $(0 : c)$ are finitely generated ideals of R for all elements $a, b, c \in R \setminus Nil(R)$.*

(2) *R is said to be a nonnil-quasi coherent ring if $a_1R \cap \cdots \cap a_nR$ and $(0 : c)$ are finitely generated ideals of R for any finite set of non-nilpotent elements c and $a_1, \ldots, a_n$ of R.*

Badawi established the concept of nonnil-Noetherian rings in [10]. Recall that a ring R is nonnil-Noetherian if every nonnil ideal of R is finitely generated. In 2004, Anderson and Badawi [3] extended the concept of Bézout domains to ϕ-Bézout rings. A ϕ-Bézout ring is a ϕ-ring R where, for every finitely generated nonnil ideal I of R, $\phi(I)$ is a principal ideal of $\phi(R)$. They proved that a ring $R \in \mathcal{H}$ is a ϕ-Bézout ring if and only if $\phi(R)$ is a Bézout ring, if and only if $R/Nil(R)$ is a Bézout domain, if and only if $\phi(R)/Nil(\phi(R))$ is a Bézout domain, if and only if every finitely generated nonnil ideal of R is principal [3, Corollary 7(2)].

Let R be a ϕ-ring. If R is a nonnil-coherent ring, then R is naturally a nonnil-quasi coherent ring (and so a nonnil-finite conductor ring), with the converse is true if R is a ϕ-Bézout ring. If R is a nonnil-finite conductor ring, then every principal nonnil-ideal of R is finitely presented, with the converse is true if R is a nonnil-Noetherian ring.

Let A be a ring and M be an A-module. Recall from [17, Corollary 2.4] that $A \ltimes M$ is a ϕ-ring if and only if A is a ϕ-ring and $sM = M$ for all $s \in A \setminus Nil(A)$.

Lemma 2.1 *Let A be a ϕ-ring and M be an A-module such that $sM = M$ for all $s \in A \setminus Nil(A)$. Let $R = A \ltimes M$. Then:*

(1) *An ideal J of R is a nonnil ideal if and only if there exists a (unique) nonnil ideal I of A such that $J = I \ltimes M$.*

(2) *A nonnil ideal $J = I \ltimes M$ of R is finitely generated if and only if I is finitely generated.*

Proof: (1) Let J be a nonnil ideal of R. Since $Nil(R)$ is a divided prime ideal of R, we get $0 \ltimes M \subseteq J$ and so $J = I \ltimes E$ with $I = \{a \in A \mid (a, 0) \in J\}$. One can see that I is a nonnil ideal of A.
(2) Assume that $J = I \ltimes E = ((a_1, e_1), \ldots, (a_n, e_n))$ is a finitely generated ideal. By projection on the first component, we deduce that $I = (a_1, \ldots, a_n)$. Conversely, assume that I is a finitely generated ideal of A. Since R is a ϕ-ring, then there exist $a_1, \ldots, a_n \in I \setminus Nil(A)$ such that $I = (a_1, \ldots, a_n)$. Let $x = (a, e) \in J$. Then $a = c_1a_1 + \cdots + c_na_n$ for some $c_1, \ldots, c_n \in A$. On the other hand, the fact that $a_1E = E$ implies that there exists $f \in E$ such that $e = a_1f$. Thus $x = (a_1, 0)(c_1, f) + (a_2, 0)(c_2, 0) + \cdots + (a_n, 0)(c_n, 0)$. It follows that $J = ((a_1, 0), \ldots, (a_n, 0))$ is a finitely generated ideal of R. □

Example 2.1 *Let D be a domain which is not a finite conductor domain (e.g. $D = k[x, yx, yw, y^2w, y^3w, \ldots]$ where k is a field, x and y are indeterminates over k, and $w = yx + 1$), Q its quotient field. Let $R = D \ltimes Q$ be the trivial extension construction. Then, every principal nonnil-ideal of R is finitely presented but R is not nonnil-finite conductor.*

Example 2.2 *Let D be a finite conductor domain which is not a coherent domain, and Q its quotient field. Let $R = D \ltimes Q$. Then R is a nonnil-finite conductor which is not nonnil-coherent ring.*

Let I be an ideal of a ring R. We denote by $\mu(I)$ the cardinality of a minimal set of generators of I. If I is not finitely generated, we consider $\mu(I) = \infty$. The following theorem gives some characterizations of nonnil-finite conductor rings.

Theorem 2.1 *Let R be a ϕ-ring. Then the following assertions are equivalent:*

(1) *R is a nonnil-finite conductor ring.*

(2) *Every nonnil ideal of R with $\mu(I) \leq 2$ is finitely presented.*

(3) *$(a : b)$ is a finitely generated ideal of R for any non-nilpotent element $b \in R$ and any element a of R.*

Proof: (1) $\Rightarrow$ (2) Let I be a finitely generated nonnil ideal of R with $\mu(I) \leq 2$. So, two cases are possible:
Case 1. If $\mu(I) = 1$, then $I = a_1R$ for some non-nilpotent element a_1 in R. Hence $(0 : a_1)$ is finitely generated and thus I is finitely presented from the following naturally exact sequence $0 \longrightarrow (0 : a_1) \longrightarrow R \longrightarrow Ra_1 \longrightarrow 0$.
Case 2. Suppose that $\mu(I) = 2$. Then $I = a_1R + a_2R$ for some non-nilpotent elements $a_1, a_2 \in R$. Consider the following exact sequence:

$$0 \longrightarrow a_1R \cap a_2R \longrightarrow a_1R \oplus a_2R \longrightarrow I \longrightarrow 0.$$

Note that $a_1R \oplus a_2R$ is finitely presented. On the other hand, since $a_1R \cap a_2R$ is finitely generated, we then have I is a finitely presented ideal.
(2) $\Rightarrow$ (3) Let b be an element of R and a be a non-nilpotent element of R. So $I := aR + bR$ is a finitely presented ideal of R. Thus, there exists an exact sequence $0 \longrightarrow K \longrightarrow R^2 \longrightarrow I \longrightarrow 0$, where K is finitely generated. Moreover, there exists a surjective homomorphism $g : K \longrightarrow (a : b)$, which shows that $(a : b)$ is a finitely generated ideal of R.
(3) $\Rightarrow$ (1) It suffices to prove that $Ra \cap Rb$ is a finitely generated ideal of R for each non-nilpotent elements $a, b \in R$. This, in turn, follows from the fact that $aR \cap bR = (a : b)b$. This completes the proof. □

Lemma 2.2 *Let R be a ϕ-ring and I be a nonnil ideal of R. Then $\mu(I) = \mu(I/Nil(R))$.*

Proof: It is clear that $\mu(I/Nil(R)) \leq \mu(I)$. If $I/Nil(R)$ is not finitely generated, then $\mu(I) = \mu(I/Nil(R)) = \infty$. Now, we assume that $I/Nil(R)$ is finitely generated. Set $I/Nil(R) = (x_1 + Nil(R), \ldots, x_n + Nil(R))$ where $\{x_1 + Nil(R), \ldots, x_n + Nil(R)\}$ is a minimal set of generators of $I/Nil(R)$. Let $x \in I$. Then there exist $c_1, \ldots, c_n \in R$ such that $c_1x_1 + \cdots + c_nx_n - x \in Nil(R)$. Since $Nil(R)$ is a divided prime ideal of R and $x_1 \in R \setminus Nil(R)$, we get $c_1x_1 + \cdots + c_nx_n - x = wx_1$ for some $w \in R$. Therefore $x = (c_1 - w)x_1 + c_2x_2 \cdots + c_nx_n \in (x_1, \ldots, x_n)$, and so $I = (x_1, \ldots, x_n)$. Thus I is a finitely generated ideal of R and $\mu(I) \leq \mu(I/Nil(R))$. This yields that $\mu(I) = \mu(I/Nil(R))$. □

Remark 2.1 *Let A be a ϕ-ring and M be an A-module such that $sM = M$ for all $s \in A \setminus Nil(A)$. Let $R = A \ltimes M$. By Lemma 2.1, we have that a nonnil ideal $J = I \ltimes M$ of R is finitely generated if and only if I is a finitely generated ideal of A. Moreover, since $J/Nil(R) = (I \ltimes E)/Nil(R) \cong I/Nil(A)$, we then have $\mu(J) = \mu(I)$.*

Theorem 2.2 *Let R be a ϕ-ring. Then R is a nonnil-finite conductor ring if and only if $R/Nil(R)$ is a finite conductor domain and $(0 : r)$ is a finitely generated ideal for every non-nilpotent element $r \in R$.*

Proof: Assume that R is a nonnil-finite conductor ring. Then $(0 : r)$ is a finitely generated ideal for every non-nilpotent element $r \in R$. Now, let J be a finitely generated ideal of $R/Nil(R)$ with $\mu(J) \leq 2$. So, $J = I/Nil(R)$ for some finitely generated nonnil ideal I of R. By Lemma 2.2, we have $\mu(I) \leq 2$. Since R is a nonnil-finite conductor ring, I is finitely presented and hence $I/\operatorname{Nil}(R)$ is a finitely presented nonzero ideal of $R/\operatorname{Nil}(R)$ by [8, Theorem 2.2]. It follows that $R/\operatorname{Nil}(R)$ is a finite conductor domain. Conversely, let a and b be two non-nilpotent elements of R. So $aR/\operatorname{Nil}(R)$ and $bR/\operatorname{Nil}(R)$ are non-zero principal ideals of $R/\operatorname{Nil}(R)$. By assumption, we get $(aR \cap bR)/\operatorname{Nil}(R) = aR/\operatorname{Nil}(R) \cap bR/\operatorname{Nil}(R)$ is finitely generated, and consequently $aR \cap bR$ is a finitely generated ideal of R by Lemma 2.2. Whence R is a nonnil-finite conductor ring. □

Corollary 2.1 *Let R be a strongly ϕ-ring. Then R is a nonnil-finite conductor ring if and only if $R/Nil(R)$ is a finite conductor domain.*

Definition 2.2 *[24] Let R be a ϕ-ring. Then R is said to be a ϕ-finite conductor ring if $\phi(R)$ is a nonnil-finite conductor ring.*

Corollary 2.2 *Let R be a ϕ-ring. Then R is a ϕ-finite conductor ring if and only if $\phi(R)/Nil(\phi(R))$ is a finite conductor domain.*

Proof: It suffices to see that $\phi(R)$ is a strongly ϕ-ring. □

Recall from [10, Lemma 1.1], $R/Nil(R) \cong \phi(R)/Nil(\phi(R))$ for every ϕ-ring R. Then we have the following corollary as a direct consequence of this result and Theorem 2.2.

Corollary 2.3 *Let R be a ϕ-ring such that $(0 : r)$ is a finitely generated ideal for every non-nilpotent element $r \in R$. Then the following statements are equivalent:*

(1) *R is a nonnil-finite conductor ring.*

(2) *R is a ϕ-finite conductor ring.*

(3) *$R/\operatorname{Nil}(R)$ is a finite conductor domain.*

(4) *$\phi(R)/\operatorname{Nil}(\phi(R))$ is a finite conductor domain.*

Let R be a ϕ-ring. Recall from [31] that the ϕ-weak global dimension of R is determined by the formulas:

$$\begin{aligned}\phi\text{-}w \cdot gl.dim(R) &= \sup\{fd_R(R/I) \mid I \text{ is a nonnil ideal of } R\} \\ &= \sup\{fd_R(R/I) \mid I \text{ is a finitely generated nonnil ideal of } R\},\end{aligned}$$

and the ϕ-global dimension of R is determined by the formula:

$$\phi\text{-}gl.dim(R) = \sup\{pd_R(R/I) \mid I \text{ is a nonnil ideal of } R\}.$$

Note that $\phi\text{-}w.gl.dim(R) \leq \phi\text{-}gl.dim(R)$. We now turn our attention to rings of small ϕ-weak dimension. Recall from [27,32] that a ϕ-ring R is called a ϕ-von Neumann regular ring if every R-module is flat; equivalently $(R, Nil(R))$ is a local ring. The rings R of $\phi\text{-}w.\dim R = 0$ are precisely the ϕ-von Neumann regular rings [31, Theorem 2.8] and so they are nonnil-coherent rings. Recall that a ring R is called nonnil-semihereditary if finitely generated nonnil ideals of R are projective. Additional information about ϕ-rings from a module-theoretic point of view can be found in the interesting survey article [26].

Proposition 2.1 *Let R be a ring of $\phi\text{-}w.gl.\dim R = 1$. The following conditions are satisfied:*

(1) *R is a nonnil-coherent ring.*

(2) *R is a nonnil-semihereditary ring.*

Proof: (1) Since ϕ-$w \cdot gl.dim(R) = \sup\{fd_R(R/I) \mid I$ is a nonnil ideal of $R\} \leq 1$, we get $fd_R(R/I) \leq 1$ for any nonnil ideal I of R and so every nonnil ideal of R is flat. Hence every ideal of R is ϕ-flat by [27, Proposition 2.6]. Consequently it follows that R is a strongly ϕ-Prüfer ring according to [25, Corollary 3.2], and hence $R/Nil(R)$ is a Prüfer domain. In particular, $R/Nil(R)$ is a coherent domain and $Z(R) = Nil(R)$, and whence R is a nonnil-coherent ring by [8, Corollary 3.2] .
(2) It is obvious. □

Theorem 2.3 *Let R be a ϕ-ring of ϕ-$w.gl.\dim R = 2$. If R is a nonnil-finite conductor ring, then R is a nonnil-quasi coherent ring.*

Before proving Theorem 2.3, we establish the following lemmas.

Lemma 2.3 *If R is a nonnil-finite conductor ring, then $I \cap J$ is a finitely generated nonnil ideal of R for any finitely generated flat nonnil ideals I and J of R.*

Proof: Set $I = (a_1, \ldots, a_n)$ and $J = (b_1, \ldots, b_m)$. Then I and J are projective and hence free (principal) at every localization of R by a prime ideal. Let P be a prime ideal of R. Then $IR_P = a_{i_1}R_P$, $JR_P = b_{j_1}R_P$ for some $i = i_1$ and $j = j_1$. Then

$$\begin{aligned} IR_P \cap JR_P &= a_{i1}R_P \cap b_{j1}R_P \\ &= (a_{i1}R \cap b_{j1}R)\, R_P \\ &\subset \left(\sum_{1 \leq i \leq n, 1 \leq j \leq m} a_iR \cap b_jR \right) R_P \\ &\subset (I \cap J)R_P. \end{aligned}$$

Thus $I \cap J = \sum_{1 \leq i \leq n, 1 \leq j \leq m} (a_iR \cap b_jR)$, and therefore $I \cap J$ is finitely generated. □

Lemma 2.4 *If R is a ϕ-ring of ϕ-$w.gl.\dim R = 2$ and for any non-nilpotent element c of R, $(0 : c)$ is a finitely generated ideal, then cR is a projective ideal of R*

Proof: Let R be a ϕ-ring with ϕ-$w.gl.\dim R = 2$. Since

$$\phi - w.gl.\dim R = sup\{fd_R(R/I) \mid I \text{ is a nonnil ideal of } R\},$$

we get $fd_R(R/cR) \leq 2$ and consequently $fd_R(cR) \leq 1$. Hence $(0 : c)$ is a finitely generated flat ideal of R. Now let P be a prime ideal of R. Then either $(0 : c)R_P = 0$ or $(0 : c)R_P \neq 0$ and so it is a projective, and hence free, ideal of R_P. On the other hand, since $c(0 : c) = 0$ we get $cR_P = 0$ or therefore $(0 : c)R_P = 0$, and so $(0 : c)R_P = R_P$ or $(0 : c)R_P = 0$, Therefore, we conclude that $(0 : c)$ is a pure ideal of R, and so $cR \cong R/(0 : c)$ is a flat ideal of R by [19, Theorem 1.2.15]. Since cR is finitely presented, then it is projective. □

Proof of Theorem 2.3: If R is a nonnil-finite conductor ring of ϕ-$w.gl \dim R = 2$, then any intersection of finitely many finitely generated flat nonnil ideals of R is a finitely generated ideal of R by Lemma 2.2. From the exact sequence $0 \to I \cap J \to I \oplus J \to (I : J) \to 0$, and the fact that any nonnil ideal of R has flat dimension at most 1, we conclude that $I \cap J$ is a flat nonnil ideal of R. The proof follows by induction on n, the number of finitely generated flat ideals intersected, with the case $n = 2$ clear from Lemma 2.2 and the fact that $I \cap J$ is a flat nonnil ideal of R for any flat nonnil ideals I and J of R. □

The following is an example of a nonnil-finite conductor (and hence nonnil-quasi coherent) ring with ϕ-$w.\dim R = 2$ that is not nonnil-coherent.

Example 2.3 *Let $S_i = Q[[t,u]]$ be countably many copies of the power series ring in two variables t and u over the field Q of rational numbers, and let $S = \Pi S_i$. According to [1], there is a localization of S, S_P, such that $S_P[x]$ is not a coherent ring. Consider the ring $A = S_P(x) = S_P[x]_{PS_P[x]}$. So, A is a finite conductor domain which is not coherent and $w.gl.\dim A = 2$. Now, we consider $R = A \ltimes Q$. By [31, Proposition 2.7], ϕ-$w.gl.\dim R = w.gl.\dim A = 2$. On the other hand, the fact that R is a strongly ϕ-ring implies that R is a nonnil-finite conductor ring which is not nonnil-coherent by Theorem 2.3 and [8, Corollary 3.2].*

As a consequence, we can clarify the relation between the three properties for rings of small ϕ-global dimensions.

Corollary 2.4 *Let R be a ϕ-ring. Then:*

1. *ϕ-$gl.\dim R = 0$ if and only if R is a ϕ-von Neumann regular ring. In this case, R is nonnil-coherent.*
2. *ϕ-$gl.\dim R = 1$ if and only if R is a strongly ϕ-Dedekind ring which is not a ϕ-von Neumann regular ring. In this case, R is nonnil-coherent.*
3. *If ϕ-$gl.\dim R = 2$, then R is a nonnil-coherent ring if and only if $(0 : c)$ is finitely generated for every non-nilpotent element c of R.*

Proof: (1) and (2) follow from [31, Theorems 3.9 and 3.10]
(3) Note that if ϕ-$gl.\dim R = 2$, then ϕ-$w.gl.\dim R \leq 2$. Thus, by Lemma 2.4, cR is a projective ideal of R for every non-nilpotent element c of R. The exact sequence $0 \longrightarrow (0 : c) \longrightarrow R \longrightarrow cR \longrightarrow 0$ splits and $(0 : c)$ is a direct summand of R and, therefore, generated by an idempotent e. Since R is a ϕ-ring, in particular $Nil(R)$ is a prime ideal of R, we conclude that $e = 0$. It follows that R is a stronlgy ϕ-ring. Let $I = (a_1, \dots, a_n)$ be a finitely generated nonnil ideal of R. Since R is a strongly ϕ-ring, we get $(0 : I) = 0$. Moreover, the fact that ϕ-$gl.\dim R = \sup\{pd_R(R/I) \mid I \text{ is a nonnil ideal of } R\} = 2$. implies that $pd_R I \leq 1$. Consider the exact sequence $0 \longrightarrow K \longrightarrow R^n \longrightarrow I \longrightarrow 0$. Note that K is a projective R-module which has rank $n-1$ at each localization. By [19, Theorem 3.2 .13], K is finitely generated, and therefore I is finitely presented. □

3. Nonnil-finite conductor properties on some ring constructions

In this section, we study the possible transfer of nonnil-finite conductor property to trivial ring extensions and amalgamation algebras along an ideal. According to [17, Corollary 2.4], the trivial ring extension $A \ltimes M$ is a ϕ-ring if and only if A is a ϕ-ring and $sM = M$ for all $s \in A \setminus Nil(A)$.

Let M be an A-module and $r \in A$. Note that $(0 :_M r)$ is a submodule of M such that $(0 : r)M \subset (0 :_M r)$, and so $(0 : r) \ltimes (0 :_M r)$ is an ideal of $A \ltimes M$.

The following theorem characterizes when a trivial ring extension is a nonnil-finite conductor ring.

Theorem 3.1 *Let A be a ϕ-ring, M be an A-module such that $aM = M$ for every $a \in A \setminus Nil(A)$ and $R = A \ltimes M$. Then, the following statements are equivalent:*

1. *R is a nonnil-finite conductor ring.*
2. *A is a nonnil-finite conductor ring and $(0 : r) \ltimes (0 :_M r)$ is a finitely generated ideal of R for each $r \in A \setminus Nil(A)$.*
3. *A is a nonnil-finite conductor ring and $R(r, 0)$ is finitely presented for all $r \in A \setminus Nil(A)$.*

Proof: (1) ⇒ (2) Assume that R is a nonnil-finite conductor ring. Let a and b be two non-nilpotent elements of A. It is easy to see that $aA \ltimes M = (a, 0)R$. Hence $aA \ltimes M$ and $bA \ltimes M$ are nonnil ideals of R. Since R is a nonnil-finite conductor ring, $(aA \ltimes M) \cap (bA \ltimes M) = (aA \cap bA) \ltimes M$ is a finitely generated ideal of R. In particular, $aA \cap bA$ is a finitely generated ideal of A by Lemma 2.2. On the other hand, let $r \in A \setminus Nil(A)$. Then, $((0,0):(r,0)) = (0:r) \ltimes (0:_M r)$ is a finitely generated ideal of R, and so $(0:r)$ is finitely generated. Therefore, A is a nonnil-finite conductor ring.

(2) $\Rightarrow$ (1) Assume that A is a nonnil-finite conductor ring and $(0:r) \ltimes (0:_M r)$ is a finitely generated ideal of R for each $r \in A \setminus Nil(A)$. Let $(a,e)R$ and $(b,f)R$ be nonnil principal ideals of R. Then, aA and bA are finitely generated nonnil ideals of A. Since A is a nonnil-finite conductor ring, $aA \cap bA$ is a finitely generated ideal of A. Thus $(a,e)R \cap (b,f)R = (aA \ltimes M) \cap (bA \ltimes M) = (aA \cap bA) \ltimes M$ is a finitely generated ideal of R. Take $(r,u) \in R \setminus Nil(R)$. Then, by hypothesis, $((0,0):(r,u)) = (0:r) \ltimes (0:_M r)$ is finitely generated. Therefore, R is a nonnil-finite conductor ring.

(2) $\Leftrightarrow$ (3) Let $r \in A \setminus Nil(A)$. It can be seen that $(0:r) \ltimes (0:Mr)$ is finitely generated if and only if $R(r,0)$ is finitely presented. □

Corollary 3.1 *Let $R = A \ltimes M$ be a ϕ-ring such that $Z(A) = \mathrm{Nil}(A)$. Then, R is a nonnil-finite conductor ring if and only if A is a nonnil-finite conductor ring and $(0:_M r)$ is a finitely generated A-submodule of M for every $r \in A \setminus Nil(A)$.*

Let A be a ring and M be an A-module. Recall from [5] that a submodule N of M is said to be an r-submodule if $rm \in N$ and $r \notin Z_A(M)$ implies that $m \in N$. Also, M is called an r-Noetherian module if every r-submodule of M is finitely generated. In particular, $(0:_M r)$ is finitely generated for every element $r \in R$.

Corollary 3.2 *Let $R = A \ltimes M$ be a strongly ϕ-ring, and M be an r-Noetherian A-module. Then, R is a nonnil-finite conductor ring if and only if A is a nonnil-finite conductor ring.*

Corollary 3.3 *Let $R = A \ltimes M$ be a ϕ-ring such that $Z(A) = \mathrm{Nil}(A) = Z_A(M)$. Then, R is a nonnil-finite conductor ring if and only if A is a nonnil-finite conductor ring*

At this point, we give an example of a ϕ-finite conductor ring which is not nonnil-finite conductor.

Example 3.1 *Let D be a finite conductor domain which is not a field, Q its quotient field and $E = \bigoplus_{i=1}^{\infty} Q/D$. Set $R = D \ltimes E$. Note that R is a ϕ-ring by [17, Corollary 2.4] and $R/\mathrm{Nil}(R) \cong D$ is a finite conductor domain, which proves that R is a ϕ-finite conductor ring. However, R is not nonnil-finite conductor. In fact, let d be a non-zero non-unit element of D. Then $(d,0)$ is a non-nilpotent element in R. One can easily check that $(0:_E d) = \bigoplus_{i=1}^{\infty} \left(\frac{1}{d} + D\right)$ is an infinitely generated D-module, and consequently $(0:d) \ltimes (0:_E d) = 0 \ltimes (0:_E d)$ is an infinitely generated ideal of R. Thus R is not a nonnil-finite conductor ring by Theorem 3.1*

Example 3.2 *Let D be a finite conductor domain which is not a field and Q its quotient field. Then $R = D \ltimes Q$ is a nonnil-finite conductor ring which is not finite conductor.*

Recall from [14], that a ϕ-ring R is called a ϕ-GCD ring if every two nonnil elements of R have a greatest common divisor; equivalently $R/Nil(R)$ is a GCD domain by [14, Theorem 3.3]. By Corollary 2.2, we conclude that every ϕ-GCD ring is a ϕ-finite conductor ring.

Example 3.3 *Let D be a finite conductor domain which is not a GCD domain and Q its quotient field. Then $R = D \ltimes Q$ is a ϕ-finite conductor ring which is not a ϕ-GCD ring.*

We next give an example of a ϕ-GCD ring which is not nonnil-finite conductor.

Example 3.4 *Let D be a GCD domain which is not a field, Q its quotient field and $E = \bigoplus_{i=1}^{\infty} Q/D$. Set $R = D \ltimes E$. Since $R/Nil(R) = D \ltimes E/0 \ltimes E \cong D$ is a GCD domain, we get R is a ϕ-GCD ring by [14, Theorem 3.3]. However R is not a nonnil-finite conductor.*

Let A and B be two rings, J be an ideal of B and $f: A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f B = \{(a, f(a)+j) \mid a \in A \text{ and } j \in J\}$$

called the amalgamation of A with B along J with respect to f. This construction has been first introduced and studied D'Anna, Finocchiaro, and Fontana in [12,13,18].

Let $R := A \bowtie^f J$ and $N(J) := Nil(B) \cap J$. Recall from [17, Theorem 2.1] that:
(1) If J is a nonnil ideal of B, then R is a ϕ-ring if and only if $f^{-1}(J) = 0$, A is an integral domain, and $N(J)$ is a divided prime ideal of $f(A) + J$.
(2) If $J \subseteq Nil(B)$, then R is a ϕ-ring if and only if A is a ϕ-ring, and for each $i, j \in J$ and each $a \in A \setminus Nil(A)$, there exist $x \in Nil(A)$ and $k \in J$ such that $xa = 0$ and $j = kf(a) + i(f(x) + k)$.
Moreover, let $\iota : A \to A \bowtie^f J$ be the natural embedding defined by $a \mapsto (a, f(a))$.

Now, we study the transfer of being ϕ-finite conductor rings in the amalgamation algebra along an ideal.

Theorem 3.2 *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nonnil ideal of B. Define $\bar{f} : A \longrightarrow B/N(J)$ by $\bar{f}(a) = f(a) + N(J)$ for any $a \in A$. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, the following statements are equivalent:*

(1) *$A \bowtie^f J$ is a ϕ-finite conductor ring.*

(2) *$A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is a finite conductor domain.*

(3) *$\bar{f}(A) + J/N(J)$ is a finite conductor domain.*

Proof: (1) $\Rightarrow$ (2) Assume that $A \bowtie^f J$ is a ϕ-finite conductor ring. Since $A \bowtie^f J$ is a ϕ-ring, it follows that A is an integral domain by [17, Theorem 2.1 (1)], and so Nil $\left(A \bowtie^f J\right) = 0 \times N(J)$. As $A \bowtie^f J$ is a ϕ-finite conductor ring, $\frac{A \bowtie^f J}{0 \times N(J)}$ is a finite conductor domain. Therefore, $A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is a finite conductor domain.

(2) $\Rightarrow$ (1) Follows directly from [29, Remark 2.6].

(2) $\Rightarrow$ (3) Assume that $A \bowtie^{\bar{f}} J/N(J)$ is a finite conductor domain. Then, by [17, Theorem 2.1 (1)], we conclude that $f^{-1}(J) = \bar{f}^{-1}(J/N(J)) = 0$, and hence $\bar{f}(A) + J/N(J)$ is an integral domain by [11, Proposition 5.2]. Also, using [11, Proposition 5.1], we obtain that $\bar{f}(A) + J/N(J) \cong A \bowtie^f J/N(J)$, as desired.

(3) $\Rightarrow$ (2) Combine [17, Theorem 2.1 (1)] and [11, Proposition 5.1]. □

Corollary 3.4 investigates the transfer of being a nonnil-finite conductor ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nonnil ideal J.

Corollary 3.4 *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nonnil ideal of B. Define $\bar{f} : A \longrightarrow B/N(J)$ by $\bar{f}(a) = f(a) + N(J)$ for any $a \in A$. Assume $A \bowtie^f J$ is a ϕ-ring. Then $A \bowtie^f J$ is a nonnil-finite conductor ring if and only if $\bar{f}(A) + J/N(J)$ is a finite conductor domain and $\left(A \bowtie^f J\right)(r, f(r) + j)$ is a finitely presented ideal for any non-nilpotent element $(r, f(r) + j)$ of $A \bowtie^f J$.*

Proof: It follows immediately from Theorem 2.2 and Theorem 3.2. □

The last theorem studies the transfer of being a ϕ-finite conductor ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Theorem 3.3 *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nil ideal of B. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, $A \bowtie^f J$ is a ϕ-finite conductor ring if and only if A is a ϕ-finite conductor ring.*

Proof: Since $J \subseteq \mathrm{Nil}(B)$, we then have $N(J) = J$ and so Nil $\left(A \bowtie^f J\right) = \mathrm{Nil}(A) \bowtie^f J$. It follows that $A \bowtie^f J$ is a ϕ-finite conductor ring and therefore $\frac{A \bowtie^f J}{\mathrm{Nil}(A) \bowtie^f J} \cong \frac{A}{\mathrm{Nil}(A)}$ is a finite conductor domain. Thus

A is a ϕ-finite conductor ring. □

Corollary 3.5 studies the transfer of being a nonnil-finite conductor ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Corollary 3.5 *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nil ideal of B. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, the following assertions are equivalent:*

(1) *$A \bowtie^f J$ is a nonnil-finite conductor ring.*

(2) *A is a ϕ-finite conductor ring and $\left(A \bowtie^f J\right)(r, f(r) + j)$ is a finitely presented ideal for any non-nilpotent element $(r, f(r) + j)$ of $A \bowtie^f J$.*

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