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On solvability of quadratic Erdélyi-Köber fractional integral equations in Orlicz space

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ABSTRACT: In this paper, we focus on providing some properties of Erdélyi-Köber fractional operators and using the operator type condensing map to find the existence of solution of Erdélyi-Köber fractional integral equation in Orlicz space. The main tool in our considerations is the technique associated with the measure of noncompactness. Lastly, we provide some examples to illustrate our main results.

Key Words: Quadratic integral equation, Erdélyi-Köber fractional integral operator, Measure of noncompactness (MNC), Fixed point theorem, Orlicz space.

Contents

1. Introduction

In the recent few decades, the study of fractional and integral equation has attracted a lot of attention from researchers. This is because of different types of fractional integral and differential equation defined over bounded and unbounded interval in recent years; and a verity of theoretical findings have been made, for example, Miller and Ross [31], Benchohra et al. [8], Diethelm and Ford [18], Furati and Tatar [20], Delboso and Rodino [17] and Momani et al. [32] as well as the monograph of Kilbas et al. [24], etc. These findings are used in a different research field of engineering disciplines and science, e.g., chemical and statistical physics, control theory, rheology, optics, viscoelasticity, robotics, acoustics, electrical and mechanical engineering, etc.

A branch of mathematical analysis known as fractional calculus investigates the various possible outcomes of applying the differentiation operator D to the powers of either real or complex numbers. The first instance is found in a letter that G.W. Leibniz wrote to Antoine de l'Hopital in the sixteenth century [23]. Fractional calculus was used in one of N. H. Abel's early papers, [21], where those elements can be taken into consideration: the definition of differentiation and integration of arbitrary real order, the strictly inverse relationship between them, the perception that differentiation and integration of arbitrary real order can be perceived as being in the same generalised operation, and in fact the coherent form for those operations. O'Neil firstly investigated the fractional calculus and convolution operators for Lorentz $L(p, q)$-space [34] and Orlicz space [35].

The principles and applications of fractional calculus significantly advanced in the nineteenth century and twentieth centuries, and innumerable authors have contributed explanations for fractional derivatives and integrals. The fact that fractional order models are frequently more accurate than integer order models is a key factor in the success of applications of fractional calculus. With the use of several kinds of fractional operators, the researchers are able to simulate the nonlocal and distributed effects frequently observed in natural and technological phenomena.

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The Erde'lyi–Köber fractional integral is applied in many areas of mathematics, including electrochemistry, viscoelasticity and porous media, see [9,22]. Several researchers have solved various types of fractional integral and differential equations; for examples, see [2,36,38].

The solution of functional integral equations relies heavily on the Darbo's fixed point theorems. The concept of the measure of compactness (MNC) is significant to the fixed point theory. This idea was pioneered by Kuratowski in his seminal paper [26]. By the middle of the nineteenth century, Darbo [14] established a theorem that guaranteed the presence of fixed points by applying the concept of the measure noncompactness.

Several kinds of integral equations were solved by using fixed point theory and the measure of noncompactness, for examples, see [3,5,6,33,37]. In [1], Abdalla and Salem investigated monotonic solutions of quadratic integral equations in Orlicz space. Also, we found discussions of the different types of functional integral equations in Orlicz space in the references [10,11,12,28,29,30]

Research works and studies on various integrals like Riemann-Liouville, tempered fractional, Hadamard fractional and many others are carried out on the Orlicz space, since nineteenth century. In comparison, works on Erdélyi-Köber fractional integral on the Orlicz space is yet to be reached a bigger arena. Hence, working on this fractional integral, we have found a solution which is unique in nature. Here we have given example of the integral on Orlicz space.

First, we examine some characteristics of Erdélyi-Köber fractional integral operators, such as boundedness, monotonicity, continuity, and acting conditions in Orlicz spaces and use these characteristics in conjunction with an appropriate measure of noncompactness to examine the fractional integral equation

$$f(t) = H_2 f(t) + \frac{H_1(f)(t)}{\Gamma(\gamma)} \beta \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}} H_3 f(s) ds \tag{1.1}$$

$t \in [0,1], 0 < \gamma < 1, \beta > 0$ in Orlicz space L_q, where $H_i, i = 1,2,3$ are general operator.
Darwish and Sadarangani [15,16] solved Erdélyi-Köber fractional integral equations. The solutions in Orlicz space grant us to examine operators with strong nonlinearity. As an illustration, exponential nonlinearities in the integral equation

$$f(t) + \int_I K(t,s)\ e^{f(s)} ds = 0.$$

This is useful in thermodynamics.

This article operates by showing some characteristics of the Erdélyi–Köber fractional integral operator and by using these to demonstrate that some general problems have monotonic solutions, but abstract, quadratic Erélyi–Köber type fractional integral equations (1.1) in Orlicz space. The uniqueness of the solution is also discussed under a broad set of assumptions. To obtain our results, we use the measure of noncompactness and the Darbo fixed point theorem.

2. Preliminaries

Notation and Auxiliary facts: Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{I} = [0,1]$. If a function $\mathbb{P} : \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function then

$$P(v) = \int_0^v u(s) ds$$

for $v \geq 0$, where $u : \mathbb{R}^+ \to \mathbb{R}^+$ is a left-continuous increasing function that is neither zero nor identically infinite on $\mathbb{R}^+$. The functions P and Q are pointed to the complementary Young functions if $Q(x) = \sup_{y\geq 0}(xy - P(x))$. In particular, if P has a finite values, where $\lim_{v\to\infty} \frac{P(v)}{v} = 0$, $\lim_{v\to\infty} \frac{P(v)}{v} = \infty$ and $P(v) > 0$ if $v > 0$ ($P(v) = 0 \Leftrightarrow v = 0$). Denoted by $L_P = L_P(I)$ Orlicz space of every bounded functions $f : I \to \mathbb{R}$ using the Luxemburg norm

$$||f||_p = \inf_{\varepsilon > 0} \left\{ \int P\left(\frac{|f(s)|}{\varepsilon}\right) ds \leq 1 \right\}.$$

Let $D_P(I)$ be closure in $L_P(I)$ of all bounded sets in the set and having absolutely continuous norms.

Proposition 2.1 *[13] Suppose that, P is the Young function, then*

(i) for $\gamma_1 \in (0,1)$ and $\int_0^t P(s^{-\gamma_1})ds$ is finite for every $t > 0$, if $\gamma_2 < \gamma_1$, then the integral

$$\int_0^t P(s^{-\gamma_2})ds$$

is finite as well.

(ii) for every $t \in \mathbb{R}^+$ and $\gamma \in (0,1)$, the set

$$P(t) = \left\{ k > 0 : \frac{1}{k^{\frac{1}{\gamma-1}}} \int_0^{tk^{\frac{1}{\gamma-1}}} P(s^{1-\gamma})ds \leq 1 \right\}$$

is a continuous function that increasing with $P(0) = 0$.

Definition 2.1 *[4] The Erdélyi-Köber fractional integral of a continuous function f of order $\beta > 0$ is defined by*

$$I_\beta^\gamma f(t) = \frac{\beta}{\Gamma(\gamma)} \int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}} f(s)ds, \beta > 0, 0 < \gamma < 1,$$

where $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt, \gamma > 0$.

Next, suppose that $(Z, ||.||_Z)$ is an arbitrary Banach space with single element θ. The space indicated by the symbol $B_r(Z)$ and $B_r = \{x \in Z : ||.||_Z) \leq r\}$, $r > 0$ is denoted. If $X \subset Z$ then $\bar{X}$ and conv$X$ represent to the clousre of X and the convex closure, correspondingly. Then symbols $\mathfrak{R}_Z$ and $\mathfrak{Q}_Z$ belong to families of any nonempty as well as bounded subsets of Z and their corresponding subfamilies are relatively compact.

Definition 2.2 *[7] A function of a mapping $\nu : \mathfrak{R}_Z \to [0,\infty)$ is considered to be the measure of non-compactness in Z if it fulfills the following axioms*

(i) $\nu(Y) = 0 \Leftrightarrow Y \in \mathfrak{Q}_Z$

(ii) $Y \subset X \Rightarrow \nu(Y) \leq \nu(X)$

(iii) $\nu(\overline{Y}) = \nu(convY) = \nu(Y)$

(iv) $\nu(\lambda Y) = |\lambda|\nu(Y), \lambda \in \mathbb{R}$

(v) $\nu(Y + X) \leq \nu(Y) + \nu(X)$,

(vi) $\nu(Y \cup X) = \max\{\nu(Y), \nu(X)\}$,

(vii) for bounded, closed, nonempty subsets Y_l of Z such that $Y_{l+1} \subset Y_l, l = 1, 2, 3, \ldots,$ and $\lim_{n\to\infty} \nu(Y_l) = 0$, then the set $Y_\infty = \bigcap_{l=1}^\infty Y_l \neq \emptyset$.

Definition 2.3 *[7] Let $X \subset Z$ be nonempty and bounded set. Then the Hausdorff measure of noncompactness $\alpha(X)$ is defined by*

$$\alpha(X) = \inf\{r > 0 : \text{there is a finite subset } Y \text{ of } Z \text{ such that } X \subset Y + B_r\}.$$

Also the equiintergrability of the set $X \in L_P(I)$ will be measured by c for any $\varepsilon > 0$ (Definition 2 in [19])

$$c(X) = \lim_{\varepsilon\to o} \sup_{mesF\leq\varepsilon} \sup_{f\in X} ||f.\chi_F||_{L_P(I)},$$

where χ_F is a measurable subsets of the characteristic function of $F \subset I$.

Lemma 2.1 *[11] Let $X \subset L_P(I)$ be bounded set. Consider a family $(\Omega_c)_{0\leq c\leq e-1} \subset I$ then meas $\Omega_c = c$ for each $c \in [0,1]$ and $f \in X$,*

$$f(t_1) \geq f(t_2), (t_1 \in \Omega_c, t_2 \notin \Omega_c).$$

As a result, X has the measure of noncompactness in $L_P(I)$.

Lemma 2.2 *[27, Theorem 10.2] Consider Q_1, Q_2, Q_3, are arbitrary Q-function. Then following criteria are equivalent:*

1. *For every $v \in L_{Q_2}(I)$ and $z \in L_{Q_3}, v, z \in L_{Q_1(I)}$.*
2. *There exists a constants $k > 0$ such that every measurable v, z on I then*

$$||vz||_{Q_1} \leq k||v||_{Q_2}||z||_{Q_3}.$$

3. *There exists numbers $C > 0, v_0 \geq 0$ such that any $s, t \geq v_0, Q_1(\frac{st}{C}) \leq Q_2(s) + Q_3(t)$.*
4. $\limsup\limits_{t\to\infty} \frac{Q_2^{-1}(t)Q_3^{-1}(t)}{Q_1(t)} < \infty$.

Lemma 2.3 *[19] Let $X \subset D_P(I)$ is a nonempty, bounded and compact set. Then*

$$\alpha(X) = c(X).$$

Theorem 2.1 *[7] Let's us assume A is a nonempty, closed, convex and bounded subset of Z and $V : A \to A$ is a continuous mapping that contracts with the measure of noncompactness ν, i.e.,*

$$\nu(V(X)) \leq k\nu(X), k \in [0,1)$$

for all nonempty $X \subset Z$. Then V has at least one fixed point in the set A.

3. New results

Now from equation (1.1) we have

$$f = A(f) = H_2(f) + W(f),$$

where

$$W(f) = H_1(f)B(f), B(f)(t) = I_\beta^\gamma H_3(f)(t)$$

such that I_β^γ is in Definition 2.1 and $H_i(f), i = 1, 2, 3$ are operators in general.

3.1. The existence of L_{Q_1}-solutions

[11,28] Suppose, that P and Q are complementary Q-functions and Q_1, Q_2, Q_3 are Q-functions. Furthermore, include the supposition

1. There exists a constant $k_1 > 0$ such that for each $v \in L_{Q_2}(I)$ and $z \in L_{Q_3}(I)$, we get

$$||vz||_{Q_1} \leq k_1||u||_{Q_2}||z||_{Q_3}$$

2. $H_1 : L_{Q_1}(I) \to L_{Q_2}(I)$, taking continuously $D_{Q_1}(I) \to D_{Q_2}(I)$ and the operator $H_2 : L_{Q_1}(I) \to L_{Q_1}(I)$, takes continuously $D_{Q_1}(I)$ into itself and the operator $H_3 : L_{Q_1}(I) \to L_Q(I)$, takes continuously $D_{Q_1}(I) \to D_Q(I)$.
3. There exist positive functions $e_1 \in L_{Q_2}(I), e_2 \in L_{Q_1}(I), e_3 \in L_Q(I)$ such that for $t \in I$, $|H_i(f)(t)| \leq e_i(t)||f||_{Q_1}$, and every H_i taking the collection of all almost everywhere nondecreasing functions into functions of the same type for $i = 1, 2, 3$. Moreover, suppose that for any $f \in D_{Q_1}(I)$ we have $H_1(f) \in D_{Q_2}(I), H_2(f) \in D_{Q_1}(I)$ and $H_3(f) \in D_Q(I)$.

4. suppose that $k(t)=\frac{1}{\varepsilon^{\frac{1}{\gamma-1}}}\int_0^{t\varepsilon^{\frac{1}{\gamma-1}}}P(s^{1-\gamma})ds\in D_{Q_3}(I)$ for a.e. $s\in I$ and $\varepsilon>0$.

Lemma 3.1 *Suppose that, P and Q are complementary Q-functions and that Q_3 is a Q-function. Further, the supposition (4) is true, then show that the operator $I^{\alpha}_{\beta}:L_Q(I)\to L_{Q_3}(I)$ is continuous.*

Proof: Assume that,

$$K(t,s)=\begin{cases}\frac{s^{\beta-1}}{(t^{\beta}-s^{\beta})^{1-\gamma}} & if s\in[0,t],t>0\\ 0, & \text{else.}\end{cases}$$

Then, given $f\in L_Q(I)$ and from Hölder inequality, we get

$$\begin{aligned}|I^{\alpha}_{\beta}f(t)|=&\left|\int_I K(s,t)f(s)ds\right|\\ \le&2||K(s,.)||_P||f||_Q\\ =&\frac{2}{\Gamma(\gamma)}\left\|\frac{s^{\beta-1}}{(.^{\beta}-s^{\beta})^{1-\gamma}}\right\|_P||f||_Q\\ =&\frac{2}{\Gamma(\gamma)}\inf_{\varepsilon>0}\left\{\int_0^t P\left(\frac{(\frac{s^{\beta+\gamma-2}}{t^{\beta}-s^{\beta}})^{1-\gamma}}{\varepsilon}\right)ds\le1\right\}||f||_Q\\ \le&\frac{2}{\Gamma(\gamma)}\inf_{\varepsilon>0}\left\{\int_0^t P\left(\frac{\frac{s^{\beta+\gamma-2}}{t^{\beta}-s^{\beta}}}{\varepsilon^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}ds\le1\right\}||f||_Q\end{aligned}$$

putting

$$v=\frac{\frac{s^{\beta+\gamma-2}}{t^{\beta}-s^{\beta}}}{\varepsilon^{\frac{1}{1-\gamma}}}$$

and from supposition (4), we have

$$\begin{aligned}||I^{\gamma}_{\beta}f(t)||_{q_2}\le&\frac{2}{\Gamma(\gamma)}\left\|\inf_{\varepsilon>0}\left\{\frac{1}{\varepsilon^{\frac{1}{\gamma-1}}}\int_0^{t\varepsilon^{\frac{1}{\gamma-1}}}P(v^{1-\gamma})dv\le1\right\}\right\|_{Q_3}||f||_Q\\ \le&\frac{2}{\Gamma(\gamma)}||k||_{Q_3}||f||_Q.\end{aligned}$$

Then, by applying Proposition 2.1 and [25, Lemma 16.3], we get $I^{\gamma}_{\beta}:L_Q(I)\to L_{q_2}(I)$ is continuous. □

Theorem 3.1 *Let the supposition (1) to (4) be fulfilled. If*

$$\left(||e_2||_{Q_1}+\frac{2k_1||k||_{Q_3}r}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q\right)<1,r>0$$

then $f\in E_{Q_1}(I)$ of (1.1) has a solution which is almost everywhere nondecreasing on I.

Proof: Step (I): First of all Lemma 3.1 implies that $I^{\gamma}_{\beta}:L_Q(I)\to L_{Q_3}(I)$ is continuous and by (2) the operator $H_3:D_{Q_1}(I)\to D_Q(I)$ is continuous. Then the operator $B=I^{\gamma}_{\beta}H_3:D_{Q_1}(I)\to D_{Q_3}(I)$ is continuous. By suppositions (2) and the operator $W:D_{Q_1}(I)\to D_{Q_1}(I)$ is continuous and the operator $A:D_{Q_1}(I)\to D_{Q_1}(I)$ is continuous.
Step (II): We will assemble a ball $B_r(D_{Q_1}(I))=\{f\in L_{Q_1}(I):||f||_{Q_1}\le r\}$, where

$$r=\frac{1-||e_2||_{Q_1}}{\frac{2k_1}{\Gamma(\gamma)}||k||_{Q_3}||e_1||_{Q_2}||e_2||_Q}. \tag{3.1}$$

Let, $f \in B_r(D_{Q_1}(I)$. By Lemma 3.1, we have

$$\begin{aligned}||A(f)||_{Q_1} \leq & ||H_2(f)||_{Q_1} + ||W(f)||_{Q_1}\\ \leq & ||e_2||f||_{Q_1}||_{Q_1} + ||H_1(f)B(f)||_{Q_1}\\ \leq & ||e_2||_{Q_1}||f||_{Q_1} + k_1||H_1(f)||_{Q_2}||B(f)||_{Q_3}\\ \leq & ||e_2||_{Q_1}||f||Q_1 + k_1||e_1||f||_{Q_1}||_{Q_2}||I_\beta^\gamma H_3(f)||_{Q_3}\\ \leq & ||e_2||_{Q_1}||f||_{Q_1} + k_1||e_1||_{Q_2}||f||_{Q_1}\frac{2}{\Gamma(\gamma)}||k||_{Q_3}||e_3||f||_{Q_1}||_Q\\ \leq & ||e_2||_{Q_1}||f||_{Q_1} + k_1||e_1||_{Q_2}||f||_{Q_1}\frac{2}{\Gamma(\gamma)}||k||_{Q_3}||e_3||_Q||f||_{Q_1}\\ \leq & ||e_2||_{Q_1}||f||_{Q_1} + \frac{2k_1||k||_{Q_3}}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q||f||^2_{Q_1}\\ \leq & ||e_2||_{Q_1}r + \frac{2k_1||k||_{Q_3}}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q r^2 \leq r.\end{aligned}$$

Then $A : B_r(D_{Q_1}(I)) \to D_{Q_1}(I)$ is continuous.

Step (III): Consider $V_r \subset B_r(D_{Q_1}(I))$ that consist of all functions that are almost everywhere nondecreasing on I. Like it was stated in [11] this set is convex ,nonempty and Bounded. Also it is closed set in $L_{Q_1}(I)$. Now from Lemma 2.1 the set V_r has compact in measure.

Step (IV): Again, we will demonstrate that A maintains the monotonicity of functions. Taking $f \in V_r$, we get f is almost everywhere nondecreasing on I. Moreover, the supposition (3), deduce that the operator $H_i(f), i = 1, 2, 3$ are almost everywhere nondecreasing on I and the Erdélyi-Köber fractional integral operator B is the same property [1], then the operator $W(f) = H_1(f)B(f)$ is almost everywhere nondecreasing on I and by above supposition it follows that $A : V_r \to V_r$ is continuous.

Step (V): Here, we'll demonstrate that A refers to the measure of noncompactness as a contraction. Let, $X \subset V_r$ is a nonempty set and consider $\varepsilon > 0$ is arbitrary. Then a set $F \subset I$ and for $f \in X$. $mesF \leq \varepsilon$. Hence supposition (2) gives that

$$||H_1(f)\chi_F||_{Q_2} \leq ||H_1(f\chi_F)||_{Q_2} \leq ||e_1||f\chi_F||_{Q_1}||_{Q_2} \leq ||e_1||_{Q_2}||f\chi_F||_{Q_1}.$$

Correspondingly,

$$||H_2(f)\chi_F||_{Q_1} \leq ||e_2||_{Q_1}||f\chi_F||_{Q_1}.$$

Then we have,

$$\begin{aligned}||A(f)\chi_F|| \leq & ||H_2(f)\chi_F||_{Q_1} + ||W(f)\chi_F||_{Q_1}\\ \leq & ||H_2(f)\chi_F||_{Q_1} + ||H_1(f)B(f)\chi_F||_{Q_1}\\ \leq & ||e_2||_{Q_1}||f\chi_F||_{Q_1} + k_1||H_1(f)\chi_F||_{Q_2}||B(f)\chi_F||_{Q_3}\\ \leq & ||e_2||_{Q_1}||f\chi_F||_{Q_1} + k_1||H_1(f\chi_F)||_{Q_2}||B(f)||_{Q_3}\\ \leq & ||e_2||_{Q_1}||f\chi_F||_{Q_1} + \frac{2k_1}{\Gamma(\gamma)}||e_1||_{Q_2}||f\chi_F||_{Q_1}||k||_{Q_3}||H_3(f)||_Q\\ \leq & ||e_2||_{Q_1}||f\chi_F||_{Q_1} + \frac{2k_1}{\Gamma(\gamma)}||e_1||_{Q_2}||f\chi_F||_{Q_1}||k||_{Q_3}||e_3||_Q||f||_{Q_1}\\ \leq & ||e_2||_{Q_1}||f\chi_F||_{Q_1} + \frac{2k_1||k||_{Q_3}r}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q||f\chi_F||_{Q_1}.\end{aligned}$$

Then, by definition of $c(X)$, we have

$$c(A(X)) \leq \left(||e_2||_{Q_1} + \frac{2k_1||k||_{Q_3}}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q\right)c(X).$$

Since, $X \subset V_1$ is a nonempty, bounded and compact of D_{Q_1}, then by using Lemma 2.3 we have

$$\alpha(A(X)) \leq \left(||e_2||_{Q_1} + \frac{2k_1||k||_{Q_3}r}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_{Q} \right)\alpha(X).$$

Since, $\left(||e_2||_{Q_1} + \frac{2k_1||k||_{Q_3}r}{\Gamma(\gamma)}||e_2||_{Q_2}||e_3||_{Q} \right) < 1$. Then by Theorem 2.1, the proof of the theorem is we have completed. □

3.2. Uniqueness of the solution:

Next we discuss that equation (1.1) has exactly one solution.

Theorem 3.2 *Assume that supposition (1) to (4) are fulfilled. If*

$$\left(||e_2||_{Q_1} + \frac{4k_1||k||_{Q_3}r}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_{Q} \right) < 1,$$

where equation (3.1) gives the value of r*. Then* $f \in L_{Q_1}$ *in* V_r *has a unique solution for (1.1).*

Proof: Suppose f and g are any two possible solutions of equation (1.1), then we get

$$\begin{aligned}
&|f(t) - g(t)|\\
&= \left| H_2(f)(t) + \frac{H_1(f)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}H_3(f)(s)ds - H_2(g)(t) - \frac{H_1(g)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}H_3(g)(s)ds \right|\\
&\leq \left| H_2(f)(t) - H_2(g)(t) \right|\\
&+ \left| \frac{H_1(f)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}H_3(f)(s)ds - \frac{H_1(g)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}H_3(g)(s)ds \right|\\
&+ \left| \frac{H_1(g)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}H_3(f)(s)ds - \frac{H_1(g)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}H_3(g)(s)ds \right|\\
&\leq \left| e_2(t)||f||_{Q_1} - e_2(t)||g||_{Q_1} \right| + \frac{|H_1(f)(t) - H_1(g)(t)|}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}|H_3(f)(s)|ds\\
&+ \frac{|H_1(g)(t)|}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma)}}|H_3(f)(s) - H_3(g)(s)|ds\\
&\leq |e_2(t)|\left| ||f||_{Q_1} - ||g||_{Q_1} \right| + \frac{\left| e_1(t)||f||_{Q_1} - e_1(t)||g||_{Q_1} \right|}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}\left| e_3(s)||f||_{Q_1} \right|ds\\
&+ \frac{|e_1(t)|||g||_{Q_1}}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}\left| e_3(t)||f||_{Q_1} - e_3(t)||g||_{Q_1} \right|ds\\
&\leq |e_2(t)|||f - g||_{Q_1} + \frac{|e_1(t)|\left| ||f||_{Q_1} - ||g||_{Q_1} \right|}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}|e_3(s)|||f||_{Q_1}ds\\
&+ \frac{|e_1(t)|||g||_{Q_1}}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}|e_3(t)|\left| ||f||_{Q_1} - ||g||_{Q_1} \right|ds\\
&\leq |e_2(t)|||f - g||_{Q_1} + \frac{|e_1(t)|||f - g||_{Q_1}}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}|e_3(s)|||f||_{Q_1}ds\\
&+ \frac{|e_1(t)|||g||_{Q_1}}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}|e_3(t)|||f - g||_{Q_1}ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
||f-g||_{Q_1} &\leq ||e_2||_{Q_1}||f-g||_{Q_1} + ||e_1||_{Q_2}||f-g||_{Q_1}\frac{2k_1\beta}{\Gamma(\gamma)}||k||_{Q_3}||e_3||_Q||f||_{Q_1} \\
&+ ||e_1||_{Q_2}||g||_{Q_1}\frac{2k_1\beta}{\Gamma(\gamma)}||k||_{Q_3}||e_3||_Q||f-g||_{Q_1} \\
&\leq ||e_2||_{Q_1}||f-g||_{Q_1} + \frac{2k_1\beta||f||_{Q_1}}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q||f-g||_{Q_1}||k||_{Q_3} \\
&+ \frac{2k_1\beta||g||_{Q_1}}{\Gamma(\gamma)}||e_1||_{Q_2}||k||_{Q_3}||e_3||_Q||f-g||_{Q_1} \\
&\leq ||e_2||_{Q_1}||f-g||_{Q_1} + \frac{2k_1\beta||k||_{Q_3}||f||_{Q_1}}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q||f-g||_{Q_1} \\
&+ \frac{2k_1\beta||k||_{Q_3}||g||_{Q_1}}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q||f-g||_{Q_1} \\
&\leq ||e_2||_{Q_1}||f-g||_{Q_1} + (2k_1+2k_1)\frac{\beta||k||_{Q_3}r}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q||f-g||_{Q_1} \\
&\leq \left(||e_2||_{Q_1} + \frac{4k_1\beta||k||_{Q_3}}{\Gamma(\gamma)}r||e_1||_{Q_2}||e_3||_Q||f-g||_{Q_1}\right)||f-g||_{Q_1}.
\end{aligned}$$

Since, $\left(||e_2||_{Q_1} + \frac{4k_1\beta||k||_{Q_3}r}{\Gamma(\gamma)}||e_1||_{Q_2}||e_3||_Q\right) < 1$, then we have $f = g$ (almost everywhere). This completes the proof. □

3.3. Illustration

We give the following examples to demonstrate our results.

Example 3.1 Let the Q-functions $P(y) = Q(y) = 2y^3$ and $Q_3(y) = y^2 + 3y + 2$. Now, we have to prove that the operator $I_\beta^\gamma : L_Q(I) \to L_{Q_3}(I)$ is continuous. Also Lemma 3.1 is satisfied.
For any $\gamma \in (0,1)$ and $t \in [0,1]$ we get

$$k(t) = \int_0^t M(s^{1-\gamma})ds = \int_t^0 (2s)^{1-\gamma}ds = 2^{1-\gamma}(\frac{t^{4-3\gamma}}{4-3\gamma})$$

which suggest that Proposition 2.1 has accomplished. Moreover,

$$\int_0^t \phi_2(k(t))ds = \int_0^t \left[\left(2^{1-\gamma}\frac{t^{4-3\gamma}}{4-3\gamma}\right)^2 + 3\left(2^{1-\gamma}\frac{t^{4-3\gamma}}{4-3\gamma}\right) + 2\right]dt < \infty.$$

Then for $f \in L_Q(I)$, we have $I_\beta^\gamma : L_Q(I) \to L_{Q_3}(I)$ is continuous.

Remark 3.1 *In Orlicz space, the acting and continuity conditions for the operators $H_i(f) = e_i(t)f(t)$ are given in [27, Theorem 18.2] (see supposition (3)(iii)).*

Example 3.2 Let $H_i(f)(t) = e_i(t)f(t), i = 1,2,3$. Then we get

$$f(t) = e_2(t)f(t) + \frac{e_1(t)(f)(t)}{\Gamma(\gamma)}\beta\int_0^t \frac{s^{\beta-1}}{(t^\beta - s^\beta)^{1-\gamma}}e_3(s)f(s)ds, \gamma \in (0,1), t \in [0,1]$$

which illustrates a specific case of equation (1.1).

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