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On Recurrent maps on Interval

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ABSTRACT: An operator T acting on a metric compact space X is said to be recurrent if, for each nonempty open subset U of X, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap U \neq \emptyset$. In this paper, we introduce and study the notion of recurrence of interval map. We investigate some properties of this class of operators and show the links between recurrence, R-mixing and weakly R-mixing of operators and interval maps.

Key Words: Recurrence, R-mixing, weakly R-mixing, interval map.

Contents

1. Introduction

Throughout this paper, X will denote a compact metric space. An interval map refers to a continuous map acting on an interval.

The most studied notion in linear dynamics is that of hypercyclicity: An operator T acting on a metric compact space is said to be hypercyclic if there exists some vector $x \in X$ whose orbit under T;

$$Orb(T, x) := \{T^n x : n \geq 0\},$$

is a dense subset of X. In this case, the vector x is called a hypercyclic vector for T, and the set of all hypercyclic vectors of T is denoted by $HC(T)$.

For the more detailed information on hypercyclicity, see [1,2,3,4,5,6,10].

Conversely, a central notion in topological dynamics is that of recurrence. This notion goes back to Poincaré and Birkhoff and it refers to the existence of points in the space for which parts of their orbits under a continuous map "return" to themselves.

A vector $x \in X$ is called recurrent for an operator T or T-recurrent vector if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that

$$T^{k_n} x \longrightarrow x \quad \text{as} \quad n \longrightarrow \infty.$$

The set of all recurrent vectors for T is denoted by $Rec(T)$.

The operator T itself is called recurrent if, for each nonempty open set U of X, there exists some $n \in \mathbb{N}$ such that

$$T^{-n}(U) \cap U \neq \emptyset.$$

Following Amouch and Benchiheb in [7], the notion of recurrent sets of operators was introduced. It was proved that a set is recurrent if and only if the set of all recurrent vectors is dense. For more information about recurrent vectors and recurrent operators, the reader may refer to [12,8,9,11,13]. In this paper, we introduce and study the links between recurrence, R-mixing and weakly R-mixing in a metric compact space X, see [14].

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In section 2, we introduce and study the notion of recurrent vector operators. We prove that an operator is recurrent if and only if the set of all recurrent vectors is dense in X. Moreover, we prove that the set of all recurrent vectors is a G_δ type.

In section 3, we introduce the notion of R-mixing and weakly R-mixing operators. We prove that if f is topologically weakly R-mixing, then the product system $(X^n, \underbrace{f \times f \times ... \times f}_{n \ \ times})$ is recurrent for all integers $n \geq 1$. Also we prove that topologically R-mixing, implies topologically weakly R-mixing and topologically weakly R-mixing, implies that f^n is topologically weakly R-mixing for all $n \geq 1$ and f is totally recurrent.

In section 4, We present some example of R-mixing operators and interval maps.

2. Some properties of recurrent operators

In this section, we give some properties of recurrent operators.

Definition 2.1 ***(Recurrent)*** *Let (X, f) be a topological dynamical system. The map f is recurrent if, for all open set $U \subset X$, there exists $n \geq 0$ such that:*

$$f^n(U) \cap U \neq \emptyset.$$

A vector $x \in X \setminus \{0\}$ is said to be recurrent for f or f-recurrent if there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers such that

$$T^{k_n}x \longrightarrow x. \quad as \quad n \longrightarrow +\infty.$$

Definition 2.2 ***(Total recurrent)*** *Let (X, f) be a topological dynamical system. The map f is totally recurrent if, f^n is recurrent for all $n \geq 1$*

Proposition 2.1 *Let (X, f) be a topological dynamical system. The following assertions are equivalents.*

i) $\overline{Rec(f)} = X$.

ii) f *is recurrent.*

Moreover, The set of recurrent vectors f is even empty or a G_δ type.

Proof: $i) \Rightarrow ii)$ Assume that f has a dense set of recurrent points and let U be an open set in X. Let $y \in U$ be a recurrent vector and $\varepsilon > 0$ such that $B := B(y, \varepsilon) \subset U$. Then there exists $k \in \mathbb{N}$ such that $d(f^k y, y) < \varepsilon$. Thus $y \in U \cap f^{-k}(U)$. So f is recurrent.

$ii) \Rightarrow i)$Let

$$B := B(x, \varepsilon)$$

be a fixed open ball for some $x \in X$ and $\varepsilon < 1$. Suppose that f is recurrent, We need to show that there is a recurrent vector in B. Since f is recurrent, it follows that there exists a positive integer k_1 such that $f^{-k_1}(B) \cap B \neq \emptyset$. Hence, there exists $x_1 \in X$ such that $x_1 \in f^{-k_1}(B) \cap B$.

Since f is continuous, there exists $\varepsilon_1 < \frac{1}{2}$ such that

$$B_2 := B(x_1, \varepsilon_1) \subset B \cap f^{-k_1}(B)$$

Now since f is recurrent, so there exists $k_2 > k_1$ such that $x_2 \in f^{-k_2}(B_2) \cap B_2$ for some $x_2 \in X$. Now we use again the continuity of f, there exists $\varepsilon_2 < \frac{1}{2^2}$ such that

$$B_3 := B(x_2, \varepsilon_2) \subset B_2 \cap f^{-k_2}(B_2).$$

Continuing inductively we construct a sequence $(x_n)_{n\in\mathbb{N}}$ of X, a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$, and a sequence of positive real numbers $\varepsilon_k < \frac{1}{2^k}$, such that

$$B(x_n, \varepsilon_n) \subset B(x_{n-1}, \varepsilon_{n-1}) \quad \text{and} \quad f^{k_n}(B(x_n, \varepsilon_n)) \subset B(x_{n-1}, \varepsilon_{n-1}).$$

Since X is complete we conclude by Cantor's theorem that

$$\bigcap_n B(x_n, \varepsilon_n) = \{y\}.$$

for some $y \in X$. It follows that $f^{k_n}y \longrightarrow y$ that is, y is a recurrent point in the original ball B. Finally observe that

$$Rec(f) = \bigcap_{s=1} \bigcup_{n=0} \left\{x \in X : d(f^n x, x) < \frac{1}{s}\right\}.$$

which shows that the set of T-recurrent vectors is a G_δ-set. Indeed:
Let $x \in Rec(f)$. Then there exist a sequence $(n)_{n\in\mathbb{N}}$ of positive integer such that $T^n x \longrightarrow x$ as $n \longrightarrow +\infty$. Hence, for all $s \geq 1$, there exists $n \geq 0$ such that $d(f^n x, x) < \frac{1}{s}$, this implies that $x \in \bigcap_{s=1} \bigcup_{n=0} \left\{x \in X : d(f^n x, x) < \frac{1}{s}\right\}$. For the converse, let $x \in \bigcap_{s=1} \bigcup_{n=0} \left\{x \in X : d(f^n x, x) < \frac{1}{s}\right\}$, then for all $s \geq 1$ there exists $n \geq 0$ such that $d(f^n x, x) < \frac{1}{s}$, this means that $f^n x \longrightarrow x$ as $n \longrightarrow +\infty$. Hence, $x \in Rec(f)$. Since for all $s \geq 1$ the set $\left\{x \in X : d(f^n x, x) < \frac{1}{s}\right\}$ is open, it follows that $Rec(f)$ is a G_δ type. □

Definition 2.3 ***(Omega-limit set.)*** *Let (X, f) be a topological dynamical system. The ω-limit set of a point $x \in X$ denoted by $\omega(x, f)$, is the set of all limit points of the trajectory of x, that is*

$$\omega(x, f) := \bigcap \overline{\{f^k(x) \quad | \ k \geq n\}}.$$

Proposition 2.2 *Let (X, f) be a topological dynamical system. If there exists a point x such that $\omega(x, f) = X$, then f is recurrent.*

Proof: Assume that $\omega(x, f) = X$ for some point $x \in X$. This means that for every nonempty open set $U \subset X$, there exists $n \geq 0$ such that $f^n(x) \in U$.

To show that f is recurrent, we need to prove that for every nonempty open set $U \subset X$, there exists $m \geq 0$ such that $f^m(x) \in U$.

Let $U \subset X$ be a nonempty open set. By the definition of $\omega(x, f) = X$, there exists $n \geq 0$ such that $f^n(x) \in U$. Choose the smallest such $n \geq 0$, and denote it as m. Then $f^m(x) \in U$. Hence, there exists $m \geq 0$ such that $f^m(x) \in U$, so f is recurrent. □

Remark 2.1 *Let (X, f) be a topological dynamical system and U be an nonempty open set of X If for all nonempty open set V of X there exists $m \geq 0$ such that: $f^m(V) = U$. Then f is recurrent and there exists a dense G_δ-set G of points, because*

$$\bigcup_{n\geq 0} f^{-(n+m)}(U) \cap V \neq \emptyset.$$

So $\bigcup_{n\geq 0} f^{-(n+m)}(U)$ is dense in X. Since X is a compact metric space, there exists a countable basis of nonempty open sets, say $(U_k)_{k\geq 0}$. Therefore for all $k \geq 0$, the set $\bigcup_{n+m\geq 0} f^{-(n+m)}(U_k)$ is a dense open set.
Let

$$G := \bigcap_{k\geq 0} \bigcup_{n\geq 0} f^{-n}(U_k).$$

Then G is a dense G_δ-set.

Definition 2.4 *A topological dynamical system (X, f) is locally eventually onto if, for every nonempty open set $U \subset X$, there exists an integer N such that $f^n(U) = X$ for all $n \geq N$.*

Proposition 2.3 *Let (X, f) be a topological dynamical system. If (X, f) is locally eventually onto, then f is recurrent.*

Proof: Assume that (X, f) is locally eventually onto. Then for every nonempty open set $U \subset X$, there exists an integer N such that $f^n(U) = X$ for all $n \geq N$. So $f^n(U) \cap U = U \neq \emptyset$. Then f is recurrent. □

3. R-mixing, weakly R-mixing

In the following, we introduced the notion of R-mixing and weakly R-mixing, also we study the link between this notions.

Definition 3.1 ***(R-mixing, weakly R-mixing))*** *Let (X, f) be a topological dynamical system. The map f is topologically R-mixing if, for every nonempty open set U in X, there exists an integer $N \geq 0$ such that:*

$$\forall n \geq N, f^n(U) \cap U \neq \emptyset.$$

The map f is topologically weakly R-mixing if $f \times f$ is recurrent, where $f \times f$ is the map

$$\begin{array}{rcrcl} f \times f & : & X \times X & \longrightarrow & X \times X \\ & & (x, y) & \longmapsto & (f(x), f(y)). \end{array}$$

In the following proposition we prove that topological R-mixing implies topological weak R-mixing. Moreover, topological weak R-mixing implies total recurrent.

Proposition 3.1 *Let (X, f) be a topological dynamical system. If f is topologically weakly R-mixing, then the product system $(X^n, \underbrace{f \times f \times ... \times f}_{n \quad times})$ is recurrent for all integers $n \geq 1$.*

Proof: Let U an open set of X, we define

$$N(U) := \{n \geq 0 / U \cap f^{-n}(U) \neq \emptyset\}.$$

Let U_1, U_2 be a nonempty open sets in X. Since $f \times f$ is recurrent, there exists an integer $n \geq 0$ such that

$$(U_1 \times U_2) \cap f^{-n}(U_1 \times U_2) \neq \emptyset,$$

that is, $U_1 \cap f^{-n}(U_1) \neq \emptyset$. Then We remark that this implies for all U_1, open sets in X, we have

$$N(U_1) \neq \emptyset. \tag{3.1}$$

Now we are going to show that there exist a nonempty open set U in X such that

$$N(U) \subset N(U_1) \cap N(U_2).$$

We set

$$U := U_1 \cap f^{-n}(U_1).$$

This set is open, and we have shown that it's not empty. Let $k \in N(U)$

This integer exists by 3.1 and satisfises

$$U_1 \cap f^{-n}(U_2) \cap f^{-k}(U_1) \cap f^{-k-n}(U_2) \neq \emptyset,$$

which implies that

$$U_1 \cap f^{-k}(U_1) \neq \emptyset \text{ and } U_2 \cap f^{-k}(U_2) \neq \emptyset,$$

and thus

$$N(U) \subset N(U_1) \cap N(U_2).$$

Then, by a straightforward induction, we see that, for all nonempty open sets $U_1, ..., U_n$, there exist nonempty open sets U, such that

$$N(U) \subset N(U_1) \cap N(U_2) ... \cap N(U_n).$$

Combined with 3.1, this implies that $(X^n, \underbrace{f \times f \times ... \times f}_{n \quad times})$ is recurrent. $\square$

Theorem 3.1 *Let (X, f) be a topological dynamical system.*

i) If f is topologically R-mixing, then it is topologically weakly R-mixing.

ii) If f is topologically weakly R-mixing, then f^n is topologically weakly R-mixing for all $n \geq 1$ and f is totally recurrent.

Proof: $i)$ Assume that f is topologically R-mixing. Let W be nonempty open sets in $X \times X$. There exists nonempty open sets U, U' in X such that $U \times U' \subset W$. Since f is topologically R-mixing, there exists $N_1 \geq 0$ and $N_2 \geq 0$ such $f^n(U) \cap U \neq \emptyset$, for all $n \geq N_1$ and $f^n(U') \cap U' \neq \emptyset$, for all $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then there exists $N \geq 0$ such that: $f^N(U) \cap U \neq \emptyset$ and $f^N(U') \cap U' \neq \emptyset$. So

$$f^N(U) \times f^N(U') \cap U \times U' \neq \emptyset,$$

Hence

$$f^N(U \times U') \cap U \times U' \neq \emptyset,$$

then

$$f^N(W) \cap W \neq \emptyset.$$

We deduce that f is topologically weakly R-mixing.
$ii)$ From now, we assume that f is topologically weakly R-mixing and we fix $n \geq 1$. Let U, U' be two open sets in X. We define

$$W = U \times f^{-1}(U) \times ... f^{-(n-1)}(U) \times U \times f^{-1}(U) \times ... f^{-(n-1)}(U).$$

The set W is open in X^{2n}. Moreover, the system $(X^{2n}, \underbrace{f \times ... \times f}_{2n \quad times})$ is recurrent by proposition 3.1. Then there exists $k \geq 0$ such that:

$$f^{-k}(W) \cap W \neq \emptyset,$$

This implies that

$$f^{-(k+i)}(U) \cap U \neq \emptyset \quad \text{and} \quad f^{-(k+i)}(U') \cap U' \neq \emptyset, \quad \text{for all} \quad i \in [0, n-1].$$

We choose $i \in [0, n-1]$ such that $k + i$ is a multiple of n, we write $k + i = np$. We deduce that

$$(f \times f)^{-np}(U \times U') \cap U \times U' \neq \emptyset.$$

Then f^n is topologically weakly R-mixing for all $n \geq 1$. This trivially implies that f^n is totally recurrent. □

Proposition 3.2 *Let $f \colon [a, b] \to [a, b]$ be an interval map. The following assertions are equivalents.*

i) f is topologically R-mixing

ii) for all $\varepsilon > 0$ and all non degenerate intervals $J \subset [a, b]$, there exists an integer N such that:

$$f^n(J) \supset [a, a + \varepsilon] \quad (res \quad [b - \varepsilon, b]) \quad for\ all \quad n \geq N$$

.

Proof: $i) \Rightarrow ii)$ Assume that f is topologically R-mixing. Let $\varepsilon > 0$. Let $U := (\varepsilon, a + \varepsilon)$ and $U' := (a, b - \varepsilon)$.

If J is a nonempty open interval, there exists N_1, N_2, N_3 such that $f^n(J) \cap J \neq \emptyset$, for all $n \geq N_1$. $f^n(U) \cap U \neq \emptyset$, for all $n \geq N_2$.

$f^n(U') \cap U' \neq \emptyset$, for all $n \geq N_3$. Since f is topologically $R-$mixing, then the system (f^3, X^3) is recurrent. Then there exists $n \geq 0$ such that $f^n(J \times U \times U') \cap J \times U \times U' \neq \emptyset$, for all $n \geq \max\{N_1, N_2, N_3\}$.

Since $f^n(J)$ meets both U and U', which implies that $f^n(J) \supset [a, a+\varepsilon]$ by connectedness.
ii) $\Rightarrow$ *i)* Suppose now that, for every $\varepsilon > 0$ and every non degenerate interval $J \subset [a, b]$, there exists an integer N such that $f^n(J) \supset [a, a+\varepsilon]$ res $[b-\varepsilon, b]$ for all $n \geq N$. Let U be a nonempty open set in $[a, b]$. We choose a nonempty open subinterval K such that $K \subset U$, and neither a nor b is an endpoint of K. There exists $\varepsilon > 0$ such that $K \subset [a, a+\varepsilon]$. By assumption, there exists N such that $f^n(J) \supset [a, a+\varepsilon] \supset K$ for all $n \geq N$. This implies that $f^n(U) \cap U \neq \emptyset$ for all $n \geq N$. We conclude that f is topologically R-mixing. □

Lemma 3.1 *Let $f : I \longrightarrow I$ be a recurrent interval map.*

i) The image of a non degenerate interval is a non degenerate interval.

ii) The map f is onto.

Proof: *i)* Let $I = [a, b]$ and J be a non degenerate interval of I. Since J is connected, $f(J)$ is also connected, that is, it is an interval. Assume that $f(J) = \{y\}$. Since f is recurrent, then there exists $n \geq 0$ such that $f^n(J) \cap J \neq \emptyset$. Then $f^{n+1}(J) \cap f(J) = y \neq \emptyset$. So $f^{n+1}(f(J)) \cap f(y) \neq \emptyset$, this is implies that $f^{n+1}(y) = f(y)$. Then $f^n(J) = f(y)$. According to the proposition 3.2, then for all $\varepsilon > 0$ and for all subinterval J of I there exists $N \geq 0$ such that $[a, a+\varepsilon] \subset f^n(J)$ for all $n \geq N$. Absurd and we conclude the result.
ii) It is enough to show that $I \subset f(I)$. Since f is a recurrent map, then for all non degenerate interval J of I, there exists an integer $n \geq 0$ such that $f^n(J) \cap J \neq \emptyset$. Then there exists a vector $x \in I$ such that $x \in f^n(J) \cap J \neq \emptyset$, so $x = f(f^{n-1}(y))$. Since $f^{n-1}(y) \in I$. Thus $I \subset f(I)$ and we conclude that $f(I) = I$. □

4. A basic example of R-mixing interval map

in this section, we give some examples of R-mixing interval map and we study some properties of total recurrent interval map.

Definition 4.1 *Let f be an interval map and $\lambda > 1$. Suppose that f has finitely or countably many critical points. The map f is called λ-expanding if, for every subinterval $[x, y]$ on which f is monotone, $|f(y) - f(x)| \geq \lambda |x - y|$.*

Lemma 4.1 *[14] Let $f : I \to I$ be a λ-expanding interval map with $\lambda > N$ where N is a positive integer. Then, for every non degenerate subinterval J, there exists an integer $n \geq 0$ such that $f^n(J)$ contains at least N distinct critical points.*

Lemma 4.2 *[14] Let $f : I \to I$ be a interval map, $\lambda > 1$ and $a, b \in I$ with $a < b$. Suppose that*

$$f(a) = a \text{ and }, f(x) - f(a) \geq \lambda(x - a), \quad \forall x \in [a, b].$$

Then, for all $\epsilon > 0$, there exists $n \geq 0$ such that $[a, b] \subset f^n([a, a+\epsilon])$.

Example 4.1 *In this example, we are going to exhibit a family of topologically mixing interval maps. We employe Lemmas 4.1 and 4.2 for piecewise linear maps or for maps $f : I \to I$ such that the interval I can be divided into countably many subintervals on each of which f is linear. In these situations, f is λ-expanding if and only if the absolute value of the slope of f is greater than or equal to λ on each interval on which f is linear.*
These maps are piecewise linear, and the absolute value of their slope is constant.
We fix an integer $p \geq 2$. We define the map $T_p : [0, 1] \to [0, 1]$ by:

$$T_p = \begin{cases} px - 2k & si\, 0 \leq k \leq \frac{p-1}{2},\ x \in [\frac{2k}{p}, \frac{2k+1}{p}] \\ -px + 2k + 2 & si\, 0 \leq k \leq \frac{p-1}{2},\ x \in [\frac{2k+1}{p}, \frac{2k+2}{p}]. \end{cases}$$

The slope of T_p is either p or $-p$ on each interval of monotonicity. The image of each interval of monotonicity is $[0,1]$.

- *If $p = 2$, T_2 is the so called tent map.*

Let J be a non degenerate interval in $[0,1]$. The image of a non degenerate interval by T_p is obviously non degenerate, then $T_p^n(J)$ is also a non degenerate interval for all $n \geq 0$. By Lemma 4.1, there exists n such that $T_p^n(J)$ contains $p-1$ distinct critical points.

- *If $p \geq 3$, $T_p^n(J)$ contains at least one critical point whose image is 0, and thus $0 \in T_p^{n+2}(J)$, because 0 is a fixed point.*
- *If $p = 2$, $T_p^n(J)$ contains the unique critical point $\frac{1}{2}$ and $T_p^2(\frac{1}{2}) = T_p(1) = 0$.*

In both cases, $T_p^{n+2}(J)$ is a non degenerate interval containing 0. Applying Lemma 4.2 with $a = 0$ and $b = \frac{1}{p}$, (T_p is of slope $\lambda = p$ on $[0, \frac{1}{P}]$). We deduce that there exists an integer $m \geq 0$ such that $[0, \frac{1}{P}] \subset T_p^{n+m+2}(J)$. Then

$$T([0, \frac{1}{P}]) \subset T_p^{n+m+3}(J).$$

which implies that

$$J \subset [0,1] \subset T_p^{n+m+3}(J).$$

Then

$$T_p^k(J) = [0,1], \forall k \geq n+m+3.$$

we deduce that T_p is topologically R-mixing.

Lemma 4.3 *[14] Let $f : I \to I$ be an interval map, $x, y \in I$ and $m, n \in \mathbb{N}$. Let J be a subinterval of I containing no periodic point and suppose that $x, y, f^n(x)$ and $f^n(y)$ belong to J. If $x < f^n(x)$ then $y < f^n(y)$.*

Proposition 4.1 *If $f : I \to I$ is a recurrent interval map, then the set of periodic points is dense in I.*

Proof: Suppose that there exist $a, b \in I$, with $a < b$, such that $J = (a,b)$ contains no periodic point. Since f is recurrent, then there exists $n \geq 0$ such that $f^n(J) \cap J \neq \emptyset$. So there exists $x, y \in J$ such that $x \in J$ and $x = f^n(y) \in J$, where $y \in J$. Then there exists integers $n > 0$ and $0 < p < q$ such that

$$x < f^n(x) < b \text{ and } a < f^q(x) < f^p(x) < x.$$

Set $y := f^p(x)$. Then we have

$$a < f^{q-p}(y) < y < x < f^n(x) < b.$$

By Lemma 4.3 applied to $J = (a,b)$ this is impossible. This concludes the proof. □

Proposition 4.2 *Let $f \colon I \to I$ be an interval map. If f is totally recurrent, then it is topologically R-mixing.*

Proof: We write $I = [a,b]$. Let J be a non degenerate subinterval of I and $\varepsilon > 0$. According to Proposition4.1, the periodic points are dense in I. Thus, there exist periodic points x, x_1, x_2 with $x \in J$, $x_1 \in (a, a+\varepsilon)$ and $x_2 \in (b-\varepsilon, b)$. Moreover, x_1 and x_2 can be chosen in such a way that their orbits are included in (a,b) because there is at most one periodic orbit containing a (resp. b). We set

$$\forall i \in \{1,2\},\ y_i := \min\{f^n(x_i) \mid n \geq 0\} \text{ and } z_i := \max\{f^n(x_i) \mid n \geq 0\}.$$

Then $y_1 \in (a, x_1] \subset (a, a+\varepsilon)$, $z_2 \in [x_2, b) \subset (b-\varepsilon, b)$ and $y_2, z_1 \in (a,b)$. Let k be a common multiple of the periods of x, y_1 and y_2. We set $g := f^k$ and

$$K := \bigcup_{n=0}^{+\infty} g^n(J).$$

The point $x \in J$ is fixed under the action of g and thus $g^n(J)$ contains x for all $n \geq 0$. This implies that K is an interval. Moreover K is dense in $[a, b]$ because g is recurrent, and hence $K \supset (a, b)$. It follows that $y_1, y_2, z_1, z_2 \in K$. For $i = 1, 2$, let p_i and q_i be non negative integers such that $y_i \in g^{p_i}(J)$ and $z_i \in g^{q_i}(J)$. We set $N := \max\{p_1, p_2, q_1, q_2\}$. Since y_1, y_2, z_1, z_2 are fixed points of g, they belong to $g^N(J)$ and thus, by the intermediate value theorem, $[y_i, z_i] \subset g^N(J) = f^{kN}(J)$ for $i = 1, 2$. According to the definition of y_i, z_i, the interval $[y_i, z_i]$ contains the whole orbit of x_i. A trivial induction shows that $[y_i, z_i] \subset f^n([y_i, z_i])$ for all $n \geq 0$. Therefore,

$$\forall n \geq kN,\ [y_1, z_1] \cup [y_2, z_2] \subset f^n(J).$$

Since $a < y_1 < a + \varepsilon$ and $b - \varepsilon < z_2 < b$, the fact that $f^n(J)$ is connected implies that $[a, a + \varepsilon] \subset f^n(J)$ res $[b, b - \varepsilon] \subset f^n(J)$, for all $n \geq kN$. We conclude that f is topologically R-mixing. □

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