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Generating New Families of Distributions Using the Exponential Reliability Method

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ABSTRACT: In this paper, a new modifying method has been introduced by adding extra parameter to generate a new family of distributions that has more flexibility and better model fitting. A special case has been considered; two parameters Weibull distribution. All the main properties of the new modified Weibull are derived, including CDF, PDF, hazard and reliability functions. The maximum likelihood estimation method is used to estimate unknown parameters. The modified Weibull distribution has been applied on two lifetime data sets after analyzing them.

Key Words: Generalized distributions, Weibull distribution, reliability, hazard rate, reliability function, application, Akaike's information criterion, Bayesian information criterion, coefficient of determination, root mean square error.

Contents

1. Introduction

In statistical distributions, introducing new extra parameters to a family of distribution functions is very familiar. Common distributions are usually strongly effective, but sometimes these distributions show lack of fit in some cases and failed horribly in others, which gave modified and more generalized distributions the opportunity to appear. The goal of this process is to find new distributions that have better fitting for data and more flexibility. In the last few decades, modifying has been widely used, for example, beta-normal distribution which was proposed by Eugene et al. (2002) [4], TX family of distributions by Aljarrah et al. (2014) [2]. In addition, many improvements were made for the Weibull distribution [11], such as the transmuted Rayleigh distribution (2013) [9], the transmuted modified Weibull distribution (2013) [6], the transmuted inverse Weibull distribution (2013) [7].

This paper aims to introduce a new parameter to a family of distributions functions to produce a new more powerful family that has better fitting and more flexibility. This new method is called the Exponential Reliability Method (ERM). ERM can be used easily and effectively for analysis of data.

First, we will present the modified family and its main properties. Then, it will be applied to the Weibull distribution with two parameters. Two lifetime data sets will be analyzed and fitted by the Weibull distribution and the modified Weibull distribution using the ERM, we will compare between them using Akaike's information criterion, Bayesian information criterion, coefficient of determination and root mean square error. The maximum likelihood estimation method is used to estimate the parameters. All calculations were done using Wolfram's Mathematica.

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2. ER Method

Let X be a continuous random variable that follows a distribution with CDF $F(x)$, then the exponential reliability function of $F(x)$ for $x \in \mathbb{R}$ is defined by:

$$F_\alpha(x) = F(x) \ \ \alpha^{R(x)}, \tag{2.1}$$

which is also a CDF, and the corresponding PDF is given by:

$$f_\alpha(x) = f(x) \ \ \alpha^{R(x)} \ (1 - F(x) \ \log(\alpha) \ ), \tag{2.2}$$

where: $0 < \alpha \leq e$, $R(x)$ is the reliability function of $F(x)$, and $f(x)$ is the PDF of the distribution defined by $F(x)$.

The ERM distributions can be useful in lifetime data analysis. The reliability function by the ERM $R_\alpha(x)$ is given by:

$$R_\alpha(x) = 1 - F(x) \ \alpha^{R(x)}, \tag{2.3}$$

and hazard rate function $H_\alpha(x)$:

$$H_\alpha(x) = \frac{f(x) \ \alpha^{R(x)} \ (1 - F(x) \log(\alpha))}{1 - F(x) \ \alpha^{R(x)}}. \tag{2.4}$$

3. Weibull Distribution

A random variable X is said to follow Weibull distribution with three parameters, $X \sim W(a, b, c)$, if its CDF is given by:

$$G(x) = 1 - e^{-((-c+x)/b)^a}, \qquad x \geq c, \tag{3.1}$$

with PDF:

$$g(x) = \frac{a e^{\left(\frac{-c+x}{b}\right)^a} \ \left(\frac{-c+x}{b}\right)^{-1+a}}{b}, \qquad x > c, \tag{3.2}$$

where a is the shape parameter, b is the scale parameter, and c is the location parameter: $a, b > 0$ and $c \geq 0$.

If $c = 0$, then (3.1) and (3.2) will transform to the CDF and the PDF, respectively, of the Weibull distribution with two parameters $W(a, b)$:

$$G(x) = 1 - e^{-(x/b)^a}, \qquad x \geq 0, \tag{3.3}$$

$$g(x) = \frac{a e^{\left(\frac{x}{b}\right)^a} \ \left(\frac{x}{b}\right)^{-1+a}}{b}, \qquad x > 0. \tag{3.4}$$

Equations (3.5) and (3.6) are the reliability and hazard functions for the three parameters Weibull distribution:

$$R(x) = e^{-\left(\frac{-c+x}{b}\right)^a}, \qquad x \geq c, \tag{3.5}$$

$$H(x) = \frac{a \ \left(\frac{-c+x}{b}\right)^{-1+a}}{b}, \qquad x > c, \tag{3.6}$$

while (3.7) and (3.8) show the reliability and hazard functions for the two parameters Weibull distribution:

$$R(x) = e^{-\left(\frac{x}{b}\right)^a}, \qquad x \geq 0, \tag{3.7}$$

$$H(x) = \frac{a \ \left(\frac{x}{b}\right)^{-1+a}}{b}, \qquad x > 0. \tag{3.8}$$

4. Modified Weibull Distribution Using ERM

Consider the two parameters Weibull distribution. In (2.1), substitute $F(x)$ with (3.3), and then $R(x)$ with (3.7) to get:

$$F_\alpha(x) = \alpha^{e^{-\left(\frac{x}{b}\right)^a}} \left(1 - e^{-\left(\frac{x}{b}\right)^a}\right), \tag{4.1}$$

which is the cumulative density function (CDF) of the ER Weibull distribution, where $0 < \alpha \le e$.

Similarly, the probability density function (PDF) of it is shown in (4.2):

$$f_\alpha(x) = -\frac{a e^{-2\left(\frac{x}{b}\right)^a} \left(\frac{x}{b}\right)^a \alpha^{e^{-\left(\frac{x}{b}\right)^a}} \left(e^{\left(\frac{x}{b}\right)^a} \left(-1 + \log(\alpha)\right) - \log(\alpha)\right)}{x}. \tag{4.2}$$

Let $R_\alpha(x)$, $H_\alpha(x)$ be the reliability and hazard functions for the Weibull distribution using ERM, respectively, then:

$$R_\alpha(x) = 1 - \alpha^{e^{-\left(\frac{x}{b}\right)^a}} \left(1 + e^{-\left(\frac{x}{b}\right)^a}\right), \tag{4.3}$$

$$H_\alpha(x) = \frac{a e^{-\left(\frac{x}{b}\right)^a} \left(\frac{x}{b}\right)^a \alpha^{e^{-\left(\frac{x}{b}\right)^a}} \left(e^{\left(\frac{x}{b}\right)^a} \left(-1 + \log(\alpha)\right) - \log(\alpha)\right)}{x\left(-e^{\left(\frac{x}{b}\right)^a} + \left(-1 + e^{\left(\frac{x}{b}\right)^a}\right) \alpha^{e^{-\left(\frac{x}{b}\right)^a}}\right)}. \tag{4.4}$$

5. Parameters' Estimation

The modified Weibull distribution using ERM has three parameters; a, b, α. To estimate these parameters, the maximum likelihood estimation (MLE) method will be used.

Let L be the likelihood function. It can be expressed in the following form:

$$\begin{aligned} L &= \prod_1^n f(x_i; a, b, c, \alpha) \\ &= (-a)^n \; e^{-2\sum_1^n \left(\frac{x_i}{b}\right)^a} \alpha^{\sum_1^n e^{-\left(\frac{x_i}{b}\right)^a}} \prod_1^n \left(\frac{{x_i}^{a-1}}{b^a} \; \left(e^{\left(\frac{x_i}{b}\right)^a} \left(1 - \log(\alpha)\right) - \log(\alpha)\right)\right). \end{aligned} \tag{5.1}$$

Taking the natural logarithm for both sides to get:

$$\begin{aligned} \log(L) =& n \log(-a) - 2\sum_1^n \left(\frac{x_i}{b}\right)^a + \log(\alpha) \sum_1^n e^{-\left(\frac{x_i}{b}\right)^a} \\ &+ \sum_1^n \log\left(\frac{{x_i}^{a-1}}{b^a} \; \left(e^{\left(\frac{x_i}{b}\right)^a} \left(1 - \log(\alpha)\right) - \log(\alpha)\right)\right). \end{aligned} \tag{5.2}$$

Differentiate (5.2) with respect to a, b and α, respectively, and equate each derivative to zero:

$$\begin{aligned} \frac{\partial}{\partial a} \log(L) =& \frac{n}{a} \; - 2\sum_1^n \left(\left(\frac{x_i}{b}\right)^a \log\left(\frac{x_i}{b}\right)\right) - \log(\alpha) \sum_1^n \left(e^{-\left(\frac{x_i}{b}\right)^a} \left(\frac{x_i}{b}\right)^a \log\left(\frac{x_i}{b}\right)\right) \\ &+ \sum_1^n \left(\log\left(\frac{x_i}{b}\right) + \frac{e^{\left(\frac{x_i}{b}\right)^a} \left(\frac{x_i}{b}\right)^a \log\left(\frac{x_i}{b}\right) \left(1 - \log(\alpha)\right)}{e^{\left(\frac{x_i}{b}\right)^a} \left(1 - \log(\alpha)\right) - \log(\alpha)}\right) = 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \frac{\partial}{\partial b} \log(L) =& -\frac{a}{b} + 2\sum_1^n \left(\frac{a x_i^a}{b^{a+1}}\right) + \log(\alpha) \sum_1^n \left(e^{-\left(\frac{x_i}{b}\right)^a} \frac{a x_i^a}{b^{a+1}}\right) \\ &- \sum_1^n \left(e^{\left(\frac{x_i}{b}\right)^a} \frac{a x_i^a}{b^{a+1}} \; \left(1 - \log(\alpha)\right) - \log(\alpha)\right) = 0, \end{aligned} \tag{5.4}$$

$$\frac{\partial}{\partial \alpha} \log (L) = \frac{1}{\alpha} \sum_{1}^{n} \left(e^{-\left(\frac{x_i}{b}\right)^a} \right) - \sum_{1}^{n} \left(\frac{e^{\left(\frac{x_i}{b}\right)^a} + 1}{\alpha \left(e^{\left(\frac{x_i}{b}\right)^a} (1 - \log(\alpha)) - \log(\alpha) \right)} \right) = 0. \tag{5.5}$$

Solving this non-linear system (5.3-5.5) iteratively provides us with MLE estimates $\hat{a}$, $\hat{b}$ and $\hat{\alpha}$.

6. Application

To show that the ERM Weibull distribution has a better fitting and more flexibility, a comparison should be held between it with the Weibull distribution. R^2, RMSE, AIC and BIC criteria will be discussed for the distributions. The lower AIC, BIC, RMSE values the better. A higher value for R^2 indicates a better distribution.

- Akaike's Information Criterion (AIC) [1] is calculated using (6.1):

$$AIC = -2 \log L \left(\hat{\theta} \right) + 2m, \tag{6.1}$$

  where $L\left(\hat{\theta}\right)$ is the estimated likelihood function, m is number of parameters to be estimated. Another form for the AIC with a correction term is called the Corrected Akaike's Information Criterion (AICC) [5] is calculated by (6.2):

$$AICC = AIC + \frac{2m(m+1)}{n-m+1}, \tag{6.2}$$

  where n is the sample size.

- Bayesian's Information Criterion (BIC) [10] is calculated by (6.3):

$$BIC = -2 \log L \left(\hat{\theta} \right) + m \log n. \tag{6.3}$$

- The Coefficient of Determination (R^2) is calculated by:

$$R^2 = \frac{\sum_{i=1}^{n} \left(\hat{F}(x_i) - \bar{F} \right)^2}{\sum_{i=1}^{n} \left(\hat{F}(x_i) - \bar{F} \right)^2 + \sum_{i=1}^{n} \left(F_n(x_i) - \hat{F}(x_i) \right)^2}, \tag{6.4}$$

  where $\hat{F}(x_i)$ is the estimated CDF, $\bar{F}$ is the average of the estimated CDF and $F_n(x)$ is the empirical distribution function with the form:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(x_i \le x), \tag{6.5}$$

  where $I(x_i \le x)$ is the indicator function, $I(x_i \le x) = 1$ if $x_i \le x$ and 0 otherwise, for values of x_i in ascending order.

- The Root Mean Square Error (RMSE) which is calculated by:

$$RMSE = \left(\frac{1}{n} \sum_{i=1}^{n} \left(F_n(x_i) - \hat{F}(x_i) \right)^2 \right)^{\frac{1}{2}}. \tag{6.6}$$

Now, we will analyze two lifetime data sets in order to show the performance of the ERM Weibull distribution compared to the Weibull distribution with two and three parameters. 72 guinea pigs that are infected with virulent tubercle bacilli are subjected to a survival-time study (in days).

The data below were observed and reported in [3]. The results are as follows: 10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

Data show right skewness (skewness=1.37 and kurtosis=2.22).

Distributions	θ	AIC	AICC	BIC	RMSE	R^2
W(a,b)	$\hat{a} = 1.825$ $\hat{b} = 199.602$	858.724	858.893	863.277	0.050	0.966
W(a,b,c)	$\hat{a} = 1.761$ $\hat{b} = 193.357$ $\hat{c} = 5.00$	860.353	860.696	867.183	0.048	0.969
ERMW(a,b,α)	$\hat{a} = 2.067$ $\hat{b} = 265.436$ $\hat{\alpha} = 2.102$	857.905	858.248	864.735	0.041	0.978

Table 1: Comparing between distributions with data set 1.

Results in table 1 prove that ERMW is better than the Weibull since it has the lowest values for AIC, AICC and RMSE, and the highest R^2 value.

Figure 1 presents plots of the estimated PDF's curves with the data set.

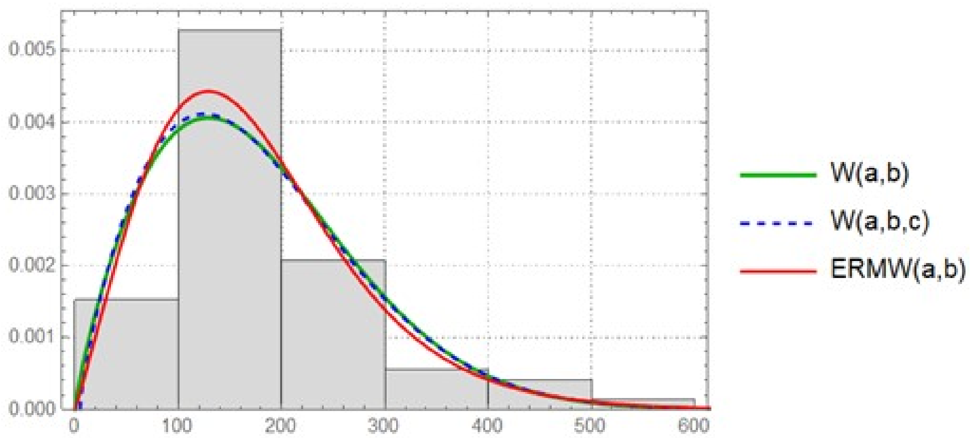

Figure 1: Estimated PDF curves with a histogram representation for data.

Figure 2 shows the reliability functions for the three distributions and Figure 3 shows their hazard functions.

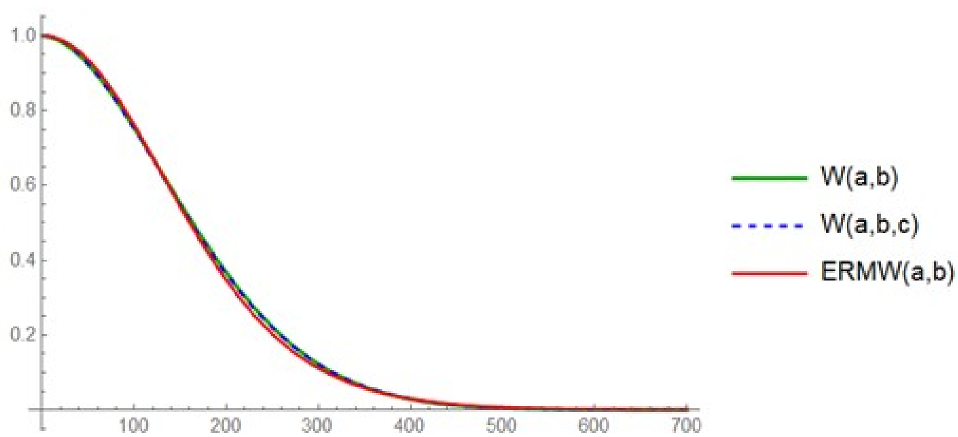

Figure 2: Reliability functions for the distributions.

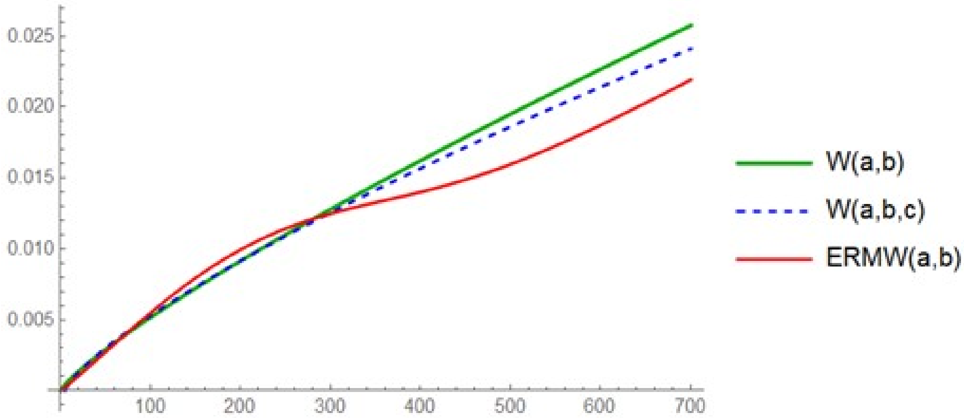

Figure 3: Hazard functions for the distributions.

The second data set is obtained from the Breast Tumors Detection Unit in Benha Hospital University, Egypt. Ages for 155 patients suffering from the breast cancer were recorded from June to November 2014. The data below were observed and reported in [8].

12, 17, 18, 18, 20, 24, 24, 25, 26, 28, 28, 28, 29, 30, 30, 30, 30, 31, 31, 31, 32, 32, 32, 32, 33, 34, 34, 35, 35, 35, 35, 35, 35, 35, 36, 36, 36, 36, 36, 36, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 38, 39, 39, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 40, 41, 42, 42, 42, 42, 42, 42, 42, 42, 43, 43, 44, 44, 45, 45, 45, 45, 45, 45, 45, 46, 46, 47, 48, 48, 48, 48, 48, 48, 48, 49, 49, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 50, 52, 52, 52, 53, 54, 54, 55, 56, 56, 58, 58, 58, 59, 60, 60, 60, 60, 60, 60, 60, 63, 65, 65, 66, 69, 80, 90, 90.

Data show almost symmetric distribution (skewness=0.685 and kurtosis=5.163).

Distributions	θ	AIC	AICC	BIC	RMSE	R^2
W(a,b)	$\hat{a} = 3.687$ $\hat{b} = 48.11$	1224.59	1224.67	1230.68	0.069	0.932
W(a,b,c)	$\hat{a} = 2.951$ $\hat{b} = 38.337$ $\hat{c} = 9.277$	1220.96	1221.12	1230.09	0.061	0.948
ERMW(a,b,α)	$\hat{a} = 4.300$ $\hat{b} = 57.670$ $\hat{\alpha} = 2.438$	1214.97	1215.12	1224.1	0.058	0.956

Table 2: Comparing between distributions with data set 2.

Results from table 2 prove that ERMW is better than the others because it shows the least values for AIC, AICC, BIC and RMSE, and the greatest value for R^2.

Figure 4 shows plots of the estimated PDF's curves with the data set.

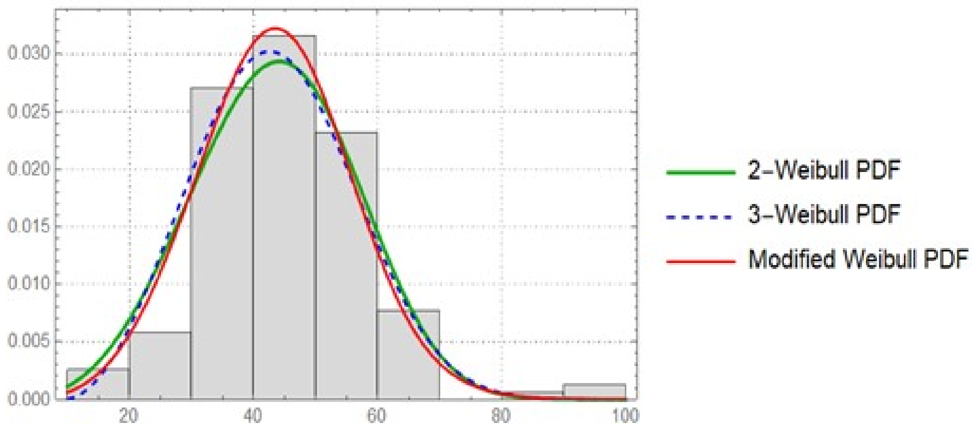

Figure 4: Estimated PDF curves with a histogram representation for data.

Figure 5 shows the reliability functions for the three distributions while Figure 6 shows their hazard functions.

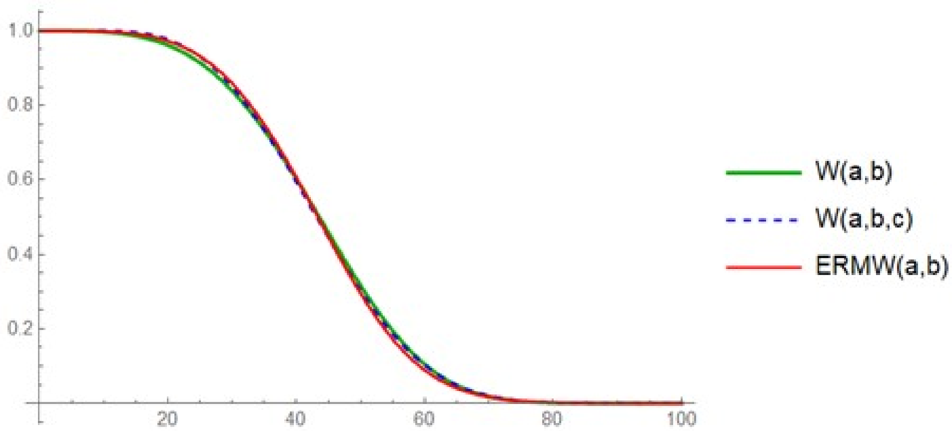

Figure 5: Reliability functions for distributions.

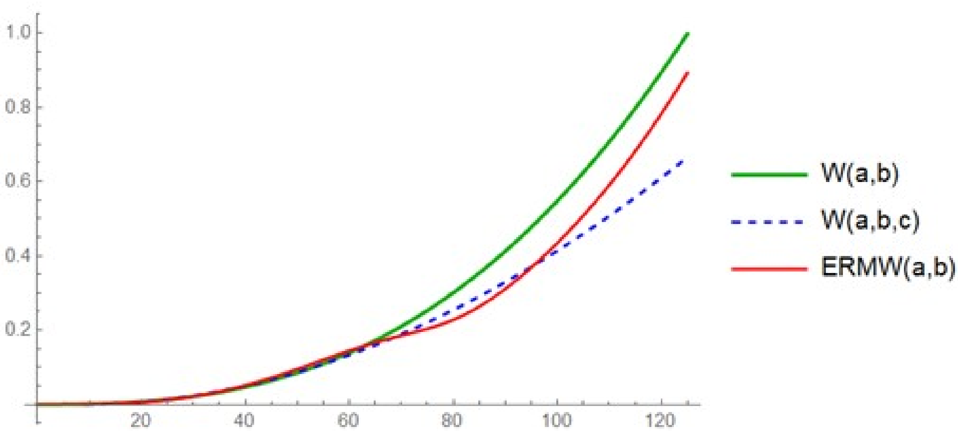

Figure 6: Hazard functions for distributions.

7. Conclusion

In this paper, we introduce a new modifying method for an old family of distributions to generate a new family that has better fit and more flexibility. We have called the Exponential Reliability Method (ERM). The ERM depends on adding an extra parameter α to the distribution which shows better properties. A special case is studied on the Weibull distribution, then a comparison is discussed, using two data sets with four criteria, between the Weibull distribution and its new form, ERMW. Results prove our assumption and show that the ERMW is better in some cases.

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