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Subclasses of Close-to-Convex Functions With Respect to Symmetric and Conjugate Points

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ABSTRACT: This paper is concerned with the estimates of initial coefficient bounds for certain subclasses of analytic functions with fixed point in the unit disc $E = \{z \in \mathbb{C} :| z |< 1\}$ and with respect to symmetric and conjugate points. The classes are defined by subordinating to Janowski function and this idea will motivate the other researchers to work in this direction.

Key Words: Univalent functions, close-to-convex functions, quasi-convex functions, symmetric points, conjugate points, subordination, coeffcient estimates.

Contents

1. Introduction

Let $\mathcal{U}$ be the class of analytic functions defined in the unit disc $E = \{z \in \mathbb{C} :| z |< 1\}$ and of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which satisfy the conditions $u(0) = 0$ and $| u(z) |< 1$. The functions in the class $\mathcal{U}$ are known as Schwarzian functions.

The concept of subordination has great importance in the theory of analytic functions. Many subclasses of analytic functions were investigated by various authors by subordinating to different type of functions. The notion of subordination was given by Littlewood [12] and defined as follows:
For two analytic functions f and g in the unit disc E, f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarzian function $u \in \mathcal{U}$ such that $f(z) = g(u(z))$.

By $\mathcal{A}$, we denote the class of analytic functions f in the unit disc $E = \{z :| z |< 1\}$ and which are of the form $f(z) = z+\sum_{k=2}^{\infty} a_k z^k$. The class of functions $f \in \mathcal{A}$, which are univalent in E, is denoted by $\mathcal{S}$.

The most remarkable result in the theory of univalent functions was Bieberbach's conjecture, established by L. Bieberbach [4]. It states that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, ...$ and it remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [6], proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were come into existence and it gave rise to some new subclasses of $\mathcal{S}$.

Firstly, let's have an overview of some fundamental classes of univalent functions, which are relevant to the study of this paper:
The well known classes $\mathcal{S}^*$ of starlike functions and $\mathcal{K}$ of convex functions are defined respectively as

$$\mathcal{S}^* = \left\{f \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in E\right\},$$

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and

$$\mathcal{K}=\left\{f\in\mathcal{A}:Re\left(\frac{(zf'(z))'}{f'(z)}\right)>0,\ z\in E\right\}.$$

The classes $\mathcal{S}^*$ and $\mathcal{K}$ are related by the Alexander relation [3] as $f\in\mathcal{K}$ if and only if $zf'\in\mathcal{S}^*$.

A function $f\in\mathcal{A}$ is said to be close-to-convex if there exists a convex function h such that $Re\left(\frac{f'(z)}{h'(z)}\right)>0$ or equivalently there exists a starlike function g such that $Re\left(\frac{zf'(z)}{g(z)}\right)>0$. The class of close-to-convex functions is denoted by $\mathcal{C}$ and was introduced by Kaplan [11].

Further, Noor [13] introduced the class $\mathcal{C}^*$ of quasi-convex functions. A function $f\in\mathcal{A}$ is said to be quasi-convex if there exists a convex function $h\in\mathcal{K}$ such that

$$Re\left(\frac{(zf'(z))'}{h'(z)}\right)>0, z\in E.$$

Every quasi-convex function is convex. Obviously $f(z)\in\mathcal{C}^*$ if and only if $zf'\in\mathcal{C}$.

Sakaguchi [16] established the class $\mathcal{S}_s^*$ of the functions $f\in\mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>0.$$

The functions in the class $\mathcal{S}_s^*$ are called starlike functions with respect to symmetric points. Clearly, the class $\mathcal{S}_s^*$ is contained in the class $\mathcal{C}$ of close-to-convex functions, as $\frac{f(z)-f(-z)}{2}$ is a starlike function [5] in E.

Later on, Das and Singh [5] introduced the class $\mathcal{K}_s$ of the functions $f\in\mathcal{A}$ which satisfy the following condition:

$$Re\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)>0.$$

The functions in the class $\mathcal{K}_s$ are called convex functions with respect to symmetric points. Clearly $f\in\mathcal{K}_s$ if and only if $zf'\in\mathcal{S}_s^*$.

El-Ashwah and Thomas [7] established the class $\mathcal{S}_c^*$, the class of starlike functions with respect to conjugate points and $\mathcal{K}_c$, the class of convex functions with respect to conjugate points, as follows:

$$\mathcal{S}_c^*=\left\{f\in\mathcal{A}:Re\left(\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}\right)>0, z\in E\right\}$$

and

$$\mathcal{K}_c=\left\{f\in\mathcal{A}:Re\left(\frac{2(zf'(z))'}{\left(f(z)+\overline{f(\overline{z})}\right)'}\right)>0, z\in E\right\}.$$

Janteng et al. [9] studied the following classes:
$\mathcal{C}_s^*=\left\{f\in\mathcal{A}:Re\left(\frac{2zf'(z)}{g(z)-g(-z)}\right)>0, g\in\mathcal{S}_s^*, z\in E\right\}$, the class of close-to-convex functions with respect to symmetric points.
$\mathcal{C}_c=\left\{f\in\mathcal{A}:Re\left(\frac{2zf'(z)}{g(z)+\overline{g(\overline{z})}}\right)>0, g\in\mathcal{S}_c^*, z\in E\right\}$, the class of close-to-convex functions with respect to conjugate points.

$\mathcal{Q}_s^* = \left\{ f \in \mathcal{A} : Re\left(\frac{2(zf'(z))'}{(h(z)-h(-z))'}\right) > 0, h \in \mathcal{K}_s, z \in E \right\}$, the class of quasi-convex functions with respect to symmetric points.

$\mathcal{Q}_c = \left\{ f \in \mathcal{A} : Re\left(\frac{2(zf'(z))'}{\left(h(z)+\overline{h(\overline{z})}\right)'}\right) > 0, h \in \mathcal{K}_c, z \in E \right\}$, the class of quasi-convex functions with respect to conjugate points.

Kanas and Ronning [10] introduced an interesting class $\mathcal{A}(w)$ of analytic functions of the form

$$f(z) = (z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k \tag{1.1}$$

and normalized by the conditions $f(w) = 0, f'(w) = 1$, where w is a fixed point in E.

Also the classes of w-starlike functions and w-convex functions were defined in [10] as follows:

$$\mathcal{S}^*(w) = \left\{ f \in \mathcal{A}(w) : Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0, z \in E \right\},$$

and

$$\mathcal{K}(w) = \left\{ f \in \mathcal{A}(w) : 1 + Re\left(\frac{(z-w)f''(z)}{f'(z)}\right) > 0, z \in E \right\}.$$

The class $\mathcal{S}^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $\mathcal{K}(w)$ is defined by the property that the image of any circular arc centered at w is convex. For $w = 0$, the classes $\mathcal{S}^*(w)$ and $\mathcal{K}(w)$ agree with the well known classes of starlike and convex functions, respectively. Also it is obvious that $f \in \mathcal{K}(w)$ if and only if $(z-w)f' \in \mathcal{S}^*(w)$. Various authors such as Acu and Owa [1], Al-Hawary et al. [2], Olatunji and Oladipo [15] and Singh and Singh [17] have worked on the classes of analytic functions with fixed point.

The class $\mathcal{P}[C,D]$ consists of the functions p analytic in E with $p(0) = 1$ and subordinate to $\frac{1+Cz}{1+Dz}$, $(-1 \le D < C \le 1)$. This class was established by Janowski [8].

For $-1 \le B < A \le 1$, Oladipo [14] studied the following subclasses of $\mathcal{A}(w)$:

$$\mathcal{S}_s^*(w;A,B) = \left\{ f \in \mathcal{A}(w) : \frac{2(z-w)f'(z)}{f(z)-f(-z)} \prec \frac{1+A(z-w)}{1+B(z-w)}, z \in E \right\},$$

$$\mathcal{K}_s(w;A,B) = \left\{ f \in \mathcal{A}(w) : \frac{2((z-w)f'(z))'}{(f(z)-f(-z))'} \prec \frac{1+A(z-w)}{1+B(z-w)}, z \in E \right\},$$

$$\mathcal{S}_c^*(w;A,B) = \left\{ f \in \mathcal{A}(w) : \frac{2(z-w)f'(z)}{f(z)+\overline{f(\overline{z})}} \prec \frac{1+A(z-w)}{1+B(z-w)}, z \in E \right\},$$

$$\mathcal{K}_c(w;A,B) = \left\{ f \in \mathcal{A}(w) : \frac{2((z-w)f'(z))'}{\left(f(z)+\overline{f(\overline{z})}\right)'} \prec \frac{1+A(z-w)}{1+B(z-w)}, z \in E \right\}.$$

For $A = 1, B = -1$, the classes $\mathcal{S}_s^*(w;A,B)$, $\mathcal{K}_s(w;A,B)$, $\mathcal{S}_c^*(w;A,B)$ and $\mathcal{K}_c(w;A,B)$, reduce to $\mathcal{S}_s^*(w)$, $\mathcal{K}_s(w)$, $\mathcal{S}_c^*(w)$ and $\mathcal{K}_c(w)$, respectively.

In the subsequent work, we make the assumptions that $-1 \le D < C \le 1$, $-1 \le B < A \le 1$, $z \in E$. Motivated by the above defined work, we now introduce the following subclasses of $\mathcal{A}(w)$, associated with Janowski function:

Definition 1.1 A function $f \in \mathcal{A}(w)$ is said to be in the class $\mathcal{C}_s^*(w; A, B; C, D)$ if

$$\frac{2(z-w)f'(z)}{g(z)-g(-z)} \prec \frac{1+C(z-w)}{1+D(z-w)},$$

where $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k(z-w)^k \in \mathcal{S}_s^*(w; A, B)$.
The following points are to be noted:
(i) $\mathcal{C}_s^*(0; A, B; C, D) \equiv \mathcal{C}_s^*(A, B; C, D)$.
(ii) $\mathcal{C}_s^*(w; 1, -1; 1, -1) \equiv \mathcal{C}_s^*(w)$.
(iii) $\mathcal{C}_s^*(0; 1, -1; 1, -1) \equiv \mathcal{C}_s^*$.

Definition 1.2 A function $f \in \mathcal{A}(w)$ is said to be in the class $\mathcal{Q}_s^*(w; A, B; C, D)$ if

$$\frac{2((z-w)f'(z))'}{(h(z)-h(-z))'} \prec \frac{1+C(z-w)}{1+D(z-w)},$$

where $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k(z-w)^k \in \mathcal{K}_s(w; A, B)$.
The following observations are obvious:
(i) $\mathcal{Q}_s^*(0; A, B; C, D) \equiv \mathcal{Q}_s^*(A, B; C, D)$.
(ii) $\mathcal{Q}_s^*(w; 1, -1; 1, -1) \equiv \mathcal{Q}_s^*(w)$.
(iii) $\mathcal{Q}_s^*(0; 1, -1; 1, -1) \equiv \mathcal{Q}_s^*$.

Definition 1.3 A function $f \in \mathcal{A}(w)$ is said to be in the class $\mathcal{C}_c(w; A, B; C, D)$ if

$$\frac{2(z-w)f'(z)}{g(z)+\overline{g(\overline{z})}} \prec \frac{1+C(z-w)}{1+D(z-w)},$$

where $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k(z-w)^k \in \mathcal{S}_c^*(w; A, B)$.
We have the following observations:
(i) $\mathcal{C}_c(0; A, B; C, D) \equiv \mathcal{C}_c(A, B; C, D)$.
(ii) $\mathcal{C}_c(w; 1, -1; 1, -1) \equiv \mathcal{C}_c(w)$.
(iii) $\mathcal{C}_c(0; 1, -1; 1, -1) \equiv \mathcal{C}_c$.

Definition 1.4 A function $f \in \mathcal{A}(w)$ is said to be in the class $\mathcal{Q}_c(w; A, B; C, D)$ if

$$\frac{2((z-w)f'(z))'}{(h(z)+\overline{h(\overline{z})})'} \prec \frac{1+C(z-w)}{1+D(z-w)},$$

where $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k(z-w)^k \in \mathcal{K}_c(w; A, B)$.
The following points are obvious:
(i) $\mathcal{Q}_c(0; A, B; C, D) \equiv \mathcal{Q}_c(A, B; C, D)$.
(ii) $\mathcal{Q}_c(w; 1, -1; 1, -1) \equiv \mathcal{Q}_c(w)$.
(iii) $\mathcal{Q}_c(0; 1, -1; 1, -1) \equiv \mathcal{Q}_c$.

In the present investigation, we seek the upper bounds of the first four coefficients for the functions belonging to the classes $\mathcal{C}_s^*(w; A, B; C, D)$, $\mathcal{Q}_s^*(w; A, B; C, D)$, $\mathcal{C}_c(w; A, B; C, D)$ and $\mathcal{Q}_c(w; A, B; C, D)$.

2. Preliminary Results

Lemma 2.1 *[15] For $u(z) = \sum_{k=1}^{\infty} c_k(z-w)^k$ and $p(z) = \dfrac{1+Cu(z)}{1+Du(z)} = 1 + \sum_{k=1}^{\infty} p_k(z-w)^k$, we have,*

$$|p_n| \leq \frac{(C-D)}{(1+d)(1-d)^n}, n \geq 1, |w| = d.$$

Lemma 2.2 *[14] If $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k (z-w)^k \in \mathcal{S}_s^*(w; A, B)$, then*

$$|b_2| \leq \frac{(A-B)}{2(1-d^2)}, \tag{2.1}$$

$$|b_3| \leq \frac{(A-B)}{2(1-d^2)(1-d)}, \tag{2.2}$$

$$|b_4| \leq \frac{(A-B)[(A-B)+2(1+d)]}{8(1-d^2)^2(1-d)}, \tag{2.3}$$

and

$$|b_5| \leq \frac{(A-B)[(A-B)+2(1+d)]}{8(1-d^2)^2(1-d)^2}. \tag{2.4}$$

Lemma 2.3 *[14] If $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k (z-w)^k \in \mathcal{K}_s(w; A, B)$, then*

$$|d_2| \leq \frac{(A-B)}{4(1-d^2)}, \tag{2.5}$$

$$|d_3| \leq \frac{(A-B)}{6(1-d^2)(1-d)}, \tag{2.6}$$

$$|d_4| \leq \frac{(A-B)[(A-B)+2(1+d)]}{32(1-d^2)^2(1-d)}, \tag{2.7}$$

and

$$|d_5| \leq \frac{(A-B)[(A-B)+2(1+d)]}{40(1-d^2)^2(1-d)^2}. \tag{2.8}$$

Lemma 2.4 *[14] If $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k (z-w)^k \in \mathcal{S}_c^*(w; A, B)$, then*

$$|b_2| \leq \frac{(A-B)}{1-d^2}, \tag{2.9}$$

$$|b_3| \leq \frac{(A-B)[(A-B)+(1+d)]}{2(1-d^2)^2}, \tag{2.10}$$

$$|b_4| \leq \frac{(A-B)[(A-B)^2+3(A-B)(1+d)+2(1+d)^2]}{6(1-d^2)^3}, \tag{2.11}$$

and

$$|b_5| \leq \frac{(A-B)[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3]}{24(1-d^2)^4}. \tag{2.12}$$

Lemma 2.5 *[14] If $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k (z-w)^k \in \mathcal{K}_c(w; A, B)$, then*

$$|d_2| \leq \frac{(A-B)}{2(1-d^2)}, \tag{2.13}$$

$$|d_3| \leq \frac{(A-B)[(A-B)+(1+d)]}{6(1-d^2)^2}, \tag{2.14}$$

$$|d_4| \leq \frac{(A-B)[(A-B)^2+3(A-B)(1+d)+2(1+d)^2]}{24(1-d^2)^3}, \tag{2.15}$$

and

$$|d_5| \leq \frac{(A-B)[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3]}{120(1-d^2)^4}. \tag{2.16}$$

3. Main Results

Theorem 3.1 *If $f \in \mathcal{C}_s^*(w; A, B; C, D)$, then*

$$|a_2| \leq \frac{(C-D)}{2(1-d^2)}, \tag{3.1}$$

$$|a_3| \leq \frac{(A-B)+2(C-D)}{6(1-d)(1-d^2)}, \tag{3.2}$$

$$|a_4| \leq \frac{(C-D)[(A-B)+2(1+d)]}{8(1-d^2)^2(1-d)}, \tag{3.3}$$

and

$$|a_5| \leq \frac{(A-B)[(A-B)+2(1+d)]+(C-D)[4(A-B)+2(1+d)]}{40(1-d^2)^2(1-d)^2}. \tag{3.4}$$

Proof: Using the concept of subordination in Definition 1.1, it yields

$$\frac{2(z-w)f'(z)}{g(z)-g(-z)} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1+\sum_{k=1}^{\infty} p_k (z-w)^k, \tag{3.5}$$

where $u(z) = \sum_{k=1}^{\infty} c_k (z-w)^k$.
Expansion of (3.5) leads to

$$\begin{aligned}(z-w) + 2a_2(z-w)^2 + 3a_3(z-w)^3 + 4a_4(z-w)^4 + 5a_5(z-w)^5 + ...\\ = (z-w) + b_3(z-w)^3 + b_5(z-w)^5 + ...\\ +p_1(z-w)^2 + p_1b_3(z-w)^4 + p_1b_5(z-w)^6 + ...\\ +p_2(z-w)^3 + p_2b_3(z-w)^5 + p_2b_5(z-w)^7 + ...\\ +p_3(z-w)^4 + p_3b_3(z-w)^6 + p_3b_5(z-w)^8 + ...\\ +p_4(z-w)^5 + p_4b_3(z-w)^7 + p_4b_5(z-w)^9 + ...\\ +p_5(z-w)^6 + p_5b_3(z-w)^8 + ...\end{aligned}$$

Comparing the coefficients of $(z-w)^2, (z-w)^3, (z-w)^4$ and $(z-w)^5$ in the above expansion, we obtain

$$2a_2 = p_1, \tag{3.6}$$

$$3a_3 = p_2 + b_3, \tag{3.7}$$

$$4a_4 = p_3 + b_3p_1, \tag{3.8}$$

and

$$5a_5 = p_4 + b_3p_2 + b_5. \tag{3.9}$$

On taking modulus and application of triangle inequality, the equations (3.6), (3.7), (3.8) and (3.9) transform to

$$2|a_2| = |p_1|, \tag{3.10}$$

$$3|a_3| \leq |p_2| + |b_3|, \tag{3.11}$$

$$4|a_4| \leq |p_3| + |b_3||p_1|, \tag{3.12}$$

and

$$5|a_5| \leq |p_4| + |b_3||p_2| + |b_5|. \tag{3.13}$$

Using Lemma 2.1 and inequality (2.1) in (3.10), the result (3.1) is obvious.
Again using Lemma 2.1 and inequality (2.2) in (3.11), the simplification leads to the result (3.2).
Further using the inequality (2.3) and applying Lemma 2.1, the result (3.3) can be easily obtained from

(3.12).
On using the inequalities (2.2) and (2.4) and application of Lemma 2.1 in (3.13), it leads to the result (3.4).

□

For $A = 1, B = -1, C = 1, D = -1$, Theorem 3.1 reduces to the following result:

Corollary 3.1 *If $f \in \mathcal{C}_s^*(w)$, then*

$$|a_2| \leq \frac{1}{1-d^2},$$

$$|a_3| \leq \frac{1}{(1-d)(1-d^2)},$$

$$|a_4| \leq \frac{2+d}{2(1-d)(1-d^2)^2},$$

and

$$|a_5| \leq \frac{7+2d}{10(1-d)^2(1-d^2)^2}.$$

Theorem 3.2 *If $f \in \mathcal{Q}_s^*(w; A, B; C, D)$, then*

$$|a_2| \leq \frac{(C-D)}{4(1-d^2)}, \tag{3.14}$$

$$|a_3| \leq \frac{(A-B)+2(C-D)}{18(1-d)(1-d^2)}, \tag{3.15}$$

$$|a_4| \leq \frac{(C-D)[(A-B)+2(1+d)]}{32(1-d^2)^2(1-d)}, \tag{3.16}$$

and

$$|a_5| \leq \frac{[(A-B)+2(1+d)][(A-B)+4(C-D)]}{200(1-d^2)^2(1-d)^2}. \tag{3.17}$$

Proof: Using concept of subordination in Definition 1.2, we have

$$\frac{2((z-w)f'(z))'}{(h(z)-h(-z))'} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1 + \sum_{k=1}^{\infty} p_k (z-w)^k, \tag{3.18}$$

where $u(z) = \sum_{k=1}^{\infty} c_k (z-w)^k$.
On expanding (3.18), it leads to

$$\begin{aligned}(z-w) + 4a_2(z-w)^2 + 9a_3(z-w)^3 + 16a_4(z-w)^4 + 25a_5(z-w)^5 + ...\\ = (z-w) + 3d_3(z-w)^3 + 5d_5(z-w)^5 + ...\\ +p_1(z-w)^2 + 3p_1d_3(z-w)^4 + 5p_1d_5(z-w)^6 + ...\\ +p_2(z-w)^3 + 3p_2d_3(z-w)^5 + 5p_2d_5(z-w)^7 + ...\\ +p_3(z-w)^4 + 3p_3d_3(z-w)^6 + 5p_3d_5(z-w)^8 + ...\\ +p_4(z-w)^5 + 3p_4d_3(z-w)^7 + 5p_4d_5(z-w)^9 + ...\\ +p_5(z-w)^6 + 3p_5d_3(z-w)^8 + ...\end{aligned}$$

On equating the coefficients of $(z-w)^2, (z-w)^3, (z-w)^4$ and $(z-w)^5$ in the above expansion, it yields

$$4a_2 = p_1, \tag{3.19}$$

$$9a_3 = p_2 + 3d_3, \tag{3.20}$$

$$16a_4 = p_3 + 3d_3p_1, \tag{3.21}$$

and

$$25a_5 = p_4 + 3d_3p_2 + 5d_5. \tag{3.22}$$

On taking modulus and application of triangle inequality, the equations (3.19), (3.20), (3.21) and (3.22) transform to

$$4|a_2| = |p_1|, \tag{3.23}$$

$$9|a_3| \leq |p_2| + 3|d_3|, \tag{3.24}$$

$$16|a_4| \leq |p_3| + 3|d_3||p_1|, \tag{3.25}$$

and

$$25|a_5| \leq |p_4| + 3|d_3||p_2| + 5|d_5|. \tag{3.26}$$

Using Lemma 2.1 and inequality (2.5) in (3.23), the result (3.14) is obvious.
Again using Lemma 2.1 and inequality (2.6) in (3.24), the simplification leads to the result (3.15).
Further using the inequality (2.6) and applying Lemma 2.1, the result (3.16) can be easily obtained from (3.25).
On using the inequalities (2.6) and (2.8) and application of Lemma 2.1 in (3.26), it leads to the result (3.17).

□

For $A = 1, B = -1, C = 1, D = -1$, Theorem 3.2 yields the following result:

Corollary 3.2 *If $f \in \mathcal{Q}_s^*(w)$, then*

$$|a_2| \leq \frac{1}{2(1-d^2)},$$

$$|a_3| \leq \frac{1}{3(1-d)(1-d^2)},$$

$$|a_4| \leq \frac{2+d}{8(1-d)(1-d^2)^2},$$

and

$$|a_5| \leq \frac{2+d}{10(1-d)^2(1-d^2)^2}.$$

Theorem 3.3 *If $f \in \mathcal{C}_c(w; A, B; C, D)$, then*

$$|a_2| \leq \frac{(A-B)+(C-D)}{2(1-d^2)}, \tag{3.27}$$

$$|a_3| \leq \frac{[(A-B)+(1+d)][(A-B)+2(C-D)]}{6(1-d^2)^2}, \tag{3.28}$$

$$|a_4| \leq \frac{1}{24(1-d^2)^3}\begin{bmatrix} (A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +3(C-D)(A-B)[(A-B)+3(1+d)]+6(1+d)^2(C-D) \end{bmatrix} \tag{3.29}$$

and

$$|a_5| \leq \frac{1}{120(1-d^2)^4}\begin{bmatrix} (A-B)[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3] \\ +4(C-D)(A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +12(C-D)(A-B)(1+d)[(A-B)+3(1+d)] \\ +24(1+d)^3(C-D) \end{bmatrix}. \tag{3.30}$$

Proof: On applying the concept of subordination in Definition 1.3, it yields

$$\frac{2(z-w)f'(z)}{g(z)+\overline{g(\overline{z})}} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1+\sum_{k=1}^{\infty} p_k(z-w)^k, \tag{3.31}$$

where $u(z)=\sum_{k=1}^{\infty} c_k(z-w)^k$.
Expansion of (3.31) leads to

$$\begin{aligned}
&(z-w)+2a_2(z-w)^2+3a_3(z-w)^3+4a_4(z-w)^4+5a_5(z-w)^5+...\\
&=(z-w)+b_2(z-w)^2+b_3(z-w)^3+b_4(z-w)^4+b_5(z-w)^5+...\\
&+p_1(z-w)^2+p_1b_2(z-w)^3+p_1b_3(z-w)^4+p_1b_4(z-w)^5+...\\
&+p_2(z-w)^3+p_2b_2(z-w)^4+p_2b_3(z-w)^5+p_2b_4(z-w)^6+p_2b_5(z-w)^7+...\\
&+p_3(z-w)^4+p_3b_2(z-w)^5+p_3b_3(z-w)^6+p_3b_4(z-w)^7+p_3b_5(z-w)^8+...\\
&+p_4(z-w)^5+...
\end{aligned} \tag{3.32}$$

On equating the coefficients of $(z-w)^2, (z-w)^3, (z-w)^4$ and $(z-w)^5$ in (3.32), it yields

$$2a_2 = p_1+b_2, \tag{3.33}$$

$$3a_3 = p_2+p_1b_2+b_3, \tag{3.34}$$

$$4a_4 = p_3+p_2b_2+b_3p_1+b_4, \tag{3.35}$$

and

$$5a_5 = p_4+p_3b_2+b_3p_2+p_1b_4+b_5. \tag{3.36}$$

Taking modulus and applying triangle inequality, the equations (3.33), (3.34), (3.35) and (3.36) transform to

$$2|a_2| \le |p_1|+|b_2|, \tag{3.37}$$

$$3|a_3| \le |p_2|+|p_1||b_2|+|b_3|, \tag{3.38}$$

$$4|a_4| \le |p_3|+|p_2||b_2|+|b_3||p_1|+|b_4|, \tag{3.39}$$

and

$$5|a_5| \le |p_4|+|b_2||p_3|+|b_3||p_2|+|b_4||p_1|+|b_5|. \tag{3.40}$$

Using Lemma 2.1 and inequality (2.9) in (3.37), the result (3.27) is obvious.
Again using Lemma 2.1 and inequalities (2.9) and (2.10) in (3.38), the simplification leads to the result (3.28).
Further using the inequalities (2.9), (2.10) and (2.11) and applying Lemma 2.1, the result (3.29) can be easily obtained from (3.39).
On using inequalities (2.9), (2.10), (2.11) and (2.12) and application of Lemma 2.1 in (3.40), it leads to the result (3.30).

□

For $A=1, B=-1, C=1, D=-1$, Theorem 3.3 yields the following result:

Corollary 3.3 *If $f \in \mathcal{C}_c(w)$, then*

$$|a_2| \le \frac{2}{1-d^2},$$

$$|a_3| \le \frac{3+d}{(1-d^2)^2},$$

$$|a_4| \leq \frac{2(2+d)(3+d)}{3(1-d^2)^3}$$

and

$$|a_5| \leq \frac{30+43d+20d^2+3d^3}{30(1-d^2)^4}.$$

Theorem 3.4 *If $f \in \mathcal{Q}_c(w; A, B; C, D)$, then*

$$|a_2| \leq \frac{(A-B)+(C-D)}{4(1-d^2)}, \tag{3.41}$$

$$|a_3| \leq \frac{[(A-B)+(1+d)][(A-B)+2(C-D)]}{18(1-d^2)^2}, \tag{3.42}$$

$$|a_4| \leq \frac{1}{96(1-d^2)^3}\begin{bmatrix} (A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +3(C-D)(A-B)[(A-B)+3(1+d)]+6(1+d)^2(C-D) \end{bmatrix}, \tag{3.43}$$

and

$$|a_5| \leq \frac{1}{600(1-d^2)^4}\begin{bmatrix} (A-B)[(A-B)^3+6(1+d)(A-B)^2+11(1+d)^2(A-B)+6(1+d)^3] \\ +4(C-D)(A-B)[(A-B)+(1+d)][(A-B)+2(1+d)] \\ +12(C-D)(A-B)(1+d)[(A-B)+3(1+d)] \\ +24(1+d)^3(C-D) \end{bmatrix}. \tag{3.44}$$

Proof: On applying the concept of subordination in Definition 1.4, it yields

$$\frac{2((z-w)f'(z))'}{(h(z)+\overline{h(\overline{z})})'} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1+\sum_{k=1}^{\infty} p_k(z-w)^k, \tag{3.45}$$

where $u(z) = \sum_{k=1}^{\infty} c_k(z-w)^k$.
Expansion of (3.45) leads to

$$\begin{aligned} &(z-w)+4a_2(z-w)^2+9a_3(z-w)^3+16a_4(z-w)^4+25a_5(z-w)^5+... \\ &= (z-w)+2d_2(z-w)^2+3d_3(z-w)^3+4d_4(z-w)^4+5d_5(z-w)^5+... \\ &+p_1(z-w)^2+2p_1d_2(z-w)^3+3p_1d_3(z-w)^4+4p_1d_4(z-w)^5+... \\ &+p_2(z-w)^3+2p_2d_2(z-w)^4+3p_2d_3(z-w)^5+4p_2d_4(z-w)^6+5p_2d_5(z-w)^7+... \\ &+p_3(z-w)^4+2p_3d_2(z-w)^5+3p_3d_3(z-w)^6+4p_3d_4(z-w)^7+5p_3d_5(z-w)^8+... \\ &+p_4(z-w)^5+... \end{aligned} \tag{3.46}$$

On equating the coefficients of $(z-w)^2, (z-w)^3, (z-w)^4$ and $(z-w)^5$ in (3.46), it yields

$$4a_2 = p_1+2d_2, \tag{3.47}$$

$$9a_3 = p_2+2p_1d_2+3d_3, \tag{3.48}$$

$$16a_4 = p_3+2p_2d_2+3d_3p_1+4d_4, \tag{3.49}$$

and

$$25a_5 = p_4+2p_3d_2+3d_3p_2+4p_1d_4+5d_5. \tag{3.50}$$

Taking modulus and applying triangle inequality, the equations (3.47), (3.48), (3.49) and (3.50) transform to

$$4|a_2| \leq |p_1| + 2|d_2|, \tag{3.51}$$

$$9|a_3| \leq |p_2| + 2|p_1||d_2| + 3|d_3|, \tag{3.52}$$

$$16|a_4| \leq |p_3| + 2|p_2||d_2| + 3|d_3||p_1| + 4|d_4|, \tag{3.53}$$

and

$$25|a_5| \leq |p_4| + 2|d_2||p_3| + 3|d_3||p_2| + 4|d_4||p_1| + 5|d_5|. \tag{3.54}$$

Using Lemma 2.1 and inequality (2.13) in (3.51), the result (3.41) is obvious.
Again using Lemma 2.1 and inequalities (2.13) and (2.14) in (3.52), the simplification leads to the result (3.42).
Further using the inequalities (2.13), (2.14) and (2.15) and applying Lemma 2.1, the result (3.43) can be easily obtained from (3.53).
On using inequalities (2.13), (2.14), (2.15) and (2.16) and application of Lemma 2.1 in (3.54), it leads to the result (3.44).

□

For $A = 1, B = -1, C = 1, D = -1$, Theorem 3.4 yields the following result:

Corollary 3.4 *If $f \in \mathcal{Q}_c(w)$, then*

$$|a_2| \leq \frac{1}{1-d^2},$$

$$|a_3| \leq \frac{3+d}{3(1-d^2)^2},$$

$$|a_4| \leq \frac{42+45d+13d^2}{24(1-d^2)^3},$$

and

$$|a_5| \leq \frac{260+473d+290d^2+63d^3}{150(1-d^2)^4}.$$

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