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Even perfect numbers in Narayana's sequence

Prasanta Kumar Ray and Kisan Bhoi*

ABSTRACT: In this note we prove that 6 and 28 are the only perfect numbers present in Narayana's sequence.

Key Words: Narayana's sequence, perfect numbers, linear forms in logarithms, reduction method.

Contents

1. Introduction

We know that a perfect number is a positive integer that equals the sum of its positive divisors excluding the number itself. Euclid proved that $2^{p-1}(2^p-1)$ is an even perfect number whenever 2^p-1 is prime. After Euclid, Euler proved that the formula $2^{p-1}(2^p-1)$ yields all even perfect numbers. Thus, Euclid-Euler theorem states that an even positive integer is perfect if and only if it has the form $2^{p-1}(2^p-1)$ where 2^p-1 is a prime.

Searching of perfect numbers in different binary recurrent sequences has been a source of attraction for many researchers. For instance, F. Luca [6] showed that there are no perfect Fibonacci or Lucas numbers. Panda and Davala [8] found that 6 is the only perfect number in balancing sequence. Perfect Pell and Pell-Lucas numbers were studied in [2]. Facó and Marques [5] extended the work of Luca by taking the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq -(k-2)}$ and they presented that there are no even perfect numbers in $(F_n^{(k)})$ when $k \not\equiv 3 (\mod 4)$. In 2021, Bravo and J. L. Herrera [1] proved that no perfect numbers are present in the generalized Pell sequence.

Inspired by the above works, we try to search a similar problem in a ternary recurrent sequence, namely Narayana's sequence. Narayana's sequence, $\{N_n\}_{n\geq 0}$ is recursively defined as $N_{n+3} = N_{n+2} + N_n$ where N_n denotes the n-th Narayana number with initial terms $N_0 = 0, N_1 = 1, N_2 = 1$. The first few terms of this sequence are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \cdots$. In this study, we show that 6 and 28 are the only perfect numbers in the Narayana's sequence. In order to prove this, we use lower bounds for linear forms in logarithms and Baker-Davenport reduction procedure and solve the Diophantine equation $N_n = 2^{p-1}(2^p-1)$.

Our main result is the following.

Theorem 1.1 *The only even perfect Narayana numbers are* 6 *and* 28*.*

* Corresponding author.
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2. Preliminaries

2.1. Some Properties of Narayana's sequence

Before proceeding the proof, we recall some properties of Narayana sequence which will be used in the next section.

The characteristic polynomial of $\{N_n\}_{n\geq 0}$ is given by $f(x) = x^3 - x^2 - 1$ and the characteristic roots are:

$$\alpha = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$
$$\beta = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$
$$\gamma = \bar{\beta} = \frac{1}{3} + w\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

where $w = \frac{-1+i\sqrt{3}}{2}$. The Binet's formula is given by

$$N_n = X\alpha^n + Y\beta^n + Z\gamma^n \quad \text{for all} \quad n \geq 0,$$

with

$$X = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)},\ Y = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)},\ Z = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}.$$

Another way to write this is as $N_n = C_\alpha \alpha^{n+2} + C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}$ for all $n \geq 0$ where $C_x = \frac{1}{x^3+2}$ for $x \in \{\alpha, \beta, \gamma\}$. The minimal polynomial of C_α is $31x^3 - 31x^2 + 10x - 1$ and all of its zeros are contained within the unit circle. The following can be approximated:

$$\alpha \approx 1.46557; \quad |\beta| = |\gamma| \approx 0.826031; \quad |C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}| < 1/2 \quad \text{for} \quad \text{all} \quad n \geq 1.$$

It is easy to establish through induction that

$$\alpha^{n-2} \leq N_n \leq \alpha^{n-1} \quad \text{holds} \quad \text{for} \quad \text{all} \quad n \geq 1. \tag{2.1}$$

2.2. Lower bound for linear forms in logarithms

Baker's theory acts as a vital role in reducing the bounds concerning linear forms in logarithms of algebraic numbers. Let η be an algebraic number with minimal primitive polynomial

$$f(X) = a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient $a_0 > 0$, and $\eta^{(i)}$'s are conjugates of η. Then, the *logarithmic height* of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{j=1}^{d} \max\{0, \log|\eta^{(j)}|\}\right).$$

The height function has the following properties which we will need later in our proof.

$$h(\eta+\gamma) \leq h(\eta) + h(\gamma) + \log 2,$$
$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma),$$
$$h(\eta^k) = |k|h(\eta), \quad k \in \mathbb{Z}.$$

We state the following theorem of Matveev (see [7] or [3, Theorem 9.4]), which provides a large upper bound for the subscript n in our main equation.

Theorem 2.1 *Let $\eta_1, \eta_2, \ldots, \eta_l$ be positive real algebraic integers in a real algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$ and $b_1, b_2, \ldots, b_l$ be non zero integers. If $\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1$ is not zero, then*

$$\log|\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \ldots A_l,$$

where $D = max\{|b_1|, |b_2|, \ldots, |b_l|\}$ and $A_1, A_2, \ldots, A_l$ are positive real numbers such that

$$A_j \geq max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l.$$

2.3. Baker-Davenort reduction method

The following is the result of Baker and Davenport due to Dujella and Pethő [4, Lemma 5], which provides a reduced bound for the subscript n.

Lemma 2.1 *Let M be a positive integer and p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|.\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Proof of Theorem 1.1

Consider the equation

$$N_n = 2^{p-1}(2^p - 1). \tag{3.1}$$

From (2.1) and (3.1) we have

$$2^{2(p-1)} < 2^{p-1}(2^p - 1) = N_n \leq \alpha^{n-1} < 2^{n-1}$$

and

$$\alpha^{n-2} \leq N_n = 2^{p-1}(2^p - 1) < 2^{2p-1}.$$

Thus

$$2p < n + 1 \text{ and } n < (2p-1)\frac{\log 2}{\log \alpha} + 2 < 4p.$$

Substituting the Binet's formula of N_n in (3.1), we have

$$C_\alpha \alpha^{n+2} + C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2} = 2^{p-1}(2^p - 1),$$

which implies

$$C_\alpha \alpha^{n+2} - 2^{2p-1} = -(C_\beta \beta^{n+2} + C_\gamma \gamma^{n+2}) - 2^{p-1}. \tag{3.2}$$

Taking absolute values and dividing on either sides of (3.2) by 2^{2p-1}, we get

$$\left| C_\alpha \alpha^{n+2} 2^{-(2p-1)} - 1 \right| < \frac{2}{2^p}. \tag{3.3}$$

Observe that, the left-hand side of the above inequality is in the form of $|\Gamma|$ as in Theorem 2.1. It is clear that $\Gamma = C_\alpha \alpha^{n+2} 2^{-(2p-1)} - 1$ is nonzero. If $\Gamma = 0$, then

$$C_\alpha \alpha^{n+2} = 2^{2p-1}. \tag{3.4}$$

Let σ be the automorphism of the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$ defined by $\sigma(\alpha) = \beta$, where $f(x) = x^3 - x^2 - 1$ is the minimal polynomial of α. The action of σ on both sides of (3.4) gives

$$|C_\beta \beta^{n+2}| = 2^{2p-1},$$

which is impossible since $|C_\beta \beta^{n+2}| < |C_\beta| \approx 0.407506\ldots < 1$, whereas $2^{2p-1} > 1$. Let

$$\eta_1 = C_\alpha,\ \eta_2 = \alpha,\ \eta_3 = 2,\ b_1 = 1,\ b_2 = n+2,\ b_3 = -(2p-1),\ l = 3,$$

with $d_{\mathbb{L}} = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. Since $2p < n+1$, $D = \max\{2p-1, n+2\} = n+2$. We compute the heights of η_1, η_2, η_3 as follows:

$$h(\eta_1) = h(C_\alpha) = \frac{\log 31}{3},\ h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3},\ h(\eta_3) = h(2) = \log 2.$$

Thus, we take

$$A_1 = \log 31, \quad A_2 = \log \alpha, \quad A_3 = 3\log 2.$$

By virtue of Theorem 2.1, we have

$$\begin{aligned}\log|\Gamma| &> -1.4 \cdot 30^6 3^{4.5} 3^2 (1+\log 3)(1+\log(n+2))(\log 31)(\log\alpha)(3\log 2)\\ &> -7.38 \cdot 10^{12} \log(1+\log(n+2)).\end{aligned}$$

From (3.3), we get

$$p\log 2 - \log 2 < 7.38 \cdot 10^{12}(1+\log(n+2)),$$

which reduces to

$$p < 1.1 \cdot 10^{13}(1+\log(n+2)).$$

Since $n < 4p$, we have

$$n < 4p < 4.4 \cdot 10^{13}(1+\log(n+2)),$$

which implies

$$n < 1.58 \cdot 10^{15}.$$

To reduce the bound, put

$$\Lambda = (n+2)\log\alpha - (2p-1)\log 2 + \log C_\alpha.$$

Then, (3.3) can be written as

$$|e^\Lambda - 1| < \frac{2}{2^p} < \frac{1}{2}.$$

Note that $\Lambda \neq 0$ as $\Gamma \neq 0$. Since $|e^z - 1| < y < \frac{1}{2}$ for real values of z and y, implies $|z| < 2y$, we obtain

$$0 < |\Lambda| < \frac{4}{2^p},$$

which implies that

$$|(n+2)\log\alpha - (2p-1)\log 2 + \log C_\alpha| < \frac{4}{2^p}.$$

Dividing both sides by $\log 2$ gives

$$\left| n\left(\frac{\log\alpha}{\log 2}\right) - (2p-1) + \left(\frac{\log(\alpha^2 C_\alpha)}{\log 2}\right)\right| < 5.78 \cdot 2^{-p}. \tag{3.5}$$

Now, with the notations of Lemma 2.1, let

$$u = n,\ \tau = \left(\frac{\log \alpha}{\log 2}\right),\ v = (2p-1),\ \mu = \left(\frac{\log(\alpha^2 C_\alpha)}{\log 2}\right),\ A = 5.78,\ B = 2,\ w = p.$$

See that $\frac{\log \alpha}{\log 2}$ is irrational otherwise we would get $2^s = \alpha^t$ for some coprime positive integers s and t. Then, applying the automorphism σ previously defined, we get $1 < 2^s = |\beta^t| < 1$, a contradiction. Chose $M = 1.58 \cdot 10^{15}$. We find that the convergent q_{41} exceeds $6M$ with $\varepsilon := \|\mu q_{41}\| - M\|\tau q_{41}\| = 0.143622$. Now, Lemma 2.1 says that there exists no solution to the inequality (3.5) if

$$p \geq \frac{\log((5.78 q_{41})/0.143622)}{\log 2} \geq 62.$$

Thus, we must have $p < 62$ and hence $n < 248$. Lastly, we execute a *Mathematica* program in the above range and obtain all the solutions mentioned in Theorem 1.1. This completes the proof of Theorem 1.1.

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Prasanta Kumar Ray,
Department of Mathematics,
Sambalpur University, Jyoti Vihar,
India.
E-mail address: prasantamath@suniv.ac.in

and

Kisan Bhoi,
Department of Mathematics GITAM (deemed to be) University,
Hyderabad India.
E-mail address: kbhoi@gitam.edu