Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.70220

Recognizing simple projective linear groups $\mathrm{PSL}(p, q)$ by their order and one conjugacy class length

Alireza Khalili Asboei* and Amirmohammad Momeni Kohestani

ABSTRACT: Let $r = \frac{q^p-1}{(q-1)(p,q-1)}$ be a prime number, where q is a prime power and p is an odd prime. In this paper, we will show that $G \cong \mathrm{PSL}(p, q)$ if and only if $|G| = |\mathrm{PSL}(p, q)|$ and G has a conjugacy class of size $\frac{|\mathrm{GL}(p,q)|}{(q^p-1)}$.

Key Words: Conjugacy class size, prime graph.

Contents

1. Introduction

In this paper, all groups are finite. Denote by $N(G)$ the set of all conjugacy class sizes of a group G. The motivation for our discussion is from a conjecture of J. G. Thompson. Thompson's conjecture which is Problem 12.38 in the Kourovka notebook [20] is as follows:

Thompson's conjecture. Let G be a group such that $Z(G) = 1$. If M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

In [1,2,13,12], Thompson's conjecture has been verified for some simple groups. Chen and his colleagues tried to contribute to Thompson's conjecture under a weak condition. They only used order and one or two special conjugacy class sizes of simple group, and characterized successfully sporadic simple groups, alternating group Alt_{10}, and projective special group $\mathrm{PSL}(4, 4)$ and $\mathrm{PSL}(2, p)$ (see [14,15,18]).

Similar characterizations have been found in [3], [4], [5], [6] and [7] for the groups: $\mathrm{PSL}(n, 2)$, ${}^2D_n(2)$, ${}^2D_{n+1}(2)$, alternating group of degree p, $p + 1$, $p + 2$, where p is a prime number and symmetric group of prime degree.

This work partially generalizes Khosravi's work [19, see Corollary 4.1]. Since recognizability by one conjugacy class makes the condition very weak, it is generally very difficult to work with the current information. So, we work on special cases. We will show that $\mathrm{PSL}(p, q)$ are uniquely determined by one conjugacy class size and order of $\mathrm{PSL}(p, q)$, where p is an odd prime and $r = \frac{q^p-1}{(q-1)(p,q-1)}$ is prime. In fact, the main theorem of our paper is as follows:

Main Theorem. Let G be a group, p an odd prime number, q a prime power and $r = \frac{q^p-1}{(q-1)(p,q-1)}$ a prime number. Then $G \cong \mathrm{PSL}(p, q)$ iff $|G| = |\mathrm{PSL}(p, q)|$ and G has a conjugacy class of size $\frac{|\mathrm{GL}(p,q)|}{(q^p-1)}$.

The *prime graph* of a group G that is denoted by $\Gamma(G)$ is a graph whose vertices are the prime divisors of $|G|$ and the prime p is defined to be adjacent to the prime q ($\neq p$) if and only if G contains an element of order pq.

We denote by $\pi(G)$ the set of prime divisors of the order of G. Also, if m is a natural number, then we consider $\pi(m)$ as the set of all prime divisors of m. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

* Corresponding author

Submitted November, 08 2023. Published March 24, 2025
2010 *Mathematics Subject Classification:* 20D08, 20D60.

Let G be a group. Then we can express $|G|$ as a product of integers m_1, m_2, $\ldots$, $m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. These numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd component of G. Write $OC(G)$ for the set $\{m_1\ , m_2, \ldots, m_{t(G)}\}$ of order components of G and $T(G)$ for the set of connected components of G. We can list the order components of finite simple groups with disconnected prime graphs according to the classification theorem of finite simple groups (see [21,23,17]). The other notation and terminology in this paper are standard, and the reader is referred to [16] if necessary.

2. Preliminary Results

Definition 2.1 *Let a and n be integers greater than 1. Then a Zsigmondy prime of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$.*

Lemma 2.1 [24] *If a and n are integers greater than 1, then there exists a Zsigmondy prime of $a^n - 1$, unless $(a, n) = (2, 6)$ or $n = 2$ and $a = 2^s - 1$ for some natural number s.*

Remark 2.1. If l is a Zsigmondy prime of $a^n - 1$, then Fermat's little theorem shows that $n \mid (l - 1)$. Put $Z_n(a) = \{l : l$ is a Zsigmondy prime of $a^n - 1\}$. If $r \in Z_n(a)$ and $r \mid a^m - 1$, then $n \mid m$.

A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. In the following lemmas, we quote some known results about Frobenius group and 2-Frobenius group which are useful in the sequel.

Lemma 2.2 [10] *Let G be a 2-Frobenius group of even order. Then:*
(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;
(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$, $(|G/K|, |K/H|) = 1$ and $G/K \lesssim \mathrm{Aut}(K/H)$.

Lemma 2.3 [10] *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$.*

Lemma 2.4 [23] *If G is a group such that $t(G) \geq 2$, then G has one of the following structures:*
(a) G is a Frobenius group or a 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.

Lemma 2.5 [19, Lemma 3.1 and Remark 3.1] *Let q be an odd prime power. Let $M = \mathrm{PSL}(p, q)$, and suppose $m_2 = (q^p - 1)/(q - 1)(p, q - 1)$. Then*

(i) If $x \in \pi_1(M)$, $x^\alpha \mid |M|$ and $x^\alpha - 1$ is congruent to 0 (mod m_2), then either $x^\alpha = q^{kp}$, where $1 \leq k \leq (p - 1)/2$ or $q \in \{3, 7, 9, t : t$ is a Fermat prime$\}$ and $x = 2$.

(ii) If $x \in \pi_1(M)$, $x^\alpha \mid |M|$ and $x^\alpha + 1$ is congruent to 0 (mod m_2), then $q \in \{3, 7, 9, t : t$ is a Fermat prime$\}$ and $x = 2$.

(iii) If $q \in \{3, 7, 9, t : t$ is a Fermat prime$\}$ and $2^\alpha \pm 1$ is congruent to 0 (mod m_2), then $2^{2\alpha} \nmid |M|$. $x \in \pi_1(M)$, then $|S_x| \leq q^{p(p-1)/2}$ where $S_x \in \mathrm{Syl}_x(M)$.

In the above Lemma and Lemma 2.7, note that by [23, Table Ib], $t(\mathrm{PSL}(p, q)) = 2$.

Lemma 2.6 [13, Lemma 8] *Let G be a group with $t(G) \geq 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G, and μ_1, μ_2, $\ldots$, μ_r are some of order components of G but not a π_i-number, then $\mu_1\mu_2...\mu_r$ is a divisor of $|N| - 1$.*

Lemma 2.7 [19, Lemma 2.13] *Let $q = 3$, 7, 9 or a Fermat prime (a prime of the form $q = 2^n + 1$) and m_2 be the odd order component of $\mathrm{PSL}(p, q)$. Then $m_2 - 1$ is not a power of 2.*

3. Proof of the main theorem

By [1, Corollary 2.11], $\mathrm{PSL}(p,q)$ has one conjugacy class of the length $\frac{|\mathrm{GL}(p,q)|}{(q^p-1)}$. Note that in [1, Corollary 2.11], it is proved that $\mathrm{PSL}(n,q)$, where n is not prime has one conjugacy class of the length $\frac{|\mathrm{GL}(p,q)|}{(q^p-1)}$, but the same proof is true when n is prime.

Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

By hypothesis, there exists an element x of order r in G such that $C_G(x)=\langle x\rangle$ and $C_G(x)$ is a Sylow r-subgroup of G. By the Sylow theorem, we have that $C_G(y)=\langle y\rangle$ for any element y in G of order r also, r is the greatest prime divisor of $|G|$ and an odd order component of G.

If $t(G)=2$, then obviously $OC(G)=OC(\mathrm{PSL}(p,q))$. So, by [19], $G\cong \mathrm{PSL}(p,q)$.

If $t(G)\geq 3$, then we will show that $(p,q)=(3,2)$ and then $G\cong \mathrm{PSL}(3,2)$.

Since $t(G)\geq 3$, Lemmas 2.2(a) and 2.3 show that G is neither a Frobenius group nor a 2-Frobenius group. So by Lemma 2.4, G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that K/H is a non-abelian simple group and r is an odd order component of K/H. Moreover, $t(K/H)\geq 3$. Since $r=\frac{q^p-1}{(q-1)(p,q-1)}$, we get that r is the greatest prime divisor of $|\mathrm{PSL}(p,q)|$. So r is the greatest prime divisor of $|G|$. Also by little Fermat theorem $p\mid (r-1)$.

According to the classification theorem of finite simple groups and the list of the groups with the disconnected prime graph obtained in [21,23,17], K/H is an alternating group, sporadic group or simple group of Lie type as follows:

Let $K/H\cong \mathrm{Alt}_n$, alternating group of degree n. Then since $t(K/H)\geq 3$, we get that n and $n-2$ are prime by table Id in [23]. Since $r=\max(\pi(G))$, we have $r=n$. This forces $r-2$ to be prime. But $\pi(G)=\{v\}\cup(\cup_{i=1}^{p}Z_i(q))$ (note that q is a power of a prime v). Thus, $r-2\in\cup_{i=1}^{p}Z_i(q)$. We show that $r-2\in\cup_{i=1}^{p}Z_i(q)$ is impossible. By [8,9], and our assumption (p is an odd prime number), we can delete the cases $p=2$, 3, 5. First, assume that $(p,q-1)\neq 1$. Let $r-2\in Z_i(q)$ for some $1\leq i\leq p$. As $r=\frac{q^p-1}{(q-1)(p,q-1)}$, $i\neq p$. Also,

$$q^{p-2}+1<q^{p-2}+q^{p-3}+\cdots+1=\frac{q^{p-1}-1}{q-1}<\frac{q^p-1}{(q-1)(q-1,p)}=r.$$

So, $q^{p-2}-1<r-2$ and so $i=p-2$. This show that $(r-2)\mid (q^{p-1}-1)$. Since $r-2\in Z_{p-1}(q)$, we have $(r-2)\nmid (q^{\frac{p-1}{2}}-1)$. Then since $p\neq 2,3$

$$(r-2)\mid \frac{q^{p-1}-1}{q^{\frac{p-1}{2}}-1}=q^{\frac{p-1}{2}}+1<q^{p-2}-1,$$

which is a contradiction.

Now, let $(p,q-1)=1$. Let $r-2\in Z_i(q)$ for some $1\leq i\leq p$. We have

$$q^{p-1}+1<q^{p-1}+q^{p-2}+\cdots+1=\frac{q^p-1}{q-1}=r.$$

So, $q^{p-1}-1<r-2$ and so $i=p-1$. This show that $(r-2)\mid (q^p-1)$. Since $r-2\in Z_p(q)$, we have $(r-2)\nmid (q^{p-1}-1)$. Then since $p\neq 2$, 3

$$(r-2)\mid \frac{q^p-1}{q^{p-1}-1}<q^{p-1}-1,$$

which is a contradiction.

Suppose that K/H is a sporadic simple group. If $K/H\cong M_{11}$, M_{12}, M_{22} or HS, then $r=11$. So, $p\mid 10$. It follows that $p=2$, 5, the cases we have already dismissed.

If $K/H\cong M_{23}$, M_{24}, Co_2, F_{23}, then $r=23$. So, $p\mid 22$. It follows that $p=2,11$. Therefore, $\frac{q^{11}-1}{(q-1)(11,q-1)}=23$. Thus, $\frac{q^{11}-1}{q-1}=23$ or $\frac{q^{11}-1}{q-1}=253$, and so $q(q^{10}-23)=-22$ or $q(q^{10}-253)=-252$, a contradiction.

If $K/H\cong J_1$, J_3, then $r=19$. So, $p\mid 18$, and hence $p=2,3$, both of which are already dismissed.

If $K/H \cong J_4$, then $r = 43$. This implies that $p \mid 42$, so $p = 7$ (as $p = 2$ and $p = 3$ are already dismissed). Therefore, $\frac{q^7-1}{(q-1)(7,q-1)} = 43$. Thus, $\frac{q^7-1}{q-1} = 43$ or $\frac{q^7-1}{q-1} = 301$, and so $q(q^7 - 43) = -42$ or $q(q^6 - 301) = -300$, a contradiction.

If $K/H \cong Suz$, then $r = 13$. So, $p \mid 12$. It follows that $p = 2, 3$, the cases we have already dismissed.

If $K/H \cong On, F_3$, then $r = 31$. So, $p \mid 30$. It follows that $p = 2, 3, 5$, the cases we have already dismissed.

If $K/H \cong Ly$, then $r = 67$. So, $p \mid 66$. It follows that $p = 2, 3, 11$. Hence, $p = 11$. Therefore, $\frac{q^{11}-1}{(q-1)(11,q-1)} = 67$. Thus, $\frac{q^{11}-1}{q-1} = 67$ or $\frac{q^{11}-1}{q-1} = 737$, and so $q(q^{10} - 67) = -66$ or $q(q^{10} - 737) = -736$, a contradiction.

If $K/H \cong F_{24}$, then $r = 29$. So, $p \mid 28$. It follows that $p = 2, 7$. Hence, $p = 7$. Therefore, $\frac{q^7-1}{(q-1)(7,q-1)} = 29$. Thus, $\frac{q^7-1}{q-1} = 29$ or $\frac{q^7-1}{q-1} = 203$, and so $q(q^6 - 29) = -28$ or $q(q^6 - 203) = -202$, a contradiction.

If $K/H \cong M$, then $r = 71$. So, $p \mid 70$, and hence $p = 2, 5$ or 7. So, $p = 7$. Then $\frac{q^7-1}{(q-1)(7,q-1)} = 71$, and so $q^7 - 1 = 497(q - 1)$ or $q^7 - 1 = 71(q - 1)$, which are impossible.

If $K/H \cong F_2$, then $r = 47$. So, $p \mid 46$, and hence $p = 2$ or 23. So, $p = 23$. Then $\frac{q^{23}-1}{(q-1)(23,q-1)} = 47$, and so $q^{23} - 1 = 47(q - 1)$ or $q^{23} - 1 = 1081(q - 1)$, which are impossible.

Similarly, we can rules out the other possibilities of K/H, when K/H is sporadic simple group.

Let K/H be a simple group of Lie type in characteristic p', q be a power of v and let

$$X = \{3, 7, 9, t : t \text{ is a Fermat prime}\}.$$

Since H is nilpotent, if $t \mid |H|$, then for $T \in \mathrm{Syl}_t(H)$, we have $T \trianglelefteq H$. So, $R \in \mathrm{Syl}_r(G)$ acts on T which this action is fixed-point-freely. Therefore, $r \mid |T| - 1 = t^u - 1$. So Lemma 2.5 shows that either $t = v$ or $q \in X$ and $t = 2$. Thus either $q \in X$ and H is a $\{2, v\}$−group or H is a $\{v\}$-group. Now let K/H be a simple group of Lie type in characteristic p'. Since $p' \mid |K/H|$ and $|K/H| \mid |\mathrm{PSL}(p, q)|$, we get that $|K/H| \mid |G|$. One of the following cases occurs:

1. $K/H \cong \mathrm{PSL}(2, q')$, where $4 \nmid q'$. So one of the following holds.

a) $r = \frac{q'-1}{2}$. Thus by Lemma 2.5 and the definition of Zsigmondy prime, we have $q' = q^p$ and $q = 3$. But $\frac{q'(q'+1)(q'-1)}{2} = |K/H| \mid |G|$ and hence $q' + 1 = q^p + 1 \mid |G|$. Therefore, for some $s \in Z_{2p}(q)$, $s \in \pi(G)$, which is a contradiction.

b) $r = \frac{q'+1}{2}$. By Lemma 2.5, we can get a contradiction.

c) $r = q'$. We have $|\mathrm{PSL}(2, r)| \mid |G|$. Thus $\frac{|G|}{|K/H|} = \frac{|G|}{|\mathrm{PSL}(2,r)|} = |H| \times |G/K| \neq 1$. By Lemma 2.4, $|G/K| \mid |\mathrm{Out}(K/H)|$. Since $q = v^n$, by [22] we have $|\mathrm{Out}(\mathrm{PSL}(2, r))| \mid 2n$, which implies that $|H| \neq 1$. Now let x be an odd prime number such that x does not divide q and $x \mid |H|$. Then let T be a x-Sylow subgroup of H. Since H is nilpotent, $T \lhd G$. Hence; $r \mid |T| - 1$, by Lemma 2.6. Therefore, $|T| = q^{kp}$, by Lemma 2.5, which is a contradiction.

2. $K/H \cong \mathrm{PSL}(2, q')$, where $4 \mid q'$. Then $r = q' - 1$ or $q' + 1$. Thus, the above argument and Lemma 2.7 lead us to a contradiction.

3. $K/H \cong G_2(q')$. So, $r = q'^2 + q' + 1$ or $q'^2 - q' + 1$. Hence, $r \mid q'^3 - 1$ or $r \mid q'^3 + 1$. Thus, Lemma 2.5 shows that either $r = q'^2 + q' + 1$ and $v \mid q'$ or $q \in X$ and $2 \mid q'$. In the former case, the definition of Zsigmondy prime forces $q'^3 = q^p$ and hence, we can see that $p = 3$ and $q' = q$. But $|G_2(q')| \mid |G|$, so $(q^2 - 1)^2 \mid |G|$, which is impossible.

Now, let $q \in X$ and $2 \mid q'$. So, by Lemma 2.5(iii), $q'^6 \nmid |G|$. On the other hand, $q'^6 \mid |G_2(q')|$. It follows that $|G_2(q')| \nmid |G|$, which is a contradiction.

4. $K/H \cong {}^2G_2(q')$, where q'^{2l+1}. Then $r = q' - \sqrt{3q'} + 1$ or $q' + \sqrt{3q'} + 1$. This forces $r \mid \frac{q'^3+1}{q'+1}$. Thus, $q'^3 + 1$ is congruent to 0 (mod r), which is a contradiction with Lemma 2.5.

5. $K/H \cong^2 D_l(3)$, where $l \geq 5$. Then $r = \frac{3^l+1}{2}$ or $\frac{3^l+4}{4}$. Thus, $3^l + 1$ is congruent to 0 (mod r), which is impossible by Lemma 2.5(ii).

6. $K/H \cong^2 D_{l+1}(2)$, where $l \geq 3$. Then $r = 2^l + 1$ or $2^{l+1} + 1$. Again by Lemma 2.5 we get a contradiction.

7. $K/H \cong F_4(q^{'})$. Then $r = q^{'4} + 1$ or $q^{'4} - q^{'2} + 1$.

If $q^{'4} + 1$ is congruent to 0 (mod r), then Lemma 2.5 forces $q \in X$, $2 \mid q'$ and $q'^8 \nmid |G|$. On the other hand, $q'^{24} \mid |F_4(q^{'})|$, so $|F_4(q^{'})| \nmid |G|$, which is a contradiction.

If $r = q^{'4} - q^{'2} + 1 = \frac{q^{'6}+1}{q^{'2}+1}$, then $q^{'6} + 1$ is congruent to 0 (mod r). Hence, arguing as the above leads us to the contradiction.

8. $K/H \cong E_7(2)$. Since r is the greatest prime divisor of $|G|$, we get that $r = 127 = \frac{q^p-1}{(q-1)(p,q-1)}$. But $p \mid r - 1 = 126$. So $p = 3$ or 7.

If $p = 3$, then $\frac{q^3-1}{(q-1)(3,q-1)} = 127$. Thus $q^3 - 1 = 127(q - 1)$ or $q^3 - 1 = 127(q - 1)$. In the former case, $q^2 + q = 126$, a contradiction. In the latter case, $q^2 + q = 380$. It follows that $q = 19$. But $43 \nmid |G| = |\mathrm{PSL}(3, 19)|$ and $43 \mid |E_7(2)|$, a contradiction.

If $p = 7$, then $\frac{q^7-1}{(q-1)(7,q-1)} = 127$. Thus,

$$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 = 127$$

or

$$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1 = 7 \times 127.$$

In the former case,

$$q(q^5 + q^4 + q^3 + q^2 + q + 1) = 126.$$

It follows that $q = 2$. So $43 \mid |K/H|$ and $43 \nmid |\mathrm{PSL}(7, 2)| = |G|$, a contradiction. In the latter case,

$$q(q^5 + q^4 + q^3 + q^2 + q + 1) = 888,$$

which is impossible.

9. $K/H \cong E_7(3)$. Arguing as above leads us to a contradiction.

10. $K/H \cong \mathrm{PSL}(3, 2)$. Then $r = 7$. So $p \mid r - 1 = 6$, and so $p = 3$. We have $\frac{q^3-1}{(q-1)(3,q-1)} = 7$. Thus $q^2 + q + 1 = 7$ or $q^2 + q + 1 = 21$. In the former case $q = 2$. Therefore, $K/H \cong \mathrm{PSL}(3, 2)$, and then $G \cong \mathrm{PSL}(3, 2)$, as wanted.

In the latter case, $q = 4$. Since $|G| = 5.9.7$, $|G|_3 > |\mathrm{Aut}(K/H)|_3$, a contradiction.

11. $K/H \cong \mathrm{PSL}(3, 4)$. Then since $r = \max(\pi(G))$, we have $r = 7$ and so $p = 3$. Since $\frac{q^3-1}{(q-1)(3,q-1)} = 7$, we have $q^2 - 7q = -6$ or $q^2 - 21q = -20$, which are impossible.

12. $K/H \cong \mathrm{PSU}(6, 4)$. Then $r = 11$. So $p = 5$. Since $\frac{q^5-1}{(q-1)(5,q-1)} = 11$, we have $q^5 - 11q = -10$ or $q^5 - 55q = -54$, which are impossible.

13. $K/H \cong {}^2F_4(q^{'})$, where q'^{2l+1}. Then $r = q^{'2} + \sqrt{2q^{'3}} + q^{'} \pm \sqrt{2q^{'}} + 1$. Thus, $r \mid q^{'6} + 1$ and so $q^{'6} + 1$ is congruent to 0 (mod r), hence Lemma 2.5(iii) shows that $q'^{12} \nmid |G|$. On the other hand, $q'^{12} \mid |{}^2F_4(q^{'})|$, so $|{}^2F_4(q^{'})| \nmid |G|$, which is a contradiction.

14. $K/H \cong {}^2B_2(q^{'})$, where q'^{2l+1}. Then $r = q^{'} \pm \sqrt{2q^{'}} + 1$ or $q^{'} - 1$. Since $q^{'} = 2^{2n+1}$, Lemma 2.7 shows that $q' - 1 \neq r$. If $r = q^{'} \pm \sqrt{2q^{'}} + 1$, then $r \mid q^{'2} + 1$, so $q^{'2} + 1$ is congruent to 0 (mod r), which is impossible by Lemma 2.5.

15. $K/H \cong {}^2E_6(2)$. Then $r = 19$. So $p = 3$. Thus, $q(q+1) = 18$ or $q(q+1) = 56$. The case $q(q+1) = 18$ is impossible. Let $q(q+1) = 56$. Then $q = 7$. Since $17 \in \pi({}^2E_6(2))$, $17 \in \pi(G) = \pi(\mathrm{PSL}(3,7))$, which is a contradiction.

16. $K/H \cong E_8(q^{'})$. Then $r = \frac{q^{'10}+q^{'5}+1}{q^{'2}+q^{'}+1}$, $q^{'8} - q^{'4} + 4$, or $\frac{q^{'10}+1}{q^{'}+1}$. Thus $r \mid q^{'15} - 1$, $q^{'12} + 1$ or $q^{'10} + 1$. If $q \in X$ and $2 \mid q'$, then Lemma 2.5(iii) forces $q'^{30} \nmid |G|$. On the other hand, $q'^{120} \mid |E_8(q')|$, so $|E_8(q')| \nmid |G|$, which is a contradiction. Thus, Lemma 2.5 forces $q^{'}$ to be a power of v and $r \mid q^{'15} - 1$. Therefore, by definition of Zsigmondy prime forces $q'^{15} = q^p$. On the other hand, $(q'^{30} - 1) \mid |E - 8(q')|$, so $Z_{2p}(q) \subset \pi(G) = \pi(\mathrm{PSL}(p,q))$, which is a contradiction.

Therefore, $G \cong \mathrm{PSL}(p,q)$, as desired. □

Corollary 3.1 *Let q be a prime power and p is an odd prime. Then Thompson's conjecture holds for the simple groups* $\mathrm{PSL}(p,q)$, *where* $(q^p - 1)/(q-1)(p, q-1)$ *is a prime number.*

Proof: Let G be a group with trivial central and $N(G) = N(\mathrm{PSL}(p,q))$. Then it is proved in [11, Lemma 1.4] that $|G| = |\mathrm{PSL}(p,q)|$. Hence; the corollary follows from the main theorem. □

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A. K. Asboei,
Department of Mathematics Education, Farhangian
University, P.O. Box 14665-889, Tehran, Iran.
E-mail address: khaliliasbo@yahoo.com

and

A. M. Kohestani,
Department of Mathematics, University of Mazandaran, Babolsar, Iran.