Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–13.
ISSN-0037-8712
doi:10.5269/bspm.70257

Generalization of Proinov Contraction in Non-Triangular Metric Space*

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ABSTRACT: The main purpose of this paper is to find the conditions under which it will be sufficient to establish the existence of a unique fixed point in the non-triangular metric space for the auxiliary functions ψ and ϕ satisfying the contractive condition $\psi(d(Sy, Sz)) \leq \phi(d(y, z))$. In 2020, Proinov [24] has proved some fixed point results using his contractive type conditions in metric space, and recently in 2022, Erdal Karpanar et. al. [16] introduced the extended Proinov contractions by avoiding the monotone condition on auxiliary function ψ in the metric space. We have generalized these in non-triangular metric space. Further, as an application, we find the existence and uniqueness of a solution of the homogeneous Fredholm integral equation in non-triangular metric space using Proinov contraction. Illustrative examples and numerical calculations are given to support the obtained results, which extend some of the theorems in recent literature.

Key Words: Non-triangular metric, Proinov contractions, extended Proinov contractions, d-Cauchy sequence, d-complete.

Contents

1. Introduction

Mathematics has consistently been of great importance in conceptualizing spaces and of further interest in generalizing metric spaces and studying their properties. It has also proven to be a fascinating area of research for mathematicians due to its applications not only in other fields of mathematics but also in a few other disciplines. According to Jleli and Samet [12], JS-metric spaces represent a generalization of metric spaces, which includes several metric spaces such as standard metric space, b-metric space, dislocated metric space, and modular space.

In 2020, Khojasteh and Khandani [17] have introduced non-triangular metric spaces. In addition, the introduction of non-triangular metric spaces has shown that there is no inherent necessity for the triangle inequality for several fixed point results to be true. Several useful properties in the squeal are established in non-triangular metric spaces.

Definition 1.1 *(Non-triangular metric)*
Let $d : X \times X \to \mathbb{R}^+$ be a mapping on a non-empty set X. d is said to be a non-triangular metric on X, if it satisfies the following conditions:

(i) $d(y, y) = 0 \ \forall \ y \in X$;

* The project is partially supported by the Human Resource Development Group, Council of Scientific and Industrial Research, India.
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2010 *Mathematics Subject Classification:* 47H10, 54H25.
Submitted November 08, 2023. Published December 04, 2025

(ii) $d(y,z) = d(z,y)\ \forall\ y,z \in X$;

(iii) for each $y,z \in X$, *and* $\{y_l\} \subset X$ *such that* $\lim_{l\to\infty} d(y_l, y) = 0$, *and* $\lim_{l\to\infty} d(y_l, z) = 0$, *then* $y = z$.

Then, (X,d) is said to be non-triangular metric space.

Definition 1.2 *Let* (X,d) *be a non-triangular metric space then the following conclusions hold:*

(i) $\{y_l\}$ *is d-convergent to y if* $\lim_{l\to\infty} d(y_l, y) = 0$;

(ii) $\{y_l\}$ *is a d-Cauchy sequence if* $\lim_{l\to\infty} \sup\{d(y_l, y_m) : m \geq l\} = 0$;

(iii) if every d-Cauchy sequence in X is d-convergent to some element in X then (X,d) *is called d-complete.*

Remark 1.1 *If* (X,d) *is a non-triangular metric space and* $y,z \in X$, *then* $d(y,z) = 0$ *implies* $y = z$.

For complementary and related results, we refer to [8,17].

Another basic element of fixed point theory is the contractivity condition which itself is an essential component of any advancement. Since 1950s, many researchers have introduced increasingly more generalized conditions for contractivity in their works. We cite here the following contributions: Boyd and Wong [3], Caristi [4], Chatterjea [5], Pant [20], Hardly and Rogers [10], Kannan [13], Ciric [6], Karapinar [14], Samet, Vetro and Vetro [27], Roldan Lopez de Hierro [26], Jleli and Samet [12], etc.

Recently Proinov [24] has introduced a remarkable result. In addition to the fact that his theorem has nicely and greatly surprised researchers in fixed point theory, he has also speculated that his theorem can be proved to be true by setting forward a series of conditions that were only weak at first but eventually became strong enough to allow him to derive existence and uniqueness of fixed point theorem. In a clear and definite way, his contractivity condition is,

$$\psi(d(Sy,Sz)) \leq \phi(d(y,z)) \qquad \forall y,z \in X \text{ with } d(Sy,Sz) > 0, \tag{1.1}$$

although it comes from the expression given by Dutta and Choudhury [9]

$$\psi'(d(Sy,Sz)) \leq \psi'(d(y,z)) - \phi'(d(y,z)) \qquad \forall y,z \in X \text{ with } d(Sy,Sz) > 0.$$

Many authors have generalized several fixed point results on the metric space using the contractivity condition (1.1). We cite here the following contributions: Amini–Harandi and Petruşel [2], Moradi [19], Ahmad, Al-Mazrooei, Cho and Yang [1], Popescu [23], Piri and Kumam [22], etc.

In fact, Proinov has introduced new contraction using $\psi = \psi'$ and $\phi = \psi' - \phi'$. According to Proinov, one of the main contributions he made was to outline how such general functions as ψ and ϕ are capable of developing all the reasonings that lead to the final objective and that these functions must satisfy certain conditions viz:

(a_1) ψ is monotone increasing;

(a_2) $\phi(\mathfrak{s}) < \psi(\mathfrak{s})\ \forall\ \mathfrak{s} > 0$;

(a_3) $\limsup_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s})\ \forall\ \mathfrak{e} > 0$.

The condition (a_1) is a restrictive condition imposed on the function ψ. Although many authors have assumed this condition in various fixed point results due to its crucial role in proofs. Some examples of fixed points can be shown using this monotone condition and satisfying the above contractivity condition (1.1) with ψ not necessarily increasing. In addition, it is possible to find such examples where ψ is not strictly increasing throughout $(0,\infty)$, this means that our main result using Proinov contractions on non-triangular metric space is not applicable.

So we generalize the Proinov contractions in terms of the auxiliary function ψ, but avoid the monotone condition on non-triangular metric space. With the constraint imposed on auxiliary functions, we can introduce a new family of intermediate contractions that map the Proinov contractions to the extended contractions derived from the constraints placed on them.

On the other hand, these ideas may be extended into non-triangular fuzzy metric space [11,18,21,30] with suitable modifications.

2. Preliminary results

A sequence $\{y_l\}$ is said to be a Picard sequence of S based on $y_0 \in X$ if $y_{l+1} = Sy_l \; \forall \; l \in \mathbb{N}$. Note that $y_l = S_l y_0 \; \forall \; l \in \mathbb{N}$, where $\{S_l : X \to X\}$ is the iteration of S defined by S_0=identity operator I, $S_1 = S$ and $S_{l+1} = S * S_l \; \forall \; l \geq 2$.

A sequence $\{y_l\}$ is infinite if $y_l \neq y_k \; \forall \; l \neq k$, and $\{y_l\}$ is almost periodic if $\exists \; n_0, N \in \mathbb{N}$ such that

$$y_{n_0+l+Np} = y_{n_0+l} \quad \forall \quad p \in \mathbb{N} \quad \text{and all } l \in \{0,1,2,\ldots,N-1\}.$$

Proposition 2.1 *[7] For the Picard sequences, which may be either infinite or almost periodic.*

Lemma 2.1 *[24] Let (X,d) be a non-triangular metric space and $\{y_l\}$ be a Picard sequence in X with $\{d(y_l,y_{l+1})\} \to 0$. If there are $l_1, l_2 \in \mathbb{N}$ with $l_1 < l_2$ and $y_{l_1} = y_{l_2}$, then there is a $l_0 \in \mathbb{N}$ and $z \in X$ such that $y_l = z \; \forall \; l \geq l_0$. Then the Picard sequence $\{y_l\}$ converges to the fixed point of S.*

Theorem 2.1 *[17] Suppose (X,d) be a complete non-triangular metric space and let $S : X \to X$ satisfies the following two conditions:*

(i) For any two sub-sequences $\{y_{n_l}\}$ and $\{y_{m_l}\}$ of $\{y_l\}$ if $\lim_{l\to\infty} d(y_{n_l}, y_{m_l}) = L$ and $\lim_{l\to\infty} d(y_{n_l-1}, y_{m_l-1}) = L$, where $L \geq 0$, and $d(y_{n_l}, y_{m_l}) > L \; \forall \; l \in \mathbb{N}$, then $L = 0$.

(ii) S is orbitally continuous, i.e, if $\lim_{l\to\infty} S_{n_l}(y_0) = y$ for some $y_0 \in X$ implies $\lim_{l\to\infty} SS_{n_l}(y_0) = S(y)$ for each $y \in X$.

If $\exists \; y_0 \in X$ such that $\{d(S_i(y_0), S_j(y_0)) : i,j \in \mathbb{N}\}$ is bounded, then S has at least a fixed point in X.

Theorem 2.2 *[15] Suppose (X,d) be a non-triangular metric space and $S : X \to X$ be a mapping such that*

$$d(Sy,Sz) \leq \lambda d(y,z) \qquad \forall y,z \in X \quad \text{and} \quad \forall \lambda \in [0,1).$$

If $\exists \; y_0 \in X$ such that $\{d(S_i(y_0), S_j(y_0)) : i,j \geq 1\}$ is bounded. Then, S has a fixed point.

For related results and extensions, see, [8,25,28,29].

In order to compare different conditions for control functions, the following three lemmas are very useful.

Lemma 2.2 *[24] Let $\psi : (0,\infty) \to \mathbb{R}$ then for any $\epsilon > 0$, the conditions below are equivalent:*

(i) $\inf_{\mathfrak{t} > \epsilon} \psi(\mathfrak{t}) > -\infty$;

(ii) $\liminf_{\mathfrak{t} \to \epsilon+} \psi(\mathfrak{t}) > -\infty$;

(iii) $\lim_{l\to\infty} \psi(\mathfrak{t}_l) = -\infty$ implies $\lim_{l\to\infty} \mathfrak{t}_l = 0$.

Lemma 2.3 *[24] Let $\phi : (0,\infty) \to (0,\infty)$ then for any $\epsilon > 0$, the conditions below are equivalent:*

(i) $\lim_{l\to\infty} \mathfrak{t}_l = \epsilon > 0$ *implies* $\liminf_{l\to\infty} \phi(\mathfrak{t}_l) > 0$;

(ii) *for a bounded sequence* $\{\mathfrak{t}_l\}$ *if* $\lim_{l\to\infty} \phi(\mathfrak{t}_l) = 0$, *then* $\mathfrak{t}_l \to 0$ *as* $l \to \infty$;

(iii) $\liminf_{\mathfrak{t}\to\epsilon} \phi(\mathfrak{t}) > 0$.

Lemma 2.4 *[24] Let* $\psi, \phi : (0,\infty) \to \mathbb{R}$ *be two functions with*

$$\limsup_{\mathfrak{t}\to\epsilon} \phi(\mathfrak{t}) < \liminf_{\mathfrak{t}\to\epsilon} \psi(\mathfrak{t}) \qquad \forall \qquad \epsilon > 0$$

and for a bounded sequence $\{\mathfrak{t}_l\}$, *the two sequences* $\psi(\mathfrak{t}_l)$ *and* $\phi(\mathfrak{t}_l)$ *converge to same limit. Then* $\mathfrak{t}_l \to 0$ *as* $l \to \infty$.

3. Main results

3.1. Proinov contraction in non-triangular metric space

Theorem 3.1 *Suppose* (X, d) *be a complete non-triangular metric space and* $S : X \to X$ *be a mapping such that*

$$\psi(d(Sy, Sz)) \le \phi(d(y, z)) \qquad \forall y, z \in X \qquad \text{with } d(Sy, Sz) > 0, \tag{3.1}$$

where $\psi, \phi : (0,\infty) \to \mathbb{R}$ *satisfy the following three conditions:*

(a_1) ψ *is monotone increasing;*

(a_2) $\phi(\mathfrak{s}) < \psi(\mathfrak{s})$ $\forall$ $\mathfrak{s} > 0$;

(a_3) $\limsup_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s})$ $\forall$ $\mathfrak{e} > 0$.

Then, S has a unique fixed point in X.

Proof: First we construct the sequence $y_{l+1} = S(y_l)$ $\forall$ $l \in \mathbb{N}$. Let $\alpha_l = d(y_{l+1}, y_l)$, if $\alpha_l = 0$ for some $l \ge 0$, then it is trivial. Suppose that $\alpha_l > 0$ $\forall$ $l \ge 0$. Now applying the relation (3.1) and taking into account the condition (a_2) we have,

$$\psi(\alpha_l) \le \phi(\alpha_{l-1}) < \psi(\alpha_{l-1}). \tag{3.2}$$

Since ψ is monotone increasing therefore, $\alpha_l < \alpha_{l-1}$. Thereby, $\{\alpha_l\}$ is a strictly decreasing sequence in $\mathbb{R}$ and bounded below by 0 and let $\lim_{l\to\infty} \alpha_l = \alpha \ge 0$. Now assume that $\alpha > 0$. Now taking limit $l \to \infty$ in (3.2) we have,

$$\psi(\alpha+) = \lim_{l\to\infty} \psi(\alpha_l) \le \limsup_{l\to\infty} \phi(\alpha_{l-1}) \le \limsup_{\mathfrak{t}\to\alpha+} \phi(\mathfrak{t}),$$

which contradicts the condition (a_3). Therefore $\alpha = \lim_{l\to\infty} d(y_l, y_{l-1}) = 0$ $\forall$ $l \in \mathbb{N}$.

Now, we have to prove that $\{y_l\}$ is a d-Cauchy sequence in the non-triangular metric space X.

Let $r_l = \sup\{d(S_i(y_0)), S_j(y_0) : i, j \in \mathbb{N}$ and $i, j \ge l\}$. Note that $0 \le r_{l+1} \le r_l$. Therefore $\{r_l\}$ is a monotone decreasing sequence and bounded below by 0, which implies convergent. Let $\lim_{l\to\infty} r_l = r$. Now applying the relation (3.1) and taking into account the condition (a_2) we have,

$$\psi\Big(d(S_{\xi_l}(y_0), S_{w_l}(y_0))\Big) \le \phi\Big(d(S_{\xi_l-1}(y_0), S_{w_l-1}(y_0))\Big) < \psi\Big(d(S_{\xi_l-1}(y_0), S_{w_l-1}(y_0))\Big).$$

Since ψ is a monotone increasing function, therefore $\{d(S_{\xi_l}(y_0), S_{w_l}(y_0))\}$ is monotone decreasing sequence in $\mathbb{R}$ and bounded below by 0, which implies convergent.

$$\lim_{l\to\infty} d(S_{\xi_l}(y_0), S_{w_l}(y_0)) = r.$$

Let $\beta_l = d(S_{\xi_l}(y_0), S_{w_l}(y_0))$, therefore $\lim_{l\to\infty} \beta_l = r$. If possible let $r > 0$, we have,

$$\psi(r+) = \lim_{l\to\infty} \psi(\beta_l) \leq \limsup_{l\to\infty} \phi(\beta_{l-1}) \leq \limsup_{\mathfrak{t}\to r+} \phi(\mathfrak{t}),$$

which contradicts the property (a_3).

Therefore $r = 0$, which implies $\lim_{l\to\infty} \sup\{d(S_i(y_0)), S_j(y_0) : i, j \in \mathbb{N} \text{ and } i, j \geq l\} = 0$.
Therefore we conclude that $\{y_l\}$ is d-Cauchy sequence in the non-triangular metric space X.

Since X is d-complete non-triangular metric space therefore $\{y_l\}$ is d-convergent to some element ξ in X. Therefore, $\lim_{l\to\infty} y_{l+1} = \lim_{l\to\infty} S(y_l) = \xi$.

If $d(Sy_l, S\xi) = 0$ for infinitely many values of l, then ξ is a fixed point of S. Now suppose that $d(Sy_l, S\xi) > 0$, then applying the relation (3.1) we have,

$$\psi(d(Sy_l, S\xi)) \leq \phi(d(y_l, \xi)).$$

Now using the condition (a_2) we have $\psi(d(Sy_l, S\xi)) < \psi(d(y_l, \xi))$.

Since ψ is an increasing function, therefore,

$$d(Sy_l, S\xi) < d(y_l, \xi). \tag{3.3}$$

Taking limit $l \to \infty$ on both sides of (3.3) we get, $d(\xi, S\xi) \leq 0$, which implies ξ is a fixed point of S.

For the uniqueness of the fixed point, let us consider S has two fixed points ξ and η, therefore, $S\xi = \xi$, $S\eta = \eta$ and $d(\xi, \eta) > 0$. Now $\psi(d(S\xi, S\eta)) \leq \phi(d(\xi, \eta)) < \psi(d(\xi, \eta))$.

Since ψ is an increasing function, therefore $d(\xi, \eta) < d(\xi, \eta)$, which is a contradiction. Therefore $d(\xi, \eta) = 0$, which implies $\xi = \eta$. Hence S has a unique fixed point.

Hence the theorem.

□

Remark 3.1 *If $\psi(\mathfrak{t}) = \mathfrak{t}$ and $\phi(\mathfrak{t}) = \lambda\mathfrak{t}$, where $\lambda \in [0, 1)$, then Theorem 3.1 reduces to the Theorem 2.2.*

Example 3.1 *Let $X = [0, 1]$, define*

$$d(y, z) = \begin{cases} \frac{(y+z)^2}{(y+z)^2+1}, & \text{if } 0 \neq y \neq z \neq 0, \\ \frac{y}{2}, & \text{if } z = 0, \\ \frac{z}{2}, & \text{if } y = 0, \\ 0, & \text{if } y = z. \end{cases}$$

Clearly $d(y, y) = 0 \;\forall\; y \in X$ and $d(y, z) = d(z, y) \;\forall\; y, z \in X$. Let $\{y_l\}$ be a sequence in X such that $d(y_l, y) \to 0$ and $d(y_l, z) \to 0$ as $l \to \infty$. It implies that

$$\lim_{l\to\infty} \frac{(y_l + y)^2}{(y_l + y)^2 + 1} = \lim_{l\to\infty} \frac{(y_l + z)^2}{(y_l + z)^2 + 1} = 0$$

and these hold iff $\lim_{l\to\infty} y_l = -y = -z$ and so $y = z$. Therefore, (X, d) is a non-triangular metric space.

Let S be an operator on X defined by $Sy = \frac{y}{2}$. Let us define $\psi, \phi : (0, \infty) \to \mathbb{R}$, by:

$$\psi(\mathfrak{s}) = \begin{cases} \mathfrak{s}, & \text{if } \mathfrak{s} \in (0, 2) \\ 1 + \frac{\mathfrak{s}}{2}, & \text{if } \mathfrak{s} \in [2, \infty) \end{cases}$$

and

$$\phi(\mathfrak{s}) = \frac{2\psi(\mathfrak{s})}{3}.$$

(i) if $y = z$ *then it is trivial.*

(ii) if $0 \neq y \neq z \neq 0$ *then* $\psi(d(Sy, Sz)) = \psi(d(\frac{y}{2}, \frac{z}{2})) = \psi\Big(\frac{(y+z)^2}{(y+z)^2+4}\Big) = \frac{(y+z)^2}{(y+z)^2+4}$ *and* $\phi(d(y, z)) = \phi\Big(\frac{(y+z)^2}{(y+z)^2+1}\Big) = \frac{2\psi\Big(\frac{(y+z)^2}{(y+z)^2+1}\Big)}{3} = \frac{2(y+z)^2}{3(y+z)^2+3}$

(iii) if $y \neq 0$ *and* $z = 0$, $\psi(d(Sy, Sz)) = \psi(\frac{y}{4}) = \frac{y}{4}$ *and* $\phi(d(y, z)) = \phi(\frac{y}{2}) = \frac{y}{3}$.

(iv) if $z \neq 0$ *and* $y = 0$, $\psi(d(Sy, Sz)) = \psi(\frac{z}{4}) = \frac{z}{4}$ *and* $\phi(d(y, z)) = \phi(\frac{z}{2}) = \frac{z}{3}$.

Therefore from the above cases, we can conclude that, $\psi(d(Sy, Sz)) \leq \phi(d(y, z))\ \forall\ y, z \in X$, *and*

(a_1) ψ *is monotone increasing on* $(0, \infty)$*;*

(a_2) $\phi(\mathfrak{s}) = \frac{2\psi(\mathfrak{s})}{3} < \psi(\mathfrak{s})\ \forall\ \mathfrak{s} > 0$;

(a_3) $\limsup\limits_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim\limits_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s})\ \forall\ \mathfrak{e} > 0$.

Therefore ψ *and* ϕ *satisfy all the conditions of Theorem 3.1. Hence,* S *has a fixed point* 0 *which is unique.*

3.2. Extended Proinov contractions in non-triangular metric space

The following is an introduction to the new contractions class in the setting of non-triangular metric spaces derived from the Proinov contraction mentioned above. These contractions are defined in the following.

Let $\mathfrak{F}$ be the set of all pairs (ψ, ϕ) with $\psi, \phi : (0, \infty) \to \mathbb{R}$ are two functions satisfying the conditions below:

(b_1) if $\{\mathfrak{t}_l\} \subset (0, \infty)$ is a sequence such that $\psi(\mathfrak{t}_{l+1}) \leq \phi(\mathfrak{t}_l)\ \forall\ l \in \mathbb{N}$, then $\{\mathfrak{t}_l\} \to 0$;

(b_2) $\limsup\limits_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim\limits_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s})\ \forall\ \mathfrak{e} > 0$;

(b_3) if $\{\mathfrak{t}_l\}, \{\mathfrak{s}_l\} \subset (0, \infty)$ are two sequences such that $\{\mathfrak{s}_l\} \to 0$ and $\psi(\mathfrak{t}_l) \leq \phi(\mathfrak{s}_l)\ \forall\ l \in \mathbb{N}$, then $\{\mathfrak{t}_l\} \to 0$.

Example 3.2 *If* $\psi, \phi : (0, \infty) \to \mathbb{R}$ *are defined by* $\psi(\mathfrak{s}) = \lambda_1\mathfrak{s}$ *and* $\phi(\mathfrak{s}) = \lambda_2\mathfrak{s}\ \forall\ \mathfrak{s} > 0$, *where* $\lambda_1, \lambda_2 \in (0, \infty)$ *are such that* $\lambda_2 < \lambda_1$, *then* $(\psi, \phi) \in \mathfrak{F}$.

In the following Lemma, we find that the pair (ψ, ϕ) of functions associated with a Proinov contraction belongs to $\mathfrak{F}$.

Lemma 3.1 *Let* $\psi, \phi : (0, \infty) \to \mathbb{R}$ *be functions satisfying the following properties:*

(a_1) ψ *is monotone increasing;*

(a_2) $\phi(\mathfrak{s}) < \psi(\mathfrak{s})\ \forall\ \mathfrak{s} > 0$;

(a_3) $\limsup\limits_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim\limits_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s})\ \forall\ \mathfrak{e} > 0$.

Then $(\psi, \phi) \in \mathfrak{F}$.

Proof: Let $\{\mathfrak{t}_l\}$ be a sequence in $(0,\infty)$ such that $\psi(\mathfrak{t}_{l+1}) \leq \phi(\mathfrak{t}_l)\ \forall\ l \in \mathbb{N}$. Since $\mathfrak{t}_l > 0$, therefore the property (a_2) leads to

$$\psi(\mathfrak{t}_{l+1}) \leq \phi(\mathfrak{t}_l) < \psi(\mathfrak{t}_l) \quad \forall\, l \in \mathbb{N}. \tag{3.4}$$

As ψ is increasing, therefore $0 < \mathfrak{t}_{l+1} < \mathfrak{t}_l\ \forall\ l \in \mathbb{N}$. Therefore $\{\mathfrak{t}_l\}$ is a decreasing sequence in $\mathbb{R}$ and bounded below by 0, which implies it is convergent and let $\mathfrak{e} = \lim_{l\to\infty} \mathfrak{t}_l$. It is obvious that $\mathfrak{e} < \mathfrak{t}_l\ \forall\ l \in \mathbb{N}$. Now we have to prove $\mathfrak{e} = 0$. If possible let $\mathfrak{e} > 0$.

Therefore,

$$\psi(\mathfrak{e}) \leq \lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s}) = \lim_{l\to\infty} \psi(\mathfrak{t}_l) = \lim_{l\to\infty} \psi(\mathfrak{t}_{l+1}).$$

Using the inequality (3.4) we get,

$$\lim_{l\to\infty} \phi(\mathfrak{t}_l) = \lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s}),$$

which contradicts (a_3) because,

$$\lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s}) = \lim_{l\to\infty} \phi(\mathfrak{t}_l) \leq \limsup_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s}).$$

Therefore $\mathfrak{e} = 0$, which implies the condition (b_1) is satisfied.

The property (b_2) is the same as the property (a_3).

Finally, to check the property (b_3), let $\{\mathfrak{t}_l\}, \{\mathfrak{s}_l\}$ be two sequences in $(0,\infty)$ such that $\{\mathfrak{s}_l\} \to 0$ and $\psi(\mathfrak{t}_l) \leq \phi(\mathfrak{s}_l)\ \forall\ l \in \mathbb{N}$. Since $\mathfrak{s}_l > 0$, therefore from property (a_2) we have $\psi(\mathfrak{t}_l) \leq \phi(\mathfrak{s}_l) < \psi(\mathfrak{s}_l)$.

Since ψ is monotone increasing function (by property (a_1)) therefore, $0 < \mathfrak{t}_l < \mathfrak{s}_l\ \forall\ l \in \mathbb{N}$. Hence by applying Sandwich theorem we can conclude that $\{\mathfrak{t}_l\} \to 0$.

Hence $(\psi, \phi) \in \mathfrak{F}$.

□

Theorem 3.2 *Suppose (X, d) be a d-complete non-triangular metric space and $S : X \to X$ be a mapping such that $\exists\ (\psi, \phi) \in \mathfrak{F}$ with*

$$\psi(d(Sy, Sz)) \leq \phi(d(y, z)) \qquad \forall y, z \in X \ \text{and} \quad d(Sy, Sz) > 0. \tag{3.5}$$

Then the Picard sequence $\{S_l y\}$ converges to a fixed point of S.

Proof: Let $y \in X$ be arbitrary. Construct $y_1 = y$ and $y_{l+1} = Sy_l\ \forall\ l \in \mathbb{N}$. If there is a $l_0 \in \mathbb{N}$ such that $y_{l_0} = y_{l_0+1}$, then y_{l_0} is a fixed point of S. In such a case, $\{d(y_l, y_{l+1})\} \to 0,\ \forall\ l \geq l_0$.

Suppose that $y_l \neq y_{l+1}\ \forall\ l \in \mathbb{N}$ then each y_l is not a fixed point of S. Therefore, $d(y_l, y_{l+1}) > 0$, which implies $d(Sy_l, Sy_{l+1}) > 0,\ \forall\ l \in \mathbb{N}$. Now applying the contractive condition (3.5) we get,

$$\psi(d(y_{l+1}, y_{l+2})) = \psi(d(Sy_l, Sy_{l+1})) \leq \phi(d(y_l, y_{l+1})).$$

If we define $\mathfrak{s}_l = d(y_l, y_{l+1})$ then $\psi(\mathfrak{s}_{l+1}) \leq \phi(\mathfrak{s}_l)$, therefore using the property (b_1), we can conclude that $\{\mathfrak{s}_l\} \to 0$.

If there are $l_1, l_2 \in \mathbb{N}$ such that $l_1 < l_2$ and $y_{l_1} = y_{l_2}$ then Lemma 2.1 ensures that $\exists\ l_0 \in \mathbb{N}$ and $\xi \in X$ such that $y_l = \xi\ \forall l \geq l_0$. Therefore, ξ is a fixed point of S, and there is an assurance of the existence of a fixed point.

Now let $y_{l_1} \neq y_{l_2}$ for all $l_1 \neq l_2$, that is, $\{y_l\}$ is an infinite sequence. In particular, $d(Sy_{l_1}, Sy_{l_2}) = d(y_{l_1+1}, y_{l_2+1}) > 0 \ \forall \ l_1 \neq l_2$.

Now we have to prove that $\{y_l\}$ is a d-Cauchy sequence in non-triangular metric space X.

Let $r_l = \sup\{d(S_i(y_0)), S_j(y_0) : i, j \in \mathbb{N} \text{ and } i, j \geq l\}$. Note that $0 \leq r_{l+1} \leq r_l$. Therefore $\{r_l\}$ is a monotone decreasing sequence and bounded below by 0, which implies convergence of the sequence. Let $\lim_{l\to\infty} r_l = r$.

Again by definition of supremum, for every $l \in \mathbb{N} \ \exists \ w_l, \xi_l$ such that $\xi_l > w_l \geq l$ and

$$r_l - \frac{1}{l} < d(S_{\xi_l}(y_0), S_{w_l}(y_0)) \leq r_l.$$

Hence by Sandwich theorem we can conclude that,

$$\lim_{l\to\infty} d(S_{\xi_l}(y_0), S_{w_l}(y_0)) = r.$$

Let $\beta_l = d(S_{\xi_l}(y_0), S_{w_l}(y_0))$, therefore $\lim_{l\to\infty} \beta_l = r$. If possible let $r > 0$, we have

$$\psi(r+) = \lim_{l\to\infty} \psi(\beta_l) \leq \limsup_{l\to\infty} \phi(\beta_{l-1}) \leq \lim_{\mathfrak{t}\to r+} \phi(\mathfrak{t}),$$

which contradicts the property (b_2).

Therefore $r = 0$, which implies $\lim_{l\to\infty} \sup\{d(S_i(y_0)), S_j(y_0) : i, j \in \mathbb{N} \text{ and } i, j \geq l\} = 0$.

Therefore we conclude that $\{y_l\}$ is d-Cauchy sequence in non-triangular metric space X.

Since X is d-complete non-triangular metric space, therefore $\exists \ \xi \in X$ such that $\{y_l\}$ d-converges to ξ. Since $\{y_l\}$ is an infinite sequence, there is a $l_0 \in \mathbb{N}$ with $y_l \neq \xi$ and $Sy_l \neq S\xi \ \forall \ l \geq l_0$. Then we have,

$$\psi(d(y_{l+1}, S\xi)) = \psi(d(Sy_l, S\xi)) \leq \phi(d(y_l, \xi)).$$

It follows from the property (b_3) that, $\{d(y_{l+1}, S\xi)\} \to 0$, so $S\xi = \xi$. This completes the proof. □

Theorem 3.3 *Suppose that the family of pairs (ψ, ϕ) satisfies the hypothesis of Theorem 3.2 with the following hypothesis:*

(b_4) there is a subset $\Omega \subseteq X$ with fixed point set of $S \subseteq X$ and $\psi(d(y,z)) > \phi(y,z) \ \forall \ y \neq z \in \Omega$.

Then S has a unique fixed point $y_0 \in X$ and the Picard sequence $\{S(y_l)\}$ converges to $y_0 \ \forall \ y \in X$.

Proof: For the uniqueness of the fixed point of S, let $y_1, y_2 \in X$ be two distinct fixed points of S. Then $d(Sy_1, Sy_2) = S(y_1, y_2) > 0$. The contractive condition (1.1) implies that

$$\psi(d(y_1, y_2)) = \psi(d(Sy_1, Sy_2)) \leq \phi(d(y_1, y_2)),$$

which contradicts the property (b_4). Hence, S has a fixed point, which is unique. □

Example 3.3 *Let $X = [0, 1]$, define*

$$d(y,z) = \begin{cases} |y-z|, & \text{if } \ 0 \neq y \neq z \neq 0, \\ \frac{y}{2}, & \text{if } \ z = 0, \\ \frac{z}{2}, & \text{if } \ y = 0, \\ 0, & \text{if } \ y = z. \end{cases}$$

Clearly, (X,d) is a non-triangular metric space.

Let S be an operator on X defined by $Sy = \frac{y}{2}$. Let us define $\psi, \phi : (0,\infty) \to \mathbb{R}$, by:

$$\psi(\mathfrak{s}) = \begin{cases} \mathfrak{s}, & \text{if } \mathfrak{s} \in (0,2) \\ 1 + \frac{2}{\mathfrak{s}}, & \text{if } \mathfrak{s} \in [2,\infty) \end{cases}$$

and

$$\phi(\mathfrak{s}) = \frac{2\psi(\mathfrak{s})}{3}.$$

Further, we find the following:

(i) *if $y = z$ then it is trivial.*

(ii) *if $y \neq z \neq 0$ then $\psi(d(Sy,Sz)) = \psi(d(\frac{y}{2},\frac{z}{2})) = \psi\Big(|\frac{u-v}{2}|\Big) = |\frac{u-v}{2}|$ and $\phi(d(y,z)) = \phi(|u-v|) = \frac{2\psi(|u-v|)}{3} = \frac{2|u-v|}{3}$*

(iii) *if $y \neq 0$ and $z = 0$ then $\psi(d(Sy,Sz)) = \psi(\frac{y}{4}) = \frac{y}{4}$ and $\phi(d(y,z)) = \phi(\frac{y}{2}) = \frac{y}{3}$*

(iv) *if $z \neq 0$ and $y = 0$ then $\psi(d(Sy,Sz)) = \psi(\frac{z}{4}) = \frac{z}{4}$ and $\phi(d(y,z)) = \phi(\frac{z}{2}) = \frac{z}{3}$*

Therefore, from the above cases, we can conclude that, $\psi(d(Sy,Sz)) \leq \phi(d(y,z))\ \forall\ y,z \in X$.

However, Theorem 3.1 is not applicable as ψ is not monotone increasing in $[2,\infty)$.

We claim that $(\psi,\phi) \in \mathfrak{F}$.

(b_1) *let $\{\mathfrak{t}_l\} \subset (0,\infty)$ be such that $\psi(\mathfrak{t}_{l+1}) \leq \phi(\mathfrak{t}_l)\ \forall\ l \in \mathbb{N}$, that is, $\psi(\mathfrak{t}_{l+1}) \leq \frac{2\psi(\mathfrak{t}_l)}{3}\ \forall\ l \in \mathbb{N}$.*

Therefore $\psi(\mathfrak{t}_l) \leq (\frac{2}{3})^l \mathfrak{t}_o\ \forall\ l \in \mathbb{N}$, which implies, $\{\psi(\mathfrak{t}_l)\} \to 0$, so $\{\mathfrak{t}_l\} \to 0$;

(b_2) *$\limsup_{\mathfrak{s}\to\mathfrak{e}+} \phi(\mathfrak{s}) < \lim_{\mathfrak{s}\to\mathfrak{e}+} \psi(\mathfrak{s})\ \forall\ \mathfrak{e} > 0$;*

(b_3) *let $\{\mathfrak{t}_l\}$ and $\{\mathfrak{s}_l\}$ be two sequences in $(0,\infty)$ such that $\{\mathfrak{s}_l\} \to 0$ and $\psi(\mathfrak{t}_l) \leq \phi(\mathfrak{s}_l)\ \forall\ l \in \mathbb{N}$. Then*

$$\psi(\mathfrak{t}_l) \leq \phi(\mathfrak{s}_l) = \frac{2\psi(\mathfrak{s}_l)}{3} = \frac{2\mathfrak{s}_l}{3}.$$

Since $\{\mathfrak{s}_l\} \to 0$ therefore, $\{\psi(\mathfrak{t}_l)\} \to 0$ as $l \to \infty$. Thus we can conclude that $\{\mathfrak{t}_l\}$ converges to 0.

Hence $(\psi,\phi) \in \mathfrak{F}$ and it satisfies all the conditions of Theorem 3.2 and so S has a fixed point.

4. An application

As an application, we find the existence and uniqueness of a solution of the following homogeneous Fredholm integral equation in the non-triangular metric space:

$$y : [0,1] \to [0,\infty)$$

$$y(x) = \int_0^1 k(x,t)y(t)dt, \tag{4.1}$$

where $k(x,t)$ is continuous on $[0,1]\times[0,1]$ and let $k(x,t) > 0$.

Let $\mathfrak{C}[0,1]$ be the set of all real continuous functions defined on $[0,1]$ with supremum norm $||y|| = \sup\{|y(x)| : x \in [0,1]\}$ and let $M = \sup\{k(x,t) : x,t \in [0,1]\}$.

Now, we define the mapping $S : \mathfrak{C}[0,1] \to \mathfrak{C}[0,1]$ by,

$$S\Big(y(x)\Big) = \int_0^1 k(x,t)y(t)dt. \tag{4.2}$$

Now, we define the non-triangular metric d on $\mathfrak{C}[0,1]$ as,

$$d\Big(y(x), z(x)\Big) = \begin{cases} \sup\{|y(x) - z(x)| : x \in [0,1]\}, & \text{if } 0 \neq y(x) \neq z(x) \neq 0, \\ \frac{1}{2}\sup\{|y(x)| : x \in [0,1]\}, & \text{if } z(x) = 0, \\ \frac{1}{2}\sup\{|z(x)| : x \in [0,1]\}, & \text{if } y(x) = 0, \\ 0, & \text{if } y(x) = z(x). \end{cases}$$

Clearly, $(\mathfrak{C}[0,1], d)$ is a d-complete non-triangular metric space.

Let us define $\psi, \phi : (0,\infty) \to \mathbb{R}$, by:

$$\psi(s) = s \quad \text{and} \quad \phi(s) = \frac{2s}{3}.$$

Further, we find the following:

(i) if $y(x) = z(x)$ then, it is trivial;

(ii) if $0 \neq y(x) \neq z(x) \neq 0$ then,

$$\begin{aligned} \psi\Big(d(Sy(x), Sz(x))\Big) = d(Sy, Sz) &= \sup\{|S(y(x)) - S(z(x))| : x \in [0,1]\} \\ &\leq \sup_{x\in[0,1]} \Big\{ \int_0^1 |k(x,t)|\, |[y(t) - z(t)]dt| \Big\} \\ &\leq \sup_{x,t\in[0,1]} |k(x,t)| \sup_{t\in[0,1]} \{|y(t) - z(t)|\} \int_0^1 dt \\ &= Md\Big(y(x), z(x)\Big) \end{aligned}$$

and

$$\phi\Big(d(y(x), z(x))\Big) = \frac{2}{3} d\Big(y(x), z(x)\Big);$$

(iii) if $y(x) \neq 0$ and $z(x) = 0$ then,

$$\begin{aligned} \psi\Big(d(Sy, Sz)\Big) = d(Sy, Sz) &= \frac{1}{2}\sup\{|S(y(x))| : x \in [0,1]\} \\ &\leq \frac{1}{2} \sup_{x\in[0,1]} \Big\{ \int_0^1 |k(x,t)|\, |y(t)dt| \Big\} \\ &\leq \sup_{x,t\in[0,1]} |k(x,t)| \sup_{t\in[0,1]} \frac{1}{2}\{|y(t)|\} \int_0^1 dt \\ &= Md\Big(y(x), z(x)\Big) \end{aligned}$$

and

$$\phi\Big(d(y(x), z(x))\Big) = \frac{2}{3} d\Big(y(x), z(x)\Big);$$

(iv) if $z(x) \neq 0$ and $y(x) = 0$ then,

$$\begin{aligned}\psi\Big(d(Sy,Sz)\Big) = d(Sy,Sz) &= \frac{1}{2}\sup\{|S(z(x))| : x \in [0,1]\} \\ &\leq \frac{1}{2}\sup_{x\in[0,1]}\Big\{\int_0^1 |k(x,t)|\ |z(t)dt|\Big\} \\ &\leq \sup_{x,t\in[0,1]}|k(x,t)| \sup_{t\in[0,1]}\frac{1}{2}\{|z(t)|\}\int_0^1 dt \\ &= Md\Big(y(x),z(x)\Big)\end{aligned}$$

and

$$\phi\Big(d(y(x),z(x))\Big) = \frac{2}{3}d\Big(y(x),z(x)\Big).$$

Now if $M = \sup\{k(x,t) : x,t \in [0,1]\} \leq \frac{2}{3}$ then, $\psi\Big(d(Sy,Sz)\Big) \leq \phi\Big(d(y,z)\Big)\ \forall\ y,z \in \mathfrak{C}[0,1]$.

Therefore ψ and ϕ satisfy all the conditions of Theorem 3.1. Therefore, the operator S defined in (4.2) has a unique fixed point. Hence, Fredhlom integral equation (4.1) has a unique solution.

5. Conclusion

In this work, we have generalized Proinov contractions in the non-triangular metric space in light of Proinov's attractive results in metric space. Also, we have introduced a family $\mathfrak{F}$ of auxiliary function (ψ,ϕ) in non-triangular metric space to avoid monotone condition on ψ. This idea may be extended to non-triangular fuzzy metric space by suitable changes.

Further, as an application, we find the existence and uniqueness of a solution of the homogeneous Fredholm integral equation in non-triangular metric space using Proinov contraction.

It will be interesting to see if the contractivity condition (1.1) could be replaced by a generalized contractivity condition. We recall that a self-map S on the non-triangular metric space (X,d) satisfies a generalized contractive type condition if

$$\psi(d(Sy,Sz)) \leq \phi(m(y,z)) \qquad \forall y,z \in X,$$

where $\psi,\phi : (0,\infty) \to \mathbb{R}$ and $m(y,z)$ is defined by

$$m(y,z) = \max\Big\{d(y,z), d(y,Sy), d(z,Sz), \frac{d(y,Sz)+d(z,Sy)}{2}\Big\}.$$

Acknowledgments

The work of the first author is financially supported by the CSIR, India with Grant No: 09/1217(13093)/ 2022-EMR-I. Furthermore, the authors would like to express their gratitude to the reviewers for their valuable suggestions to enhance the quality of the paper.

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