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On Generalised Almost r-Contact Structure in a GF Manifold

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ABSTRACT: The focus of this research is to study properties and theorems of a generalised almost r-contact structure in a GF manifold. GF structure endowed with a metric tensor, an affine connection, Nijenhuis tensor and π plane field of a generalised almost r-contact structure are also a part of the study. Contact structures has application in different branches of mathematics and physics which can be explored further with the help of examples presented in standard and generalised form in this research.

Key Words: Generalised r-Contact Structure, GF structure, Metric tensor, Affine connection, Nijenhuis tensor, π plane field.

Contents

1. Introduction

Contact geometry is an important tool to study odd dimensional manifolds and is a basis for many physical phenomenon. Contact structures first appeared in the work of Sophus Lie. He worked on partial differential equations and gathered attention of the researchers. Many researchers have worked on it till date to explore properties of different geometric structures.

Boothby and Wang [2] discussed the properties of different types of manifolds along with contact manifolds. Yano [21] and Hit [10], studied affine connexions in an almost product space and product manifold respectively. Sasaki [17] extended the work to study properties of almost contact structure. An almost contact structure in a complex manifold and in a product manifold has been studied by Mishra [14], Upadhyay & Dube [18] respectively, by making use of index free notations. Dube and Nivas [6] further extended the study and studied the almost r-contact structure in a product manifold. Yano and Kon [22] gave a detailed study of different types of structures in the manifolds. Upadhyay and Agarwal [19] studied generalised r- contact manifold in a complex manifold.

Geiges [8] studied contact structures in (2n+1) manifolds and showed that contact structures exist on simply connected 5-manifolds by applying results of Eliashberg and Weinstein on contact surgery. Pandey and Dasila [16] discussed properties of different structures on differentiable manifold. Later Chandra and Lal [3] discussed the almost product structure in a differentiable manifold. Eum [7] studied curvature

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tensors of 3-dimensional almost contact metric manifolds whereas Blair et. al [1] studied the properties of concircular curvature tensor of a contact metric manifold. Generalisation of almost contact structures in manifold is done by Cho [4]. Recently Khan at. el. [11,12,13] studied different structures on manifold.

In this research we have extended the work of [19] and [9] to a structure. In the first section we gave the definition followed with the properties associated with the generalised structure. At the end we have supported our study with multiple examples which shows its relevance.

Let M^{n+r} be an (n+r)- dimensional differentiable manifold of differentiability class C^{∞}. Let there exist in M^{n+r} a (1, 1) tensor field F of class C^{∞}, r - contravariant vector fields, $T^1, T^2, ..., T^r$ of class C^{∞} and r 1-forms $A_1, A_2, ..., A_r$ of class C^{∞} satisfying the following [6].

By Nivas and Suab [15] this generalised structure can be categorised into different particular forms as given below:

Case 1: Contact structure for $\lambda = \pm 1$

Case 2: Complex structure for $\lambda = \pm i$

Case 3: Tangent structure for $\lambda = 0$

$$\bar{\bar{X}} = \lambda^2 X + \sum_{p=1}^{r} A_p(X) T^p, \tag{1.1}$$

$$\bar{X} \stackrel{\text{def}}{=} F(X) \tag{1.2}$$

$$\bar{T}^p = 0 for 1 \leq p \leq r \tag{1.3}$$

$$A_p(\bar{X}) = 0 \tag{1.4}$$

for arbitrary vector field X for $1 \leq p \leq r$

$$A_q(T^p) = \lambda^2 \delta_q^p \tag{1.5}$$

δ_q^p denotes the Kronecker delta.

Thus using equations (1.1), (1.2), (1.3), (1.4) and (1.5), M^{n+r} is said to possess a generalised almost r-contact structure in a GF manifold by [19].

Theorem 1.1 *Let M^{n+r} be a generalised almost r-contact in a GF manifold. Then there are r-eigen values where 0 is corresponding to eigen vector $T^p (1 \leq p \leq r)$ and if k values correspond to 1 , then (n-k) eigen values correspond to -1.*

Proof: Let λ be an eigen value of F and L be the corresponding eigen vector. Then

$$\bar{L} = \lambda L \tag{1.6}$$

Barring (1.6) and using (1.1) and (1.6), we get

$$(\lambda^2 - 1)L = \sum_{p=1}^{r} A_p(L) T^p \tag{1.7}$$

Case 1. Let $L = T^p$ then from (1.6) and (1.7) we get $\lambda = 0$. This gives the eigen value corresponding to eigen vector T^p.

Case 2. Let L and T^p be linearly independent, then from (1.7) we have
$A_p(L) = 0$ and $\lambda = \pm 1$.

If k values correspond to the eigen value 1, then n-k values will correspond to the eigen value -1.

□

2. The metric tensor

Let the generalised almost r-contact GF manifold be endowed with the non-singular metric tensor g. From (1.1) and (1.4), we have

$$\bar{\bar{\bar{X}}} = \lambda^2 \bar{X} \tag{2.1}$$

By [21] let us restrict g such that

$$g(\bar{\bar{X}}, \bar{\bar{Y}}) = g(\lambda^2 X, \lambda^2 Y) = \lambda^4 g(X, Y) \tag{2.2}$$

and

$$g(T^p, X) = A_p(X) \tag{2.3}$$

On putting $X = T^p$ and $Y = T^p$ in equation (2.3) we have,
$g(T^p, Y) = A_p(Y)$ and $g(X, T^p) = A_p(X)$

Barring Y and X in above equation we have,

$$\begin{aligned} g(T^p, \bar{Y}) = 0 \\ g(\bar{X}, T^p) = 0 \end{aligned} \tag{2.4}$$

Equation (2.2) in view of (1.1), (1.4), (2.1) and (2.3) yield

$$g(X, \bar{Y}) = g(\bar{X}, Y) = 0 \tag{2.5}$$

Thus we have

$$g(\bar{X}, \bar{Y}) = g(\bar{\bar{X}}, Y) = g(X, \bar{\bar{Y}}) \tag{2.6}$$

Using (1.1) and (2.3), equation (2.5) is equivalent to

$$g(\bar{X}, \bar{Y}) = \lambda^2 g(X, Y) + \sum_{p=1}^{r} A_p(X) A_p(Y) \tag{2.7}$$

We call a generalised almost r - contact GF manifold M^{n+r} endowed with non singular metric tensor g satisfying (2.7) as a generalised almost r - contact GF metric manifold.

Let us define a tensor $'F$ by

$$'F(X, Y) \stackrel{\text{def}}{=} \lambda^2 g(\bar{X}, Y) \tag{2.8}$$

From (2.6) and (2.8), we have

$$'F(X, \bar{Y}) = \lambda^2 g(\bar{X}, \bar{Y}) \tag{2.9}$$

$$= \lambda^2 g(\bar{\bar{X}}, Y)$$

$$= \lambda^{2\prime} F(\bar{X}, Y)$$

Then we have,

$$'F(X, \bar{Y}) - \lambda^{2\prime} F(\bar{X}, Y) = 0$$

$$'F(\bar{X}, \bar{Y}) - \lambda^{2\prime} F(X, Y) = 0$$

Putting T^p for X in (2.9) and using (1.3), we get

$$\begin{aligned} 'F(T^p, Y) = 0 \\ 'F(X, T^p) = 0 \end{aligned} \tag{2.10}$$

Theorem 2.1 *The tensor $'F$ defined by (2.8) is hybrid in both the slots and is also symmetric.*

Proof: Barring X in (2.8) and using(1.1) and (2.9), we get

$$'F(\bar{X}, \bar{Y}) =' F(X, Y)$$

Also from (2.5) and (2.8) we have

$$'F(X, Y) =' F(Y, X)$$

which proves the theorem. □

3. Generalised almost r contact GF manifold with specified affine connection

Lemma 3.1 *Let us take the affine connection D in M^{n+r} such that it satisfies the following:*

$$D_X T^p = \bar{X} \tag{3.1}$$

$$D_X \bar{Y} = D_Y \bar{X} + \overline{[X, Y]} + X A_p(Y) \tag{3.2}$$

$$(D_X A_p)(Y) + (D_Y A_p)(X) = 0 for 1 \leq p \leq r \tag{3.3}$$

Let S be the torsion tensor of D, K the curvature tensor with respect to D and Ric be the corresponding Ricci tensor.

Then,

$$S(X, Y) \stackrel{\text{def}}{=} D_X Y - D_Y X - [X, Y] \tag{3.4}$$

$$K(X, Y, Z) \stackrel{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \tag{3.5}$$

$$Ric(Y, Z) \stackrel{\text{def}}{=} (C_1^1 K)(Y, Z) \tag{3.6}$$

where $C_1^1 K$ denotes contraction of K.

Now we shall prove the following theorems:

Theorem 3.1 *Let $1 \leq p \leq r$, then we have*

$$S(\bar{X}, T^p) = \frac{1}{\lambda^2}\overline{[X, T^p]} - [\bar{X}, T^p], \tag{3.7}$$

$$\overline{D_{T^p}\bar{X}} = \bar{X} - \frac{1}{\lambda^2}[X, T^p] - \frac{1}{\lambda^2}\sum_{p=1}^{r} A_p([X, T^p])T^p, \tag{3.8}$$

$$(D_X F)(\bar{Y}) - \overline{(D_X F)(Y)} = X\sum_{p=1}^{r} A_p(Y)T^p + A_p(Y)\bar{X} - \sum_{p=1}^{r} A_p(D_X Y)T^p \tag{3.9}$$

Proof: Putting T^p for Y in (3.4) and making use of (3.1), we get

$$S(X, T^p) = \bar{X} - D_{T^p}X - [X, T^p] \tag{3.10}$$

Barring X in (3.2) and making use of (1.1) and (3.1) we obtain,

$D_{\bar{X}}\bar{Y} = \lambda^2 D_Y X + D_Y(\sum_{p=1}^{r} A_p(X)T^p) + \overline{[\bar{X}, Y]} + \bar{X}A_p(Y),$

$D_{\bar{X}}\bar{Y} = \lambda^2 D_Y X + \sum_{p=1}^{r} D_Y(A_p(X))T^p$

$+ \sum_{p=1}^{r} A_p(X)D_Y(T^p) + \overline{[\bar{X}, Y]} + \bar{X}A_p(Y)$

Putting T^p for Y in the above equation and making use of (1.3), (1.5) and (3.1), we get

$D_{\bar{X}}\bar{T}^p = \lambda^2 D_{T^p}X + \sum_{p=1}^{r} D_{T^p}(A_p(X))T^p$

$+ \sum_{p=1}^{r} A_p(X)D_{T^p}(T^p) + \overline{[\bar{X}, Y]} + \bar{X}A_p(T^p)$

$\lambda^2 D_{T^p}X = D_{\bar{X}}\bar{T}^p - \sum_{p=1}^{r} D_{T^p}(A_p(X))T^p$

$- \sum_{p=1}^{r} A_p(X)D_{T^p}(T^p) - \overline{[\bar{X}, Y]} - \bar{X}A_p(T^p)$

$\lambda^2 D_{T^p}X = \lambda^2\bar{X} - \overline{[\bar{X}, T^p]} - T^p\sum_{p=1}^{r} A_p(D_{T^p}(X))$

$$D_{T^p}X = \bar{X} - \frac{1}{\lambda^2}\overline{[\bar{X}, T^p]} - \frac{1}{\lambda^2}T^p\sum_{p=1}^{r} A_p(D_{T^p}(X)) \tag{3.11}$$

Therefore from (3.10) and (3.11), we have

$S(\bar{X}, T^p) = \frac{1}{\lambda^2}\bar{[}\bar{\bar{X}}, T^p] + T^p\sum_{p=1}^{r} A_p(D_{T^p}(\bar{X})) - [\bar{X}, T^p],$

which from (1.1), (1.4) and (3.3) yields (3.7).

Barring X in (3.11) and making use of (1.1) ,(1.4) and (3.2), we obtain

$$D_{T^p}\bar{X} = \bar{\bar{X}} - \frac{1}{\lambda^2}\overline{[\bar{\bar{X}}, T^p]} - \frac{1}{\lambda^2}\bar{T}^p\sum_{p=1}^{r} A_p(D_{T^p}(\bar{X})) \tag{3.12}$$

$D_{T^p}\bar{X} = X + \frac{1}{\lambda^2}\sum_{p=1}^{r} A_p(X)T^p - \frac{1}{\lambda^2}\overline{[X, T^p]}$

Barring (3.11) throughout and making use of (1.1) and (1.3) we get (3.8).

We know that

$$(D_X F)Y = D_X \bar{Y} + \overline{D_X Y} \tag{3.13}$$

Barring Y in (3.13) and making use of (1.1) and(3.2)we obtain

$(D_X f)\bar{Y} = D_X Y + D_X(\sum_{p=1}^{r} A_p(Y)T^p) + \overline{D_X Y}$,

$$(D_X f)\bar{Y} = D_X Y + X(\sum_{p=1}^{r} A_p(Y)T^p) + A_p(Y)\bar{X} + \overline{D_X Y} \tag{3.14}$$

Barring (3.13) throughout and making use of (1.1) we get,

$$\overline{(D_X f)Y} = \overline{D_X Y} + D_X Y + \sum_{p=1}^{r} A_p(D_X Y)T^p \tag{3.15}$$

Subtracting (3.15) from (3.14) we have (3.9). Thus the theorem is proved. □

Theorem 3.2 *Let G be the Einstein Tensor. Then we have*

$$G(\bar{X}, T^p) = 0 \tag{3.16}$$

Proof: From [2] we have

$$G(X, Y) = Ric(X, Y) - \frac{1}{2}kg(X, Y) \tag{3.17}$$

putting T^p for Y in (3.17) and making use of (2.4) we get,

$$G(X, T^p) = \{(n-1) - \frac{1}{2}k\}A_p(X) \tag{3.18}$$

From [3]

$$Ric(X, T^p) = (n-1)A_p(X)$$

Barring X in (3.18) and using (1.4) we obtain (3.16). □

4. The Nijenhuis Tensor

Let $N(X, Y)$ be the Nijenhuis tensor of F, then we have

$$N(X, Y) = [\bar{X}, \bar{Y}] - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]} + \overline{\overline{[X, Y]}} \tag{4.1}$$

Using (1.1) above equation becomes

$$N(X, Y) = [\bar{X}, \bar{Y}] - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]} + \lambda^2[X, Y] + \sum_{p=1}^{r} A_p([X, Y])T^p \tag{4.2}$$

Theorem 4.1 *In a generalised almost r- contact GF manifold, we have*

$$\overline{N(\bar{X},Y)} = -N(X,Y) + \sum_{p=1}^{r} A_p(X)N(T^p,Y) + \sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p, \tag{4.3}$$

$$\begin{aligned} N(\bar{X},Y) - N(X,\bar{Y}) = &\sum_{p=1}^{r} A_p(X)([T^p,\bar{Y}] - \overline{[T^p,Y]}) \\ &+\sum_{p=1}^{r} A_p(Y)(\overline{[X,T^p]} - [\bar{X},T^p]) \\ &+\sum_{p=1}^{r} A_p([\bar{X},Y])T^p - \sum_{p=1}^{r} A_p([X,\bar{Y}])T^p \end{aligned} \tag{4.4}$$

Proof: Barring X in (4.2) and then barring throughout, we obtain

$$\overline{N(\bar{X},Y)} = \overline{[\bar{\bar{X}},\bar{Y}]} - \overline{\overline{[\bar{\bar{X}},Y]}} - \overline{\overline{[\bar{X},\bar{Y}]}} + \lambda^2\overline{[\bar{X},Y]} + \sum_{p=1}^{r} \overline{A_p([\bar{X},Y])T^p}, \tag{4.5}$$

Using (1.1) and (1.3) it becomes

$$\begin{aligned} \overline{N(\bar{X},Y)} = &\overline{[\lambda^2 X + \sum_{p=1}^{r} A_p(X)T^p,\bar{Y}]} - \lambda^2[\bar{\bar{X}},Y] \\ &-\sum_{p=1}^{r} A_p([\bar{\bar{X}},Y])T^p - \lambda^2[\bar{X},\bar{Y}] \\ &-\sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p + \lambda^2\overline{[\bar{\bar{X}},Y]}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} = &\lambda^2\overline{[X,\bar{Y}]} + \sum_{p=1}^{r} A_p(X)\overline{[T^p,\bar{Y}]} - \lambda^2[X,Y] - \sum_{p=1}^{r} A_p(X)[T^p,Y] \\ &-\sum_{p=1}^{r} A_p([X,Y]) + \sum_{p=1}^{r} A_q([T^q,Y])T^p \\ &-\lambda^2[\bar{X},\bar{Y}] - \sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p + \lambda^2\overline{[\bar{X},Y]}, \end{aligned} \tag{4.7}$$

$= -\{\lambda^2[\bar{X},\bar{Y}] - \lambda^2\overline{[\bar{X},Y]} - \lambda^2\overline{[X,\bar{Y}]} + \lambda^2[X,Y]$

$+\sum_{p=1}^{r} A_p([X,Y])T^p + \sum_{p=1}^{r} A_p(X)([T^p,Y] - \overline{[T^p,\bar{Y}]}) + \sum_{p=1}^{r} A_(([T^p,Y])T^q$

$+\sum_{p=1}^{r} A_p([\bar{X},\bar{Y}])T^p\}$

Putting T^p for X in (4.2) and using (1.3) we get,

$$N(T^p,Y) = \lambda^2[T^p,Y] - \lambda^2\bar{[}T^p,\bar{Y}] + \sum_{p=1}^{r} A_q([T^p,Y])T^q \tag{4.8}$$

Equation (4.5) in (4.2) and (4.6) we get (4.5).

Barring X and Y respectively in (4.1) and using (1.1) we get (4.4). □

Theorem 4.2 *In a generalised almost r - contact GF manifold, we have*

$$A_pN(X,Y) = -\sum_{q=1}^{r} A_q(X)A_p(N(T^q,Y)) + A_p([\bar{X},\bar{Y}]), \tag{4.9}$$

$$\overline{N(\bar{X},T^p)} = -N(X,T^p) \tag{4.10}$$

Proof: Operating (4.3) by A_p and using (1.5) and (1.4) we obtain (4.7).

Replacing Y by T^p in (4.3) and using (1.3) and (4.1) we get (4.8). □

5. π - plane field

Walker [20] studied the partial fields of partially null vectors. Similarly we will define the π - plane field.
Operating (1.1) with F and using (1.1) ..(1.5), we have,

$$F(X) = \lambda^2 F(X) \tag{5.1}$$

Let V be the matrix defined by

$$V = F - \lambda I \tag{5.2}$$

where I is the identity tensor field and let W be the vector field such that

$$V(W) = 0 \tag{5.3}$$

Let us consider a field of planes over M^{n+r} spanned by vector field W and call it π - plane field. We can consider following theorems:

Theorem 5.1 *If the manifold M^{n+r} admits a π - plane field then its vectors are null vectors which are orthogonal to T^p.*

Proof: Using equations (5.2) and (5.3), we get

$$F(W) = \bar{W} = \lambda W \tag{5.4}$$

Operating this equation with F and using (1.1) we have

$F(F(W)) = F(\bar{W}) = F(\lambda W)$

$F^2(W) = \lambda^2 W$

Using (1.1) we have

$$\sum_{p=1}^{r} A_p(W)T^p = 0 \tag{5.5}$$

Thus $A_p(W) = 0$ or using (2.3)

$$g(T^p, W) = 0 \tag{5.6}$$

This proves theorem (5.1). □

Theorem 5.2 *In the manifold M^{n+r}, the π plane field is parallel if F is covariant constant.*

Proof: From [20] Let W be the basic vector of the π - plane field such that

$$(D_X W) = B(X)W \tag{5.7}$$

where D is Riemannian connection and B is covariant tensor on M^{n+r}. Since F is a covariant constant on M, we have

$$(D_X F)(W) = 0 \tag{5.8}$$

Differentiating (5.4) covariantly we have,

$$(D_X F)(W) + F(D_X W) = \lambda D_X W \tag{5.9}$$

Using (5.7) we have $(F - \lambda I)(D_X W) = 0$ or $V(D_X W) = 0$

This proves that F is a covariant constant and hence the theorem. □

6. Example of generalised almost contact r-structure in a GF manifold in 4 dimensional Euclidean space

Example 6.1: By Debnath [5] let R_5 be any 5 dimensional Euclidean space and let us define

$$F(X) = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let I is 5×5 identity matrix.

$$A^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \lambda \end{bmatrix}, A^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \lambda^{\frac{1}{2}} \end{bmatrix}, A^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\lambda^{\frac{1}{3}} \end{bmatrix}, A^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\lambda^{\frac{1}{4}} \end{bmatrix}, A^5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda \end{bmatrix},$

$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda^{\frac{3}{2}} \end{bmatrix}$,

$T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda^{\frac{5}{3}} \end{bmatrix}$,

$T_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda^{\frac{7}{4}} \end{bmatrix}$,

$T_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\lambda^2 \end{bmatrix}$,

$$A_1(X)T^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 \end{bmatrix}$$

$$A_2(X)T^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 \end{bmatrix}$$

$$A_3(X)T^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda^2 \end{bmatrix}$$

$$A_4(X)T^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda^2 \end{bmatrix}$$

$$A_5(X)T^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda^2 \end{bmatrix}$$

$$\sum_{p=1}^{5} A_p(X)T^p = -\lambda^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F(T^1) = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} = F(T^2) = F(T^3)$$

Similarly $F(A^1) = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = F(A^2) = F(A^3)$

Thus, $F^2(X) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$= \lambda^2 \cdot I_5 - \lambda^2 \sum_{p=1}^{5} A_p(X)T^p$. This verifies theorem (1.1).

Example 6.2: We can also prove theorem (1.1) with another example in 5 dimension. We define

$$F(X) = \begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem (1.1) can be verified similar to example (5.2) using same values of variables $A_1, A_2, A_3, A_4, A_5, T^1, T^2, T^3, T^4, T^5$ as in (6.1).

Example 6.3: Generalised example of theorem (1.1) based on example (6.1) can be done in $2n+1$ dimension. Let us define F as:

$$F(X) = \begin{bmatrix} \lambda \cdot I_n & 0 & 0 \\ 0 & -\lambda \cdot I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $A_1 = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \lambda \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \lambda^{\frac{1}{2}} \end{bmatrix}$, ...

$A_{n+1} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ -\lambda \end{bmatrix}$ $A_{n+2} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ -\lambda^{\frac{1}{2}} \end{bmatrix}$ .... $A_{2n+1} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$

$T^1 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \lambda \end{bmatrix}$, $T^2 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \lambda^{\frac{3}{2}} \end{bmatrix}$ ...

$T^{n+1} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \lambda \end{bmatrix}$, $T^{n+2} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \lambda^{\frac{3}{2}} \end{bmatrix}$ ...

$T^{2n+1} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & -\lambda^2 \end{bmatrix}$

Thus, $F^2(X) = \lambda^2 \begin{bmatrix} 1 & 0 & 0... & 0 \\ 0 & 1 & 0... & 0 \\ .... & ... & .... & .. \\ ... & ... & .... & .. \\ 0 & 0 & 0.. & 1 \end{bmatrix}$

$= \lambda^2 \cdot I_{2n+1} - \lambda^2 \sum_{p=1}^{2n+1} A_p(X)T^p$. This proves the theorem.

Similarly we can also write:

Example 6.4: Second generalised example of theorem (1.1) based on example (6.2) can be done in $2n+1$ dimension similar to example (6.3). Let us define F as:

$$F(X) = \begin{bmatrix} -\lambda \cdot I_n & 0 & 0 \\ 0 & \lambda \cdot I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem (1.1) can be verified similar to example (6.3) using same values of variables $A_1, A_2, A_3, ...T^1$, T^2, T^3... as in (6.3).

7. Conclusion

The main focus of this research is to discuss different properties of generalised almost contact r structure in a GF manifold. More recently, contact structures are seen to have relations in fluid mechanics, Riemannian geometry and low dimensional topology which can also be explored on the structure given in the paper. We have proved that if $M^{(n+r)}$ manifold has 0 eigen values for T^p and n values corresponding to 1 then $n-k$ values correspond to -1. We have also discussed properties of metric tensor in generalised almost r contact manifold and also discussed properties and theorems on specified affine connections, properties on Nijenhuis tensor and π plane field. At the end we have verified theorem 1.1 with some examples. Similar examples cab be developed for other theorems as well.

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