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Spectral Expansion for Impulsive q-Dirac System on the Whole Line

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ABSTRACT: In this study, we consider a impulsive q-Dirac system on the whole line. We show the existence of a spectral function of this system. Also we establish a Parseval equality and expansion formula in eigenfunctions in terms of the spectral function.

Key Words: Discontinuous equations, q-Dirac system, spectral function, eigenfunction expansion.

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1. Introduction

Spectral expansion theorems are intensively researched as they are encountered in various problems in physics and mathematics. Spectral expansion theorems are needed, especially when solving problems expressed with partial differential equations by the method of separation variables. There are many studies on this subject in the literature (see [1,3,4,19,26]).

Impulsive differential equations are extensively researched today due to their applications in various fields in science and engineering ([10,11,17,18,20,21,22,23,25,27]). These equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states.

On the other hand, q-calculus is a type of calculus that allows working with non-differentiating functions. It has extensive applications in several disciplines such as mechanics, the calculus of variations, orthogonal polynomials, and the theory of relativity (see [9]). Recently, Annaby and Mansour [4] applied q-calculus to classical Sturm–Liouville problems and investigated q-Sturm–Liouville problems. In 2017, Allahverdiev and Tuna [2] introduced q-analogue of the one-dimensional Dirac operator defined as

$$\begin{cases} -y_2' + p(x)\, y_1 = \lambda y_1, \\ y_1' + r(x)\, y_2 = \lambda y_2. \end{cases}$$

Later, q-Sturm–Liouville problems were studied by putting impulsive boundary conditions. In [8], the author studied discontinuous q-Sturm–Liouville problems with eigenparameter-dependent boundary conditions. In [14,15,16], Karahan and Mamedov investigated a q-Sturm–Liouville problem with discontinuity conditions. In [6], the authors studied the properties of the scattering function of an impulsive q-difference equation. In [7], Bohner and Cebesoy investigated the locations of the eigenvalues and spectral singularities of an operator corresponding to impulsive q-difference equations.

In this paper, a spectral function for impulsive q-Dirac system on the interval $(-\infty, \infty)$ is constructed. Later, we establish a Parseval equality and expansion formula in eigenfunctions in terms of the spectral function.

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2. Preliminaries

In this section, the basic concepts of q-calculus that will be used in the article will be given. For more detailed information, the following sources can be examined, [13,5,9].

Let $q \in (0,1)$ and let $A \subset \mathbb{R} := (-\infty, \infty)$ be a q-geometric set, i.e., if $q\zeta \in A$ for all $\zeta \in A$. We begin by defining the *operator* D_q by

$$D_q h(\zeta) = \begin{cases} \frac{h(q\zeta)-h(\zeta)}{(q-1)\zeta}, & \zeta \neq 0 \\ \lim_{n\to\infty} \frac{h(q^n \varsigma)-h(0)}{q^n \varsigma}, & \zeta = 0, \end{cases}$$

where $\zeta, \varsigma \in A$. When it is required, q will be replaced by q^{-1}. The following facts, which will be frequently used, can be verified directly from the definition:

$$D_{q^{-1}} h(\zeta) = (D_q h)(q^{-1}\zeta),$$
$$(D_q^2 h)(q^{-1}\zeta) = q D_q[D_q h(q^{-1}\zeta)] = D_{q^{-1}} D_q h(\zeta).$$

Related to this operator there exists a non-symmetric formula for the q-differentiation of a product

$$D_q[h(\zeta)\omega(\zeta)] = \omega(\zeta) D_q h(\zeta) + h(q\zeta) D_q \omega(\zeta).$$

We define the *Jackson q-integration* by

$$\int_0^\zeta h(\gamma)\, d_q\gamma = \zeta(1-q) \sum_{n=0}^{\infty} q^n h(q^n \zeta) \ \ (\zeta \in A),$$

provided that the series converges, and

$$\int_a^b h(\gamma)\, d_q\gamma = \int_0^b h(\gamma)\, d_q\gamma - \int_0^a h(\gamma)\, d_q\gamma,$$

where $a, b \in A$. From [12], we have

$$\int_0^\infty \sigma(\gamma)\, d_q\gamma = \sum_{n=-\infty}^{\infty} q^n \sigma(q^n).$$

Through the remainder of the paper, we deal only with functions q-regular at zero, i.e., functions satisfying

$$\lim_{n\to\infty} h(\zeta q^n) = h(0),$$

for every $\zeta \in A$.

3. Main Results

Let us consider the following impulsive q-Dirac system

$$\begin{cases} -\frac{1}{q} D_{q^{-1}} y_2 + p(\zeta) y_1 = \lambda y_1, \\ D_q y_1 + r(\zeta) y_2 = \lambda y_2, \end{cases} \quad \zeta \in I, \tag{3.1}$$

$$y_2\left(-\frac{1}{q^n}, \lambda\right) \cos\beta + y_1\left(-\frac{1}{q^n}, \lambda\right) \sin\beta = 0, \tag{3.2}$$

$$y_1(d-) - k_1 y_1(d+) = 0, \tag{3.3}$$

$$y_2\left(q^{-1}d-\right) - k_2 y_2\left(q^{-1}d+\right) = 0, \tag{3.4}$$

$$y_2\left(\frac{1}{q^n}, \lambda\right) \cos\gamma + y_1\left(\frac{1}{q^n}, \lambda\right) \sin\gamma = 0, \tag{3.5}$$

where $k_1,k_2,\gamma,\beta\in\mathbb{R},\ n\in\mathbb{N},\ I_1:=[-\frac{1}{q^n},d),\ I_2:=(d,\frac{1}{q^n}],\ -\frac{1}{q^n}<d<\frac{1}{q^n},\ I:=I_1\cup I_2$, and λ is a complex eigenvalue parameter.

Our basic assumptions throughout the paper are the following:

(A_1) Let $q\in(0,1)$ and $k_1k_2=\alpha>0$.

(A_2) $p,\ r:I\to\mathbb{R}$ are continuous functions and have finite limits $p(d\pm),\ r(d\pm)$.

A similar problem has been investigated in [1] without impulsive conditions.

Let $H_1=L_q^2(I_1)\dot{+}L_q^2(I_2)$ be a Hilbert space endowed with the following inner product

$$\langle h,\omega\rangle_{H_1}:=\int_{-\frac{1}{q^n}}^{d}(h,\omega)_{\mathbb{C}^2}\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^n}}(h,\omega)_{\mathbb{C}^2}\,d_q\zeta,$$

where

$$h(\zeta)=\begin{pmatrix}h_1(\zeta,\lambda)\\ h_2(\zeta,\lambda)\end{pmatrix},\ h_1(\zeta,\lambda)=\begin{cases}h_{11}(\zeta,\lambda), & \zeta\in I_1\\ h_{12}(\zeta,\lambda), & \zeta\in I_2,\end{cases}$$

$$h_2(\zeta,\lambda)=\begin{cases}h_{21}(\zeta,\lambda), & \zeta\in I_1\\ h_{22}(\zeta,\lambda), & \zeta\in I_2,\end{cases}$$

and

$$\omega(\zeta)=\begin{pmatrix}\omega_1(\zeta,\lambda)\\ \omega_2(\zeta,\lambda)\end{pmatrix},\ \omega_1(\zeta,\lambda)=\begin{cases}\omega_{11}(\zeta,\lambda), & \zeta\in I_1\\ \omega_{12}(\zeta,\lambda), & \zeta\in I_2,\end{cases}$$

$$\omega_2(\zeta,\lambda)=\begin{cases}\omega_{21}(\zeta,\lambda), & \zeta\in I_1\\ \omega_{22}(\zeta,\lambda), & \zeta\in I_2.\end{cases}$$

We will denote by

$$\phi_1(\zeta,\lambda)=\begin{pmatrix}\phi_{11}(\zeta,\lambda)\\ \phi_{12}(\zeta,\lambda)\end{pmatrix},\ \phi_{11}(\zeta,\lambda)=\begin{cases}\phi_{11}^{(1)}(\zeta), & \zeta\in I_1\\ \phi_{11}^{(2)}(\zeta), & \zeta\in I_2,\end{cases}$$

$$\phi_{12}(\zeta,\lambda)=\begin{cases}\phi_{12}^{(1)}(\zeta), & \zeta\in I_1\\ \phi_{12}^{(2)}(\zeta), & \zeta\in I_2,\end{cases}$$

and

$$\phi_2(\zeta,\lambda)=\begin{pmatrix}\phi_{21}(\zeta,\lambda)\\ \phi_{22}(\zeta,\lambda)\end{pmatrix},\ \phi_{21}(\zeta,\lambda)=\begin{cases}\phi_{21}^{(1)}(\zeta), & \zeta\in I_1\\ \phi_{21}^{(2)}(\zeta), & \zeta\in I_2,\end{cases}$$

$$\phi_{22}(\zeta,\lambda)=\begin{cases}\phi_{22}^{(1)}(\zeta), & \zeta\in I_1\\ \phi_{22}^{(2)}(\zeta), & \zeta\in I_2,\end{cases}$$

the solutions of (3.1) which satisfy conditions

$$\begin{gathered}\phi_{11}^{(1)}\left(-\tfrac{1}{q^n},\lambda\right)=-\cos\gamma,\ \phi_{12}^{(1)}\left(-\tfrac{1}{q^n},\lambda\right)=\sin\gamma,\\ \phi_{21}^{(2)}\left(\tfrac{1}{q^n},\lambda\right)=\cos\alpha,\ \phi_{22}^{(2)}\left(\tfrac{1}{q^n},\lambda\right)=-\sin\alpha,\end{gathered}\tag{3.6}$$

and impulsive conditions (3.3)-(3.4).

Let

$$G(\zeta,t,\lambda)=\begin{cases}\frac{\phi_2(\zeta,\lambda)\phi_1^T(t,\lambda)}{W_q(\phi_1,\phi_2)}, & -\frac{1}{q^n}\le t\le\zeta,\ \zeta\ne d,\ t\ne d\\ \frac{\phi_1(\zeta,\lambda)\phi_2^T(t,\lambda)}{W_q(\phi_1,\phi_2)}, & \zeta<t\le\frac{1}{q^n},\ \zeta\ne d,\ t\ne d,\end{cases}\tag{3.7}$$

where

$$W_q(\phi_1,\phi_2)=\phi_{11}(\xi,\lambda)\,\phi_{22}\left(q^{-1}\xi,\lambda\right)-\phi_{21}(\xi,\lambda)\,\phi_{12}\left(q^{-1}\xi,\lambda\right).$$

Theorem 3.1 *Suppose that λ is not an eigenvalue of (3.1)-(3.5). Then $G(\zeta,t,\lambda)$ defined as (3.7) is a q-Hilbert–Schmidt kernel, i.e.,*

$$\int_{-\frac{1}{q^n}}^{d}\int_{-\frac{1}{q^n}}^{d}\|G(\zeta,t,\lambda)\|_{\mathbb{C}^2}^2\,d_q\zeta d_qt<\infty,\ \ \int_{d}^{\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}}\|G(\zeta,t,\lambda)\|_{\mathbb{C}^2}^2\,d_q\zeta d_qt<\infty.$$

Proof: By (3.7), we find

$$\int_{-\frac{1}{q^n}}^{d}d_q\zeta\int_{-\frac{1}{q^n}}^{d}\|G(\zeta,t,\lambda)\|_{\mathbb{C}^2}^2\,d_qt<\infty,\ \ \int_{d}^{\frac{1}{q^n}}d_q\zeta\int_{d}^{\frac{1}{q^n}}\|G(\zeta,t,\lambda)\|_{\mathbb{C}^2}^2\,d_qt<\infty,$$

since $\phi_1(.,\lambda),\phi_2(.,\lambda)\in H_1$. Hence

$$\int_{-\frac{1}{q^n}}^{d}\int_{-\frac{1}{q^n}}^{d}\|G(\zeta,t,\lambda)\|_{\mathbb{C}^2}^2\,d_q\zeta d_qt<\infty,\ \ \int_{d}^{\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}}\|G(\zeta,t,\lambda)\|_{\mathbb{C}^2}^2\,d_q\zeta d_qt<\infty. \tag{3.8}$$

□

Theorem 3.2 ([24]) *Let $A\{t_i\}=\{z_i\},\ i\in\mathbb{N}:=\{1,2,3,...\}$, where*

$$z_i=\sum_{k=1}^{\infty}\eta_{ik}t_k. \tag{3.9}$$

If

$$\sum_{i,k=1}^{\infty}|\eta_{ik}|^2<\infty, \tag{3.10}$$

then the operator A is compact in l^2.

Theorem 3.3 *Suppose that $\lambda=0$ is not an eigenvalue of (3.1)-(3.5). The operator $\mathbf{K}$ defined as*

$$y(\zeta)=(\mathbf{K}\omega)(\zeta):=\int_{-\frac{1}{q^n}}^{d}G(\zeta,t,0)\,\omega(t)\,d_qt+\alpha\int_{d}^{\frac{1}{q^n}}G(\zeta,t,0)\,\omega(t)\,d_qt$$

is compact and self-adjoint in H_1.

Proof: Let $\omega,\sigma\in\mathcal{H}$ and let $\phi_i=\phi_i(\zeta)$ $(i\in\mathbb{N})$ be a complete, orthonormal basis of H_1. By Theorem 3.1, one can define

$$y_i=\langle\omega,\phi_i\rangle_{H_1}=\int_{-\frac{1}{q^n}}^{d}(\omega(t),\phi_i(t))_{\mathbb{C}^2}\,d_qt+\alpha\int_{d}^{\frac{1}{q^n}}(\omega(t),\phi_i(t))_{\mathbb{C}^2}\,d_qt,$$

$$z_i=\langle\sigma,\phi_i\rangle_{H_1}=\int_{-\frac{1}{q^n}}^{d}(\sigma(t),\phi_i(t))_{\mathbb{C}^2}\,d_qt+\alpha\int_{d}^{\frac{1}{q^n}}(\sigma(t),\phi_i(t))_{\mathbb{C}^2}\,d_qt,$$

$$\eta_{ik}=\int_{-\frac{1}{q^n}}^{\frac{1}{q^n}}\int_{-\frac{1}{q^n}}^{\frac{1}{q^n}}(G(\zeta,t,0)\,\phi_i(\zeta),\phi_k(t))_{\mathbb{C}^2}\,d_q\zeta d_qt,$$

where $i,k\in\mathbb{N}$. Then H_1 is mapped isometrically l^2. $\mathbf{K}$ transforms into A defined as (3.9) in l^2. (3.8) is translated into (3.10). By Theorem 3.2, A is compact. Thus, $\mathbf{K}$ is compact.

Since $G(\zeta,t,0)=G^T(t,\zeta,0)$ and $G(\zeta,t,0)$ is a matrix-valued function in $\mathbb{C}^2$ defined on $I\times I$, we find

$$\langle \mathbf{K}\omega,\sigma\rangle_{H_1}=\int_{-\frac{1}{q^n}}^{d}\left(\left(\mathbf{K}\omega\right)(\zeta),\sigma(\zeta)\right)_{\mathbb{C}^2}d_q\zeta+\alpha\int_{d}^{\frac{1}{q^n}}\left(\left(\mathbf{K}\omega\right)(\zeta),\sigma(\zeta)\right)_{\mathbb{C}^2}d_q\zeta$$

$$=\int_{-\frac{1}{q^n}}^{d}\int_{-\frac{1}{q^n}}^{d}\left(G(\zeta,t,0)\,\omega(t),\sigma(\zeta)\right)_{\mathbb{C}^2}d_qtd_q\zeta$$

$$+\alpha^2\int_{d}^{\frac{1}{q^n}}\int_{d}^{\frac{1}{q^n}}\left(G(\zeta,t,0)\,\omega(t),\sigma(\zeta)\right)_{\mathbb{C}^2}d_qtd_q\zeta$$

$$=\int_{-\frac{1}{q^n}}^{d}\left(\omega(t),\int_{-\frac{1}{q^n}}^{d}G(t,\zeta,0)\,\sigma(\zeta)\right)_{\mathbb{C}^2}d_q\zeta d_qt$$

$$+\alpha^2\int_{d}^{\frac{1}{q^n}}\left(\omega(t),\int_{d}^{\frac{1}{q^n}}G(t,\zeta,0)\,\sigma(\zeta)\right)_{\mathbb{C}^2}d_q\zeta d_qt=\langle\omega,\mathbf{K}\sigma\rangle_{H_1}.$$

□

By Theorem 3.3 and the Hilbert–Schmidt theorem, we conclude that there is an orthonormal system $\{\varphi_n\}$ $(n\in\mathbb{Z})$ of eigenvectors of (3.1)-(3.5) with corresponding nonzero eigenvalues λ_n such that

$$\int_{-\frac{1}{q^n}}^{d}\|\omega(\zeta)\|_{\mathbb{C}^2}^2\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^n}}\|\omega(\zeta)\|_{\mathbb{C}^2}^2\,d_q\zeta=\sum_{n=-\infty}^{\infty}|a_n|^2\,, \tag{3.11}$$

which is called the Parseval equality, where $\omega\in H_1$, $a_n=\langle\omega,\varphi_n\rangle_{H_1}$ and $n\in\mathbb{Z}$.

Denote by

$$\psi_1(\zeta,\lambda)=\begin{pmatrix}\psi_{11}(\zeta,\lambda)\\ \psi_{12}(\zeta,\lambda)\end{pmatrix},\ \psi_{11}(\zeta,\lambda)=\begin{cases}\psi_{11}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{11}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$
$$\psi_{12}(\zeta,\lambda)=\begin{cases}\psi_{12}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{12}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

(where $I_3:=(-\infty,d)$, $I_4:=(d,\infty)$) and

$$\psi_2(\zeta,\lambda)=\begin{pmatrix}\psi_{21}(\zeta,\lambda)\\ \psi_{22}(\zeta,\lambda)\end{pmatrix},\ \psi_{21}(\zeta,\lambda)=\begin{cases}\psi_{21}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{21}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$
$$\psi_{22}(\zeta,\lambda)=\begin{cases}\psi_{22}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{22}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

the solutions of (3.1) $(\zeta\in I_3\cup I_4)$ which satisfy conditions

$$\begin{array}{l}\psi_{11}^{(1)}(c,\lambda)=1,\ \psi_{12}^{(1)}(c,\lambda)=0,\\ \psi_{21}^{(1)}(c,\lambda)=0,\ \psi_{22}^{(1)}(c,\lambda)=1,\ -\frac{1}{q^n}<c<d.\end{array} \tag{3.12}$$

and impulsive conditions (3.3)-(3.4).

Let λ_s $(s \in \mathbb{Z})$ be the eigenvalues and y_s $(s \in \mathbb{Z})$ be the corresponding eigenfunctions of the self-adjoint problem (3.1)-(3.5), where

$$y_s(\zeta) = \begin{pmatrix} y_{s1}(\zeta) \\ y_{s2}(\zeta) \end{pmatrix}, \; y_{s1}(\zeta) = \begin{cases} y_{s1}^{(1)}(\zeta), & \zeta \in I_1 \\ y_{s1}^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$
$$y_{s2}(\zeta) = \begin{cases} y_{s2}^{(1)}(\zeta), & \zeta \in I_1 \\ y_{s2}^{(2)}(\zeta), & \zeta \in I_2. \end{cases}$$

Since the solutions $\psi_1(\zeta,\lambda)$ and $\psi_2(\zeta,\lambda)$ of the system (3.1) are linearly independent, we find

$$y_s(\zeta) = u_s \psi_1(\zeta,\lambda_s) + v_s \psi_2(\zeta,\lambda_s),$$

where $s \in \mathbb{Z}$. Without loss of generality, we can assume that $|u_s| \le 1$ and $|v_s| \le 1$ $(s \in \mathbb{Z})$. Write

$$\alpha_s^2 = \int_{-\frac{1}{q^n}}^{d} \|y_s(\zeta)\|_{\mathbb{C}^2}^2 \, d_q\zeta + \alpha \int_d^{\frac{1}{q^n}} \|y_s(\zeta)\|_{\mathbb{C}^2}^2 \, d_q\zeta \; (s \in \mathbb{Z}).$$

Let

$$\omega(.) = \begin{pmatrix} \omega_1(.) \\ \omega_2(.) \end{pmatrix} \in H_1,$$

is a real vector-valued function, where

$$\omega_1(\zeta) = \begin{cases} \omega_1^{(1)}(\zeta), & \zeta \in I_1 \\ \omega_1^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \text{ and } \omega_2(\zeta) = \begin{cases} \omega_2^{(1)}(\zeta), & \zeta \in I_1 \\ \omega_2^{(2)}(\zeta), & \zeta \in I_2. \end{cases}$$

From (3.11), we find

$$\begin{aligned}
&\int_{-\frac{1}{q^n}}^{d} \|\omega(\zeta)\|_{\mathbb{C}^2}^2 \, d_q\zeta + \alpha \int_d^{\frac{1}{q^n}} \|\omega(\zeta)\|_{\mathbb{C}^2}^2 \, d_q\zeta \\
&= \sum_{s=-\infty}^{\infty} \frac{1}{\alpha_s^2} \left\{ \int_{-\frac{1}{q^n}}^{d} (\omega(\zeta), y_s(\zeta))_{\mathbb{C}^2} \, d_q\zeta + \alpha \int_d^{\frac{1}{q^n}} (\omega(\zeta), y_s(\zeta))_{\mathbb{C}^2} \, d_q\zeta \right\}^2 \\
&= \sum_{s=-\infty}^{\infty} \frac{1}{\alpha_s^2} \left\{ \begin{array}{c} \int_{-\frac{1}{q^n}}^{d} (\omega(\zeta), u_s\psi_1(\zeta,\lambda_s) + v_s\psi_2(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \\ +\alpha \int_d^{\frac{1}{q^n}} (\omega(\zeta), u_s\psi_1(\zeta,\lambda_s) + v_s\psi_2(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \end{array} \right\}^2 \\
&= \sum_{s=-\infty}^{\infty} \frac{u_s^2}{\alpha_s^2} \left\{ \begin{array}{c} \int_{-\frac{1}{q^n}}^{d} (\omega(\zeta), \psi_1(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \\ +\alpha \int_d^{\frac{1}{q^n}} (\omega(\zeta), \psi_1(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \end{array} \right\}^2 \\
&+2 \sum_{s=-\infty}^{\infty} \frac{u_s v_s}{\alpha_s^2} \left\{ \begin{array}{c} \int_{-\frac{1}{q^n}}^{d} (\omega(\zeta), \psi_1(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \\ +\alpha \int_d^{\frac{1}{q^n}} (\omega(\zeta), \psi_1(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \end{array} \right\} \\
&\times \left\{ \begin{array}{c} \int_{-\frac{1}{q^n}}^{d} (\omega(\zeta), \psi_2(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \\ +\alpha \int_d^{\frac{1}{q^n}} (\omega(\zeta), \psi_2(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \end{array} \right\} \\
&+ \sum_{s=-\infty}^{\infty} \frac{v_s^2}{\alpha_s^2} \left\{ \begin{array}{c} \int_{-\frac{1}{q^n}}^{d} (\omega(\zeta), \psi_2(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \\ +\alpha \int_d^{\frac{1}{q^n}} (\omega(\zeta), \psi_2(\zeta,\lambda_s))_{\mathbb{C}^2} \, d_q\zeta \end{array} \right\}^2 .
\end{aligned} \tag{3.13}$$

The step function $\mu_{ij,[-\frac{1}{q^n},\frac{1}{q^n}]}$ $(i,j=1,2)$ on $\mathbb{R}$ is defined by

$$\mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda)=\begin{cases}-\sum\limits_{\lambda<\lambda_s<0}\frac{u_s^2}{\alpha_s^2}, & \text{for } \lambda\leq 0\\ \sum\limits_{0\leq\lambda_s<\lambda}\frac{u_s^2}{\alpha_s^2} & \text{for } \lambda>0,\end{cases}$$

$$\mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda)=\begin{cases}-\sum\limits_{\lambda<\lambda_s<0}\frac{u_s v_s}{\alpha_s^2}, & \text{for } \lambda\leq 0\\ \sum\limits_{0\leq\lambda_s<\lambda}\frac{u_s v_s}{\alpha_s^2} & \text{for } \lambda>0,\end{cases}$$

$$\mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda)=\mu_{21,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda),$$

$$\mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda)=\begin{cases}-\sum\limits_{\lambda<\lambda_s<0}\frac{u_s^2}{\alpha_s^2}, & \text{for } \lambda\leq 0\\ \sum\limits_{0\leq\lambda_s<\lambda}\frac{u_s^2}{\alpha_s^2}, & \text{for } \lambda>0.\end{cases}$$

It follows from (3.13) that

$$\int_{-\frac{1}{q^n}}^{d}\|\omega(\zeta)\|_{\mathbb{C}^2}^2\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^n}}\|\omega(\zeta)\|_{\mathbb{C}^2}^2\,d_q\zeta$$
$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\Omega_j(\lambda)\,d\mu_{ij,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda), \tag{3.14}$$

where

$$\Omega_1(\lambda)=\int_{-\frac{1}{q^n}}^{d}(\omega(\zeta),\psi_1(\zeta,\lambda))_{\mathbb{C}^2}\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^n}}(\omega(\zeta),\psi_1(\zeta,\lambda))_{\mathbb{C}^2}\,d_q\zeta,$$

and

$$\Omega_2(\lambda)=\int_{-\frac{1}{q^n}}^{d}(\omega(\zeta),\psi_2(\zeta,\lambda))_{\mathbb{C}^2}\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^n}}(\omega(\zeta),\psi_2(\zeta,\lambda))_{\mathbb{C}^2}\,d_q\zeta.$$

Lemma 3.1 *For any positive ξ, there is a positive constant $\Lambda=\Lambda(\xi)$ not depending on $\frac{1}{q^n}$ such that*

$$\bigvee_{-\xi}^{\xi}\left\{\mu_{ij,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda)\right\}<\Lambda\ \ (i,j=1,2). \tag{3.15}$$

Proof: By (3.12), we find $\psi_{ij}^{(1)}(d_0,\lambda)=\delta_{ij}$, where δ_{ij} $(i,j=1,2)$ is the Kronecker delta. It is clear that $\psi_{ij}^{(1)}(\zeta,\lambda)$ $(i,j=1,2)$ are continuous both with respect to $\zeta\in[-\frac{1}{q^n},d)$ and $\lambda\in\mathbb{R}$. Then for every $\varepsilon>0$ there is a $d_0<k<d$ such that

$$\left|\psi_{ij}^{(1)}(\zeta,\lambda)-\delta_{ij}\right|<\varepsilon,\ \ |\lambda|<\xi,\ \text{ where } \zeta\in[d_0,k]. \tag{3.16}$$

Let

$$\omega_k(\zeta)=\begin{pmatrix}\omega_{k1}(\zeta)\\ \omega_{k2}(\zeta)\end{pmatrix}$$

be a nonnegative vector-valued function such that $\omega_{k1}(\zeta)$ vanishes outside the interval $[d_0,k]$ with

$$\int_{d_0}^{k}\omega_{k1}(\zeta)\,d_q\zeta=1, \tag{3.17}$$

and $\omega_{k2}(\zeta) = 0$. Write

$$\Omega_{ik}(\lambda) = \int_{d_0}^{k} (\omega_k(\zeta), \psi_i)_{\mathbb{C}^2} \, d_q\zeta = \int_{d_0}^{k} \omega_{k1}(\zeta) \psi_{i1}^{(1)}(\zeta,\lambda) \, d_q\zeta,$$

where $i = 1, 2$. From (3.16) and (3.17), we see that

$$|\Omega_{1k}(\lambda) - 1| < \varepsilon, \ |\Omega_{2k}(\lambda)| < \varepsilon, \text{ and } |\lambda| < \xi. \tag{3.18}$$

By (3.14), we find

$$\begin{aligned} \int_{d_0}^{k} \omega_{k1}^2(\zeta) \, d_q\zeta \geq \int_{-\xi}^{\xi} \Omega_{1k}^2(\lambda) \, d\mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \\ +2\int_{-\xi}^{\xi} \Omega_{1k}(\lambda) \Omega_{2k}(\lambda) \, d\mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \\ + \int_{-\xi}^{\xi} \Omega_{2k}^2(\lambda) \, d\mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \geq \int_{-\xi}^{\xi} \Omega_{1k}^2(\lambda) \, d\mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \\ -2\int_{-\xi}^{\xi} |\Omega_{1k}(\lambda)| \, |\Omega_{2k}(\lambda)| \left| d\mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \right|. \end{aligned}$$

It follows from (3.18) that

$$\begin{aligned} \int_{d_0}^{k} \omega_{k1}^2(\zeta) \, d_q\zeta > \int_{-\xi}^{\xi} (1-\varepsilon)^2 \, d\mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \\ -2\int_{-\xi}^{\xi} \varepsilon(1+\varepsilon) \left| d\mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \right| \\ = (1-\varepsilon)^2 \left(\mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \right) \\ -2\varepsilon(1+\varepsilon) \overset{\xi}{\underset{-\xi}{V}} \left\{ \mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \right\}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \int_{d_0}^{k} \omega_{k1}^2(\zeta) \, d_q\zeta > (1-3\varepsilon) \left\{ \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \right\} \\ - \varepsilon(1+\varepsilon) \left\{ \mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \right\} \end{aligned} \tag{3.19}$$

due to

$$\begin{aligned} \bigvee_{-\xi}^{\xi} \left\{ \mu_{12,[-\frac{1}{q^n},\frac{1}{q^n}]}(\lambda) \right\} \\ \leq \frac{1}{2} \left[\begin{array}{c} \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \\ +\mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \end{array} \right]. \end{aligned} \tag{3.20}$$

Let

$$\sigma_k(\zeta) = \begin{pmatrix} \sigma_{k1}(\zeta) \\ \sigma_{k2}(\zeta) \end{pmatrix}$$

be a nonnegative vector-valued function such that $\sigma_{k2}(\zeta)$ vanishes outside the interval $[d_0, k]$ with $\int_{d_0}^{k} \sigma_{k2}(\zeta) \, d_q\zeta = 1$, and $\sigma_{k1}(\zeta) = 0$. Similarly, we get

$$\begin{aligned} \int_{d_0}^{k} \sigma_{k2}^2(\zeta) \, d_q\zeta > (1-3\varepsilon) \left\{ \mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{22,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \right\} \\ - \varepsilon(1+\varepsilon) \left\{ \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(\xi) - \mu_{11,[-\frac{1}{q^n},\frac{1}{q^n}]}(-\xi) \right\}. \end{aligned} \tag{3.21}$$

By virtue of (3.19) and (3.21), we conclude that

$$\int_{d_0}^{k}\left\{\omega_{k1}^2(\zeta)+\sigma_{k2}^2(\zeta)\right\}d_q\zeta>\left(1-4\varepsilon-\varepsilon^2\right)$$

$$\times\left\{\begin{array}{c}\mu_{11,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(\xi)-\mu_{11,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(-\xi)\\ +\mu_{22,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(\xi)-\mu_{22,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(-\xi)\end{array}\right\}.$$

If the number $\varepsilon>0$ is selected such that $1-4\varepsilon-\varepsilon^2>0$, then the statement follows for the functions $\mu_{11,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(-\xi)$ and $\mu_{22,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(-\xi)$, relying on their monotonicity. For the function $\mu_{12,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(-\xi)$, it follows from the Cauchy–Schwarz inequality. □

Now let's define the following spaces.

$H:=L^2\left(I_3;\mathbb{C}^2\right)\dot{+}L^2\left(I_4;\mathbb{C}^2\right)$, be a Hilbert space endowed with the following inner product

$$\langle\omega,\sigma\rangle_H:=\int_{-\infty}^{d}(\omega(\zeta),\sigma(\zeta))_{\mathbb{C}^2}\,d_q\zeta+\alpha\int_{d}^{\infty}(\omega(\zeta),\sigma(\zeta))_{\mathbb{C}^2}\,d_q\zeta,$$

where $I_3=(-\infty,d),\ I_4=(d,\infty)$,

$$\omega(\zeta)=\begin{pmatrix}\omega_1(\zeta)\\ \omega_2(\zeta)\end{pmatrix},\ \sigma(\zeta)=\begin{pmatrix}\sigma_1(\zeta)\\ \sigma_2(\zeta)\end{pmatrix},$$

$$\omega_1(\zeta)=\begin{cases}\omega_1^{(1)}(\zeta), & \zeta\in I_3\\ \omega_1^{(2)}(\zeta), & \zeta\in I_4,\end{cases}\quad \omega_2(\zeta)=\begin{cases}\omega_2^{(1)}(\zeta), & \zeta\in I_3\\ \omega_2^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

$$\sigma_1(\zeta)=\begin{cases}\sigma_1^{(1)}(\zeta), & \zeta\in I_3\\ \sigma_1^{(2)}(\zeta), & \zeta\in I_4,\end{cases}\quad \sigma_2(\zeta)=\begin{cases}\sigma_2^{(1)}(\zeta), & \zeta\in I_3\\ \sigma_2^{(2)}(\zeta), & \zeta\in I_4.\end{cases}$$

Let ϱ be any non-decreasing function on $-\infty<\lambda<\infty$. $L^2_\varrho(\mathbb{R})$ be a Hilbert space of all functions $\phi:\mathbb{R}\to\mathbb{R}$ which are measurable with respect to the Lebesque–Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty}\phi^2(\lambda)\,d\varrho(\lambda)<\infty,$$

with the inner product

$$(\phi,\sigma)_\varrho:=\int_{-\infty}^{\infty}\phi(\lambda)\,\sigma(\lambda)\,d\varrho(\lambda).$$

Theorem 3.4 *Let ω is a real vector-valued function and $\omega\in H$. There are two monotone functions $\mu_{11}(\lambda)$ and $\mu_{22}(\lambda)$, and a function $\mu_{12}(\lambda)$ with variation bounded in each finite interval, none of which depends ω, and such that the following Parseval equality holds*

$$\int_{-\infty}^{d}\|\omega(\zeta)\|^2_{\mathbb{C}^2}\,d_q\zeta+\alpha\int_{d}^{\infty}\|\omega(\zeta)\|^2_{\mathbb{C}^2}\,d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\Omega_j(\lambda)\,d\mu_{ij}(\lambda), \tag{3.22}$$

where

$$\Omega_i(\lambda)=\lim_{n\to\infty}\left\{\begin{array}{c}\int_{-\frac{1}{q^n}}^{d}(\omega(\zeta),\psi_i(\zeta,\lambda))_{\mathbb{C}^2}\,d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^n}}(\omega(\zeta),\psi_i(\zeta,\lambda))_{\mathbb{C}^2}\,d_q\zeta\end{array}\right\}\ (i=1,2).$$

We note that the matrix-valued function $\mu = (\mu_{ij})_{i,j=1}^{2}$ $(\mu_{12} = \mu_{21})$ is called a *spectral function* for the system (3.1), (3.3), (3.4).

Proof: Let

$$\omega_m(\zeta) = \begin{pmatrix} \omega_{1m}(\zeta) \\ \omega_{2m}(\zeta) \end{pmatrix} \ (m \in \mathbb{Z}),$$

where

$$\omega_{1m}(\zeta) = \begin{cases} \omega_{1m}^{(1)}(\zeta), & \zeta \in I_3 \\ \omega_{1m}^{(2)}(\zeta), & \zeta \in I_4, \end{cases} \quad \omega_{2m}(\zeta) = \begin{cases} \omega_{2m}^{(1)}(\zeta), & \zeta \in I_3 \\ \omega_{2m}^{(2)}(\zeta), & \zeta \in I_4, \end{cases}$$

satisfies the following conditions:

1) $\omega_m(\zeta)$ vanishes outside the interval $\left[-\frac{1}{q^m}, d) \cup (d, \frac{1}{q^m}\right]$, where $-\frac{1}{q^n} < -\frac{1}{q^m} < d < \frac{1}{q^m} < \frac{1}{q^n}$.

2) The real vector-valued functions $\omega_m(\zeta)$ and $D_q\omega_m(\zeta)$ are q-regular at zero.

3) $\omega_m(\zeta)$ satisfies conditions (3.2)-(3.5).

From (3.11), we find

$$\begin{aligned} &\int_{-\frac{1}{q^m}}^{d} \|\omega_m(\zeta)\|_{\mathbb{C}^2}^2 d_q\zeta + \alpha \int_{d}^{\frac{1}{q^m}} \|\omega_m(\zeta)\|_{\mathbb{C}^2}^2 d_q\zeta \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{\alpha_k^2} \left\{ \begin{array}{c} \int_{-\frac{1}{q^n}}^{d} (\omega_m(\zeta), y_k(\zeta))_{\mathbb{C}^2} d_q\zeta \\ +\alpha \int_{d}^{\frac{1}{q^n}} (\omega_m(\zeta), y_k(\zeta))_{\mathbb{C}^2} d_q\zeta \end{array} \right\}^2. \end{aligned} \tag{3.23}$$

q-integrating by parts gives

$$\begin{aligned} &\int_{-\frac{1}{q^n}}^{d} (\omega_m(\zeta), y_k(\zeta))_{\mathbb{C}^2} d_q\zeta + \alpha \int_{d}^{\frac{1}{q^n}} (\omega_m(\zeta), y_k(\zeta))_{\mathbb{C}^2} d_q\zeta \\ &= \frac{1}{\lambda_k} \int_{-\frac{1}{q^n}}^{d} \omega_{1m}^{(1)}(\zeta) \left[-\frac{1}{q} D_{q^{-1}} y_{k2}^{(1)}(\zeta) + p(\zeta) y_{k1}^{(1)}(\zeta)\right] d_q\zeta \\ &+ \frac{1}{\lambda_k}\alpha \int_{d}^{\frac{1}{q^n}} \omega_{1m}^{(2)}(\zeta) \left[-\frac{1}{q} D_{q^{-1}} y_{k2}^{(2)}(\zeta) + p(\zeta) y_{k1}^{(2)}(\zeta)\right] d_q\zeta \\ &+ \frac{1}{\lambda_k} \int_{-\frac{1}{q^n}}^{d} \omega_{2m}^{(1)}(\zeta) \left[D_q y_{k1}^{(1)}(\zeta) + r(\zeta) y_{k2}^{(1)}(\zeta)\right] d_q\zeta \\ &+ \frac{1}{\lambda_k}\alpha \int_{d}^{\frac{1}{q^n}} \omega_{2m}^{(2)}(\zeta) \left[D_q y_{k1}^{(2)}(\zeta) + r(\zeta) y_{k2}^{(2)}(\zeta)\right] d_q\zeta \\ &= \frac{1}{\lambda_k} \int_{-\frac{1}{q^n}}^{d} \left[-\frac{1}{q} D_{q^{-1}} \omega_{2m}^{(1)}(\zeta) + p(\zeta) \omega_{1m}^{(1)}(\zeta)\right] y_{k1}^{(1)}(\zeta) d_q\zeta \\ &+ \frac{1}{\lambda_k}\alpha \int_{d}^{\frac{1}{q^n}} \left[-\frac{1}{q} D_{q^{-1}} \omega_{2m}^{(2)}(\zeta) + p(\zeta) \omega_{1m}^{(2)}(\zeta)\right] y_{k1}^{(2)}(\zeta) d_q\zeta \\ &+ \frac{1}{\lambda_k} \int_{-\frac{1}{q^n}}^{d} \left[D_q \omega_{1m}^{(1)}(\zeta) + r(\zeta) \omega_{2m}^{(1)}(\zeta)\right] y_{k2}^{(1)}(\zeta) d_q\zeta \\ &+ \frac{1}{\lambda_k}\alpha \int_{d}^{\frac{1}{q^n}} \left[D_q \omega_{1m}^{(2)}(\zeta) + r(\zeta) \omega_{2m}^{(2)}(\zeta)\right] y_{k2}^{(2)}(\zeta) d_q\zeta. \end{aligned}$$

By condition 1, we find

$$\sum_{|\lambda_k| \geq s} \frac{1}{\alpha_k^2} \left\{ \begin{array}{c} \int_{-\frac{1}{q^m}}^{d} (\omega_m(\zeta), y_k(\zeta))_{\mathbb{C}^2} d_q\zeta \\ +\alpha \int_{d}^{\frac{1}{q^m}} (\omega_m(\zeta), y_k(\zeta))_{\mathbb{C}^2} d_q\zeta \end{array} \right\}^2$$

$$\leq \frac{1}{s^{2}}\sum_{|\lambda_k|\geq s}\frac{1}{\alpha_k^{2}}\left\{\begin{array}{c}\int_{-\frac{1}{q^m}}^{d}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(1)}(\zeta)+p(\zeta)\omega_{1m}^{(1)}(\zeta)\right]y_{k1}^{(1)}(\zeta)\,d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^m}}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(2)}(\zeta)+p(\zeta)\omega_{1m}^{(2)}(\zeta)\right]y_{k1}^{(2)}(\zeta)\,d_q\zeta\\ +\int_{-\frac{1}{q^m}}^{d}\left[D_q\omega_{1m}^{(1)}(\zeta)+r(\zeta)\omega_{2m}^{(1)}(\zeta)\right]y_{k2}^{(1)}(\zeta)\,d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^m}}\left[D_q\omega_{1m}^{(2)}(\zeta)+r(\zeta)\omega_{2m}^{(2)}(\zeta)\right]y_{k2}^{(2)}(\zeta)\,d_q\zeta\end{array}\right\}^{2}$$

$$\leq \frac{1}{s^{2}}\sum_{k=-\infty}^{\infty}\frac{1}{\alpha_k^{2}}\left\{\begin{array}{c}\int_{-\frac{1}{q^m}}^{d}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(1)}(\zeta)+p(\zeta)\omega_{1m}^{(1)}(\zeta)\right]y_{k1}^{(1)}(\zeta)\,d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^m}}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(2)}(\zeta)+p(\zeta)\omega_{1m}^{(2)}(\zeta)\right]y_{k1}^{(2)}(\zeta)\,d_q\zeta\\ +\int_{-\frac{1}{q^m}}^{d}\left[D_q\omega_{1m}^{(1)}(\zeta)+r(\zeta)\omega_{2m}^{(1)}(\zeta)\right]y_{k2}^{(1)}(\zeta)\,d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^m}}\left[D_q\omega_{1m}^{(2)}(\zeta)+r(\zeta)\omega_{2m}^{(2)}(\zeta)\right]y_{k2}^{(2)}(\zeta)\,d_q\zeta\end{array}\right\}^{2}$$

$$=\frac{1}{s^{2}}\left\{\begin{array}{c}\int_{-\frac{1}{q^m}}^{d}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(1)}(\zeta)+p(\zeta)\omega_{1m}^{(1)}(\zeta)\right]^{2}d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^m}}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(2)}(\zeta)+p(\zeta)\omega_{1m}^{(2)}(\zeta)\right]^{2}d_q\zeta\\ +\int_{-\frac{1}{q^m}}^{d}\left[D_q\omega_{1m}^{(1)}(\zeta)+r(\zeta)\omega_{2m}^{(1)}(\zeta)\right]^{2}d_q\zeta\\ +\alpha\int_{d}^{\frac{1}{q^m}}\left[D_q\omega_{1m}^{(2)}(\zeta)+r(\zeta)\omega_{2m}^{(2)}(\zeta)\right]^{2}d_q\zeta\end{array}\right\}.$$

From (3.23), we conclude that

$$\left|\begin{array}{c}\int_{-\frac{1}{q^m}}^{d}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^{2}\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^m}}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^{2}\,d_q\zeta\\ -\sum_{-N\leq\lambda_k\leq N}\frac{1}{\alpha_k^{2}}\left\{\langle\omega_m(.),y_k(.)\rangle_H\right\}^{2}\end{array}\right|$$

$$\leq\sum_{-N\leq\lambda_k\leq N}\frac{1}{\alpha_k^{2}}\left\{\langle\omega_m(.),(u_k\psi_1(.,\lambda_k)+v_k\psi_2(.,\lambda_k))\rangle_H\right\}^{2}$$

$$=\int_{-N}^{N}\sum_{i,j=1}^{2}\Omega_{in}(\lambda)\,\Omega_{in}(\lambda)\,d\mu_{ij,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(\lambda),$$

where

$$\Omega_{im}(\lambda)=\langle\omega_m(.),\psi_i(.,\lambda)\rangle_H\ \ (i=1,2).$$

Thus, we find

$$\begin{aligned}&\left|\begin{array}{c}\int_{-\frac{1}{q^m}}^{d}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^{2}\,d_q\zeta+\alpha\int_{d}^{\frac{1}{q^m}}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^{2}\,d_q\zeta\\ -\int_{-N}^{N}\sum_{i,j=1}^{2}\Omega_{im}(\lambda)\,\Omega_{im}(\lambda)\,d\mu_{ij,\left[-\frac{1}{q^n},\frac{1}{q^n}\right]}(\lambda)\end{array}\right|\\ &\leq\frac{1}{N^2}\int_{-\frac{1}{q^m}}^{d}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(1)}(\zeta)+p(\zeta)\omega_{1m}^{(1)}(\zeta)\right]^{2}d_q\zeta\\ &+\alpha\frac{1}{N^2}\int_{d}^{\frac{1}{q^m}}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(2)}(\zeta)+p(\zeta)\omega_{1m}^{(2)}(\zeta)\right]^{2}d_q\zeta\\ &+\frac{1}{N^2}\int_{-\frac{1}{q^m}}^{d}\left[D_q\omega_{1m}^{(1)}(\zeta)+r(\zeta)\omega_{2m}^{(1)}(\zeta)\right]^{2}d_q\zeta\\ &+\frac{1}{N^2}\alpha\int_{d}^{\frac{1}{q^m}}\left[D_q\omega_{1m}^{(2)}(\zeta)+r(\zeta)\omega_{2m}^{(2)}(\zeta)\right]^{2}d_q\zeta.\end{aligned}\tag{3.24}$$

From Helly's theorems and Lemma 3.1, we can find sequences $\left\{-\frac{1}{q^{n_s}}\right\}$ and $\left\{\frac{1}{q^{n_s}}\right\}$ such that the functions $\mu_{ij,[-\frac{1}{q^{n_s}},\frac{1}{q^{n_s}}]}(\lambda)$ converge ($n_s \to \infty$) to a function $\mu_{ij}(\lambda)$ $(i,j=1,2)$. By (3.24), we conclude that

$$\left| \begin{array}{c} \int_{-\frac{1}{q^m}}^{d} \left\|\omega_m(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta + \alpha \int_d^{\frac{1}{q^m}} \left\|\omega_m(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta \\ -\int_{-N}^{N} \sum_{i,j=1}^{2} \Omega_{im}(\lambda)\,\Omega_{jm}(\lambda)\, d\mu_{ij}(\lambda) \end{array} \right|$$

$$\leq \frac{1}{N^2}\int_{-\frac{1}{q^m}}^{d}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(1)}(\zeta)+p(\zeta)\,\omega_{1m}^{(1)}(\zeta)\right]^2 d_q\zeta$$

$$+\alpha\frac{1}{N^2}\int_d^{\frac{1}{q^m}}\left[-\frac{1}{q}D_{q^{-1}}\omega_{2m}^{(2)}(\zeta)+p(\zeta)\,\omega_{1m}^{(2)}(\zeta)\right]^2 d_q\zeta$$

$$+\frac{1}{N^2}\int_{-\frac{1}{q^m}}^{d}\left[D_q\omega_{1m}^{(1)}(\zeta)+r(\zeta)\,\omega_{2m}^{(1)}(\zeta)\right]^2 d_q\zeta$$

$$+\frac{1}{N^2}\alpha\int_d^{\frac{1}{q^m}}\left[D_q\omega_{1m}^{(2)}(\zeta)+r(\zeta)\,\omega_{2m}^{(2)}(\zeta)\right]^2 d_q\zeta.$$

As $N\to\infty$, we obtain

$$\int_{-\frac{1}{q^m}}^{d}\left\|\omega_m(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_d^{\frac{1}{q^m}}\left\|\omega_m(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_{im}(\lambda)\,\Omega_{jm}(\lambda)\,d\mu_{ij}(\lambda).$$

Now let

$$\omega(\zeta)=\begin{cases}\omega^{(1)}(\zeta), & \zeta\in I_3\\ \omega^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

is a real vector-value function and $\omega\in H$. Choose vector-valued functions

$$\omega_\eta(\zeta)=\begin{cases}\omega_\eta^{(1)}(\zeta), & \zeta\in I_3\\ \omega_\eta^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

satisfying conditions 1-3 and such that

$$\lim_{\eta\to\infty}\int_{-\infty}^{d}\left\|\omega^{(1)}(\zeta)-\omega_\eta^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta$$

$$+\alpha\lim_{\eta\to\infty}\int_d^{\infty}\left\|\omega^{(2)}(\zeta)-\omega_\eta^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta=0.$$

Let

$$\Omega_{i\eta}(\lambda)=\int_{-\infty}^{d}\left(\omega_\eta^{(1)}(\zeta),\psi_i(\zeta,\lambda)\right)_{\mathbb{C}^2} d_q\zeta$$

$$+\alpha\int_d^{\infty}\left(\omega_\eta^{(2)}(\zeta),\psi_i(\zeta,\lambda)\right)_{\mathbb{C}^2} d_q\zeta\ \ (i=1,2).$$

Then, we obtain

$$\int_{-\infty}^{d}\left\|\omega_\eta^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_d^{\infty}\left\|\omega_\eta^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_{i\eta}(\lambda)\,\Omega_{j\eta}(\lambda)\,d\mu_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{d}\left\|\omega_{\eta_1}^{(1)}(\zeta)-\omega_{\eta_2}^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_{d}^{\infty}\left\|\omega_{\eta_1}^{(2)}(\zeta)-\omega_{\eta_2}^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta\to 0$$

as $\eta_1,\eta_2\to\infty$, we get

$$\int_{-\infty}^{\infty}\sum_{i=1}^{2}\left(\Omega_{i\eta_1}(\lambda)\,\Omega_{j\eta_1}(\lambda)-\Omega_{i\eta_2}(\lambda)\,\Omega_{j\eta_2}(\lambda)\right)d\mu_{ij}(\lambda)$$

$$=\int_{-\infty}^{d}\left\|\omega_{\eta_1}^{(1)}(\zeta)-\omega_{\eta_2}^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_{d}^{\infty}\left\|\omega_{\eta_1}^{(2)}(\zeta)-\omega_{\eta_2}^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta\to 0$$

as $\eta_1,\eta_2\to\infty$. Therefore, there is a limit function $\Omega_i\ (i=1,2)$ which satisfies

$$\int_{-\infty}^{d}\left\|\omega^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_{d}^{\infty}\left\|\omega^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\Omega_j(\lambda)\,d\mu_{ij}(\lambda),$$

by the completeness of the space $L_{\mu}^2(\mathbb{R})$.

Now, we shall prove that the sequence

$$K_{\eta i}(\lambda)=\int_{-\eta}^{d}(\omega^{(1)}(\zeta),\psi_i(\zeta,\lambda))_{\mathbb{C}^2}d_q\zeta$$

$$+\int_{d}^{\eta}(\omega^{(2)}(\zeta),\psi_i(\zeta,\lambda))_{\mathbb{C}^2}d_q\zeta\ \ (i=1,2)$$

converges as $\eta\to\infty$ to $\omega_i\ (i=1,2)$ in $L_{\mu}^2(\mathbb{R})$. Let σ be another real vector-valued function in H. $\Sigma(\lambda)$ can be defined by σ. It is immediate that

$$\int_{-\infty}^{d}\left\|\omega^{(1)}(\zeta)-\sigma^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_{d}^{\infty}\left\|\omega^{(2)}(\zeta)-\sigma^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{\left(\Omega_i(\lambda)-\Sigma_i(\lambda)\right)\left(\Omega_j(\lambda)-\Sigma_j(\lambda)\right)\right\}d\mu_{ij}(\lambda).$$

Let

$$\sigma(\zeta)=\begin{cases}\omega(\zeta), & \zeta\in\left[-\frac{1}{q^{\eta}},d)\cup(d,\frac{1}{q^{\eta}}\right]\\ 0, & \text{otherwise.}\end{cases}$$

Then, we have

$$\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{\left(\Omega_i(\lambda)-K_{\eta i}(\lambda)\right)\left(\Omega_j(\lambda)-K_{\eta j}(\lambda)\right)\right\}d\mu_{ij}(\lambda)$$

$$=\int_{-\infty}^{-\frac{1}{q^{\eta}}}\left\|\omega^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta+\alpha\int_{\frac{1}{q^{\eta}}}^{\infty}\left\|\omega^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^2 d_q\zeta\to 0\ \ (\eta\to\infty),$$

which proves that $(K_{\eta i})$ converges to $\Omega_i\ (i=1,2)$ in $L_{\mu}^2(\mathbb{R})$ as $\eta\to\infty$. □

Theorem 3.5 *Suppose that the real vector-valued functions*

$$\omega(\zeta)=\left\{\begin{array}{ll}\omega^{(1)}(\zeta), & \zeta\in I_3\\ \omega^{(2)}(\zeta), & \zeta\in I_4,\end{array}\right. \quad \sigma(\zeta)=\left\{\begin{array}{ll}\sigma^{(1)}(\zeta), & \zeta\in I_3\\ \sigma^{(2)}(\zeta), & \zeta\in I_4,\end{array}\right.$$

$\omega,\sigma\in H$, and $\Omega_i(\lambda)$, $\Sigma_i(\lambda)$ $(i=1,2)$ are their Fourier transforms. Then, the following generalized Parseval equality holds

$$\int_{-\infty}^{d}\left(\omega^{(1)}(\zeta),\sigma^{(1)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta+\alpha\int_{d}^{\infty}\left(\omega^{(2)}(\zeta),\sigma^{(2)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Sigma_j(\lambda)\,d\mu_{ij}(\lambda).$$

Proof: Since $\Omega\mp\Sigma$ are transforms of $\omega\mp\sigma$, we find

$$\int_{-\infty}^{d}\left\|\omega^{(1)}(\zeta)+\sigma^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^{2}d_q\zeta+\alpha\int_{d}^{\infty}\left\|\omega^{(2)}(\zeta)+\sigma^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^{2}d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left(\Omega_i(\lambda)+\Sigma_i(\lambda)\right)\left(\Omega_j(\lambda)+\Sigma_j(\lambda)\right)d\mu_{ij}(\lambda) \tag{3.25}$$

and

$$\int_{-\infty}^{d}\left\|\omega^{(1)}(\zeta)-\sigma^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^{2}d_q\zeta+\alpha\int_{d}^{\infty}\left\|\omega^{(2)}(\zeta)-\sigma^{(2)}(\zeta)\right\|_{\mathbb{C}^2}^{2}d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left(\Omega_i(\lambda)-\Sigma_i(\lambda)\right)\left(\Omega_j(\lambda)-\Sigma_j(\lambda)\right)d\mu_{ij}(\lambda). \tag{3.26}$$

From (3.25) and (3.26), we get the desired result. □

Theorem 3.6 *Let ω is a real vector-valued function and*

$$\omega(\zeta)=\left\{\begin{array}{ll}\omega^{(1)}(\zeta), & \zeta\in I_3\\ \omega^{(2)}(\zeta), & \zeta\in I_4,\end{array}\right. \quad \omega\in H.$$

Then, the integrals

$$\int_{-\infty}^{\infty}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda)\ \ (i,j=1,2)$$

converge in H. Thus, we obtain the spectral expansion formula

$$\omega(\zeta)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda).$$

Proof: Define

$$\omega_s(\zeta)=\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda),$$

where $s>0,\ \omega_s\in H$ and

$$\omega_s(\zeta)=\left\{\begin{array}{ll}\omega_s^{(1)}(\zeta), & \zeta\in I_3\\ \omega_s^{(2)}(\zeta), & \zeta\in I_4.\end{array}\right.$$

Let

$$\sigma(\zeta)=\begin{cases}\sigma^{(1)}(\zeta), & \zeta\in I_3\\ \sigma^{(2)}(\zeta), & \zeta\in I_4,\end{cases}\quad \sigma\in H$$

be a real vector-valued function which is equal to zero outside the finite interval $\left[-\frac{1}{q^{\tau}},d\right)\cup\left(d,\frac{1}{q^{\tau}}\right]$, where $\frac{1}{q^{\tau}}<\frac{1}{q^{n}}$. Hence we get

$$\int_{-\frac{1}{q^{\tau}}}^{d}\left(\omega_s^{(1)}(\zeta),\sigma^{(1)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta+\alpha\int_{d}^{\frac{1}{q^{\tau}}}\left(\omega_s^{(2)}(\zeta),\sigma^{(2)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta$$

$$=\int_{-\frac{1}{q^{\tau}}}^{d}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda),\sigma^{(1)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta$$

$$+\alpha\int_{d}^{\frac{1}{q^{\tau}}}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda),\sigma^{(2)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\left\{\begin{array}{c}\int_{-\tau}^{d}\left(\sigma^{(1)}(\zeta),\psi_j(\zeta,\lambda)\right)_{\mathbb{C}^2}d_q\zeta\\ +\alpha\int_{d}^{\tau}\left(\sigma^{(2)}(\zeta),\psi_j(\zeta,\lambda)\right)_{\mathbb{C}^2}d_q\zeta\end{array}\right\}$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Sigma_j(\lambda)\,d\mu_{ij}(\lambda).\tag{3.27}$$

By Theorem 3.5, we deduce that

$$\int_{-\infty}^{d}\left(\omega^{(1)}(\zeta),\sigma^{(1)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta+\alpha\int_{d}^{\infty}\left(\omega^{(2)}(\zeta),\sigma^{(2)}(\zeta)\right)_{\mathbb{C}^2}d_q\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Sigma_j(\lambda)\,d\mu_{ij}(\lambda).\tag{3.28}$$

By (3.27) and (3.28), we have

$$\langle\omega-\omega_s,\sigma\rangle_H=\int_{|\lambda|>s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Sigma_j(\lambda)\,d\mu_{ij}(\lambda).\tag{3.29}$$

Let

$$\sigma(\zeta)=\begin{cases}\omega(\zeta)-\omega_s(\zeta), & \zeta\in[-s,d)\cup(d,s]\\ 0, & \text{otherwise.}\end{cases}$$

From (3.29), we obtain

$$\|\omega-\omega_s\|_H^2=\int_{|\lambda|>s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Omega_j(\lambda)\,d\mu_{ij}(\lambda).$$

Letting $s\to\infty$ gives the desired result. □

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