Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–20.
ISSN-0037-8712
doi:10.5269/bspm.70402

Time-Ordered Evolutions on the Feynman-Dyson Hilbert Space *

E. Gül†, T.L. Gill, Ö. Bakşi, S.K. Çalışkan, A.K. Hansen, S. Karayel and Y. Sezer

ABSTRACT: In this work, we consider hyperbolic and parabolic evolution problems on the Feynman-Dyson Hilbert space, $\mathcal{FD}_{\otimes}^2$. We use the possible opportunities given in $\mathcal{FD}_{\otimes}^2$ to find solutions for both homogeneous and non-homogeneous cases. Therefore, we first focus the structure of the Feynman-Dyson Hilbert space from a mathematical perspective in terms of the construction of this space and the lifting of operator theory to this time-ordered setting. We then observe that $\mathcal{FD}_{\otimes}^2$ allows operators acting at different times to commute, while maintain their relative position on paper. We also deal with a time-ordered version of the Hille-Yosida theorem for semigroups of operators. This approach has the added advantage of requiring the weakest known domain and continuity conditions. We show these advantages for the generic classes of time-dependent homogeneous hyperbolic and parabolic problems. We also see that the theory has advantages for operators with no time dependence.

Key Words: Feynman-Dyson space, time-ordering, semi-group, evolution system.

Contents

1. Introduction

This paper is devoted to study evolution equations using Feynman's time-ordered operator calculus, ([1,2]), as developed by Gill and Zachary (see [3,4]). Feynman's basic idea was to first imagine that physical reality is a three-dimension motion picture in which time directs motion from the past to the present and to the future.

Using a fixed Hilbert space $\mathcal{H}$ as their base, Gill and Zachary took Feynman literally and first constructed a film (Feynman-Dyson Hilbert space, $\mathcal{FD}_{\otimes}^2$). This film allowed operators acting at different times to commute while still maintaining their position on paper as required mathematically (time-ordering). This approach also allowed them to lift all of operator theory to this setting and to prove a time-ordered version of the Hille-Yosida Theorem for semigroups of operators. The original purpose of their work was to prove the last two remaining conjectures of Dyson concerning the foundations for quantum electrodynamics (see [5]).

Mathematical research on the Feynman calculus can be traced from the early works of Fujiwara [6], Miranker and Weiss [7], Nelson [8], Araki [9], Maslov [10] and Johnson and Lapidus [11]. References to all contributions can be found in the book of Gill and Zachary, [4]. A more recent study using Feynman's operator calculus was done by Gill and Parga, [12]. In connection with our work, we study hyperbolic and parabolic evolution problems on Feynman-Dyson space. Hyperbolic and parabolic equations have been studied in many works previously, for example, [13], [14], [15], [16], [17], [18], [19], [20] and [21].

* The project is partially supported by Research Fund of the Yildiz Technical University. Project No:FBA-2023-5765
† Corresponding author
2010 *Mathematics Subject Classification:* 47D06, 34G10, 35L03.
Submitted November 17, 2023. Published April 19, 2024

The purpose of this paper is to study the hyperbolic and parabolic evolution equations by applying the basic ideas and foundations for the general study of evolution equations on Feynman-Dyson space based on a separable Hilbert space $\mathcal{H}$. This paper is organized as follows. In the first section we would like to explain the mathematical film given in the works of Gill and Zachary, [3] , [4] and [22], which is a fiber bundle over a fixed time-interval $I = [a, b]$ and each fiber is a Hilbert space attached to a fixed time point in I. In the second section, we construct hyperbolic evolution equations on $\mathcal{FD}^2_\otimes$, study their homogeneous and inhomogeneous cases and give solutions for both cases, respectively. In the third section, we study parabolic evolution equations on $\mathcal{FD}^2_\otimes$ in the sense of Fattorini's theorem, [23] and [4].

2. On the Feynman-Dyson Hilbert Space

In this section, we review briefly the construction of Feynman-Dyson Hilbert space based on a separable Hilbert space $\mathcal{H}$. The results, along with additional details and history, can be found in [3], [4] and [22].

We first need to give the definition of infinite products of uncountably many complex numbers and, in order to avoid trivialities, here it is assumed that all terms in any product are nonzero.

Definition 2.1. *If $\{z_\nu\}$ is a sequence of complex numbers indexed by $\nu \in I$,*

1. *We say that the product $\prod_{\nu\in I} z_\nu$ is convergent with limit z if, for every $\varepsilon > 0$, there is a finite set $J(\varepsilon)$ such that for all finite sets $J \subset I$ with $J(\varepsilon) \subset J$, we have $|\prod_{\nu\in J} z_\nu - z| < \varepsilon$.*
2. *We say that the product $\prod_{\nu\in I} z_\nu$ is quasi-convergent if $\prod_{\nu\in I} |z_\nu|$ is convergent.*

If the product is quasi-convergent, but not convergent, we assign it the value zero. Since I is uncountable and $0 < |\prod_{\nu\in I} z_\nu| < \infty$ if and only if $\sum_{\nu\in I} |1 - z_\nu| < \infty$, it follows that convergence implies that at most a countable number of the $z_\nu \neq 1$.

Let $\mathcal{H}$ be a infinite-dimensional separable Hilbert space, and $I = [0, T], 0 < T \leq \infty$. For each $\nu \in I$, let $\mathcal{H}_\nu = \mathcal{H}$ and for $\{\varphi_\nu\} \in \prod_{\nu\in I} \mathcal{H}_\nu$ let

$$\Delta_I = \{\{\varphi_\nu\} : \sum_{\nu\in I} |\ ||\varphi_\nu||_\nu - 1\ | < \infty\}.$$

Define a functional on Δ_I by

$$\Phi(\psi) = \sum_{k=1}^{n} \prod_{\nu\in I} \langle \varphi^k_\nu, \psi_\nu\rangle_\nu,$$

where $\psi = \{\psi_\nu\}$, $\{\varphi^k_\nu\} \in \Delta_I$, for $1 \leq k \leq n$. This functional is linear in each component. Let us denote Φ by

$$\Phi = \sum_{k=1}^{n} \otimes_{\nu\in I} \varphi^k_\nu,$$

and define the algebraic tensor product, $\otimes_{\nu\in I}\mathcal{H}_\nu$, by

$$\otimes_{\nu\in I}\mathcal{H}_\nu = \{\sum_{k=1}^{n} \otimes_{\nu\in I} \varphi^k_\nu : \{\varphi^k_\nu\} \in \Delta_I, 1 \leq k \leq n, n \in \mathbb{N}\}.$$

Lemma 2.1. *The following is a well-defined linear functional on $\otimes_{\nu\in I}\mathcal{H}_\nu$:*

$$(\sum_{k=1}^{n} \otimes_{\nu\in I} \varphi^k_\nu, \sum_{l=1}^{m} \otimes_{\nu\in I} \psi^l_\nu)_\otimes = \sum_{l=1}^{m} \sum_{k=1}^{n} \prod_{\nu\in I} \langle \varphi^k_\nu, \psi^l_\nu\rangle_\nu.$$

For the construction of $\mathcal{FD}^2_{\otimes}$, we need to understand the structure of $\otimes_{\nu\in I}\mathcal{H}_\nu$.

Definition 2.2. *Let $\phi = \otimes_{\nu\in I}\phi_\nu$ and $\psi = \otimes_{\nu\in I}\psi_\nu$ be in $\otimes_{\nu\in I}\mathcal{H}_\nu$.*

1. The vector ϕ is said to be strongly equivalent to ψ (i.e; $\phi \equiv^s \psi$) if and only if

$$\sum_{\nu\in I}|1 - \langle\phi_\nu, \psi_\nu\rangle_\nu| < \infty.$$

2. The vector ϕ is said to be weakly equivalent to ψ (i.e; $\phi \equiv^w \psi$) if and only if

$$\sum_{\nu\in I}|1 - |\langle\phi_\nu, \psi_\nu\rangle_\nu|\ | < \infty.$$

Theorem 2.2. *The relations of strong and weak are equivalence relations on $\otimes_{\nu\in I}\mathcal{H}_\nu$, which decompose $\otimes_{\nu\in I}\mathcal{H}_\nu$ into disjoint (orthogonal) equivalence classes.*

Lemma 2.3. *We have $\phi \equiv^\omega \psi$ if and only if there exist z_ν, $|z_\nu| = 1$, such that $\otimes_{\nu\in I}z_\nu\phi_\nu \equiv^s \otimes_{\nu\in I}\psi_\nu$.*

Lemma 2.4. *Let $\otimes_{\nu\in I}\varphi_\nu$ be in $\otimes_{\nu\in I}\mathcal{H}_\nu$. Then:*

(1) The product $\prod_{\nu\in I}\|\varphi_\nu\|_\nu$ converges if and only if $\prod_{\nu\in I}\|\varphi_\nu\|^2_\nu$ converges.

(2) If $\prod_{\nu\in I}\|\varphi_\nu\|_\nu$ and $\prod_{\nu\in I}\|\psi_\nu\|_\nu$ converge, then $\prod_{\nu\in I}\langle\varphi_\nu, \psi_\nu\rangle_\nu$ is quasi-convergent.

(3) If $\prod_{\nu\in I}\langle\varphi_\nu, \psi_\nu\rangle_\nu$ is quasi-convergent, then there exist complex numbers $\{z_\nu\}, |z_\nu| = 1$, such that $\prod_{\nu\in I}\langle z_\nu\varphi_\nu, \psi_\nu\rangle_\nu$ converges.

Definition 2.3. *For $\varphi = \otimes_{\nu\in I}\varphi_\nu \in \otimes_{\nu\in I}\mathcal{H}_\nu$, $\mathcal{H}^2_{\otimes}(\varphi)^s$ is defined to be the closed subspace generated by the span of all $\psi \equiv^s \varphi$ and* **it is called the strong partial tensor product space** *generated by the vector φ. Likewise, $\mathcal{H}^2_{\otimes}(\varphi)^\omega$ is defined to be the closed subspace generated by the span of all $\psi \equiv^\omega \varphi$ and* **it is called the weak partial tensor product space** *generated by the vector φ.*

Theorem 2.5. *For the partial tensor product spaces, we have the following:*

(1) If $\psi_\nu \neq \varphi_\nu$ occurs for at most a finite number of ν then

$$\psi = \otimes_{\nu\in I}\psi_\nu \equiv^s \varphi = \otimes_{\nu\in I}\varphi_\nu.$$

(2) The space $\mathcal{H}^2_{\otimes}(\varphi)$ is the closure of the linear span of $\psi = \otimes_{\nu\in I}\psi_\nu$ such that $\psi_\nu \neq \varphi_\nu$ occurs for at most a finite number of ν.

(3) If $\Phi = \otimes_{\nu\in I}\psi_\nu$ and $\Psi = \otimes_{\nu\in I}\psi_\nu$ are in different equivalence classes of $\otimes_{\nu\in I}\mathcal{H}_\nu$, then

$$(\Phi, \Psi)_\otimes = \prod_{\nu\in I}\langle\varphi_\nu, \psi_\nu\rangle_\nu = 0.$$

(4) $\mathcal{H}^2_{\otimes}(\varphi)^\omega = \bigoplus_{\psi\equiv^\omega\phi}[\mathcal{H}^2_{\otimes}(\varphi)^s]$.

Definition 2.4. *Denote by $\mathcal{H}^2_{\otimes} = \hat{\otimes}_{\nu\in I}\mathcal{H}_\nu$, the completion of the linear space $\otimes_{\nu\in I}\mathcal{H}_\nu$ relative to the inner product $(\cdot,\cdot)_\otimes$.*

The final tool to understand the construction of the mathematical film, is the orthonormal basis for each strong partial tensor product space $\mathcal{H}^2_{\otimes}(\varphi)$, where $0 \neq \varphi = \otimes_{\nu\in I}\varphi_\nu$. Anticipating the possibility that, in the general case, a different (but equivalent) norm on each $\mathcal{H}_\nu$ may be needed, the assumption that all $\mathcal{H}_\nu$ are identical is dropped.

Let $\mathbf{N}$ be the natural numbers, and let $\{e_n^\nu,\ n \in \mathbb{N} = \mathbf{N} \cup \{0\}\}$ be a complete orthonormal basis for $\mathcal{H}_\nu$, for each ν. Let e_0^ν be a fixed unit vector in $\mathcal{H}_\nu$ and set $E = \otimes_{\nu \in I} e_0^\nu$. Let $\mathbf{F}$ be the set of all functions $f:\ I \to \mathbb{N}$ such that $f(\nu) = 0$ for all but a finite number of ν. Let $F(f)$ be the image of $f \in \mathbf{F}$ (i.e., $F(f) = \{f(\nu),\ \nu \in I\}$), and set $E_{F(f)} = \otimes_{\nu \in I} e_{\nu, f(\nu)}$, where $f(\nu) = 0$ implies that $e_{\nu,0} = e_0^\nu$ and $f(\nu) = n$ implies $e_{\nu,n} = e_n^\nu$.

Theorem 2.6. *The set $\{E_{F(f)},\ f \in \mathbf{F}\}$ is a complete orthonormal basis for $\mathcal{H}_\otimes^2(E)$.*

Let $\{e_\nu^i : i \in \mathbb{N}\}$ be a complete orthonormal basis for $\mathcal{H}_\nu$. Replacing ν by t, for each $i \in \mathbb{N}$ and $t \in I$, and set $E^i = \otimes_{t \in I} e_t^i$. We define $\mathcal{FD}_2^i = \mathcal{H}_\otimes^2(E^i)$ to be the strong partial tensor product space generated by the vector E^i. The Feynman-Dyson space $\mathcal{FD}_\otimes^2$, is defined by:

$$\mathcal{FD}_\otimes^2 = \oplus_{i=1}^\infty \mathcal{FD}_2^i \subset \mathcal{H}_\otimes^2.$$

This space provides the mathematical film over the interval $[0, T]$. Since I is uncountable, $\mathcal{FD}_\otimes^2$ is not separable. However, each time slice (fiber over t) is isometrically isomorphic to $\mathcal{H}_t$.

If $\mathcal{H}_\otimes^2 = \hat{\otimes}_{t \in I} \mathcal{H}(t)$, let $L(\mathcal{H}_\otimes^2)$ be the set of bounded linear operators on $\mathcal{H}_\otimes^2$ and let $\{A(t), t \in I\} \subset L(\mathcal{H})$. Define $L(\mathcal{H}(t)) \subset L(\mathcal{H}_\otimes^2)$ by:

$$L(\mathcal{H}(t)) = \left\{ \mathcal{A}(t) = (\hat{\otimes}_{b \geqslant s > t} \mathrm{I}_s) \otimes A(t) \otimes (\otimes_{t > s \geqslant a} \mathrm{I}_s), \forall A(t) \in L(\mathcal{H}) \right\}, \tag{2.1}$$

where $\mathrm{I_s}$ is the identity operator. Let $L^\#(\mathcal{H}_\otimes^2)$ be the uniform closure of the algebra generated by $\{L(\mathcal{H}(t)),\ t \in I\}$. For the family $\{A(t),\ t \in I\} \subset L(\mathcal{H})$, the the family $\{\mathcal{A}(t),\ t \in I\} \subset L^\#(\mathcal{H}_\otimes^2)$ commute when acting at different times:

$$\mathcal{A}(t)\mathcal{A}(\tau) = \mathcal{A}(\tau)\mathcal{A}(t) \text{ for } t \neq \tau.$$

Let

$$\mathcal{A}_z(t) = z\mathcal{A}(t)\mathbf{R}(z, \mathcal{A}(t)),$$

where $\mathbf{R}(z, \mathcal{A}(t))$ is the resolvent of $\mathcal{A}(t)$. By Fundamental Theorem for Time-Ordered Integrals in [4], $A_z(t)$ generates a uniformly bounded semigroup and $\lim_{z \to \infty} A_z(t)\phi = A(t)\phi$ for $\phi \in D(A(t))$.

Theorem 2.7. *The time ordered Yosida approximator $\mathcal{A}_z(t)$ is a bounded linear operator and for each $\Phi \in D$, $\mathcal{A}(t)\mathcal{A}_z(t)\Phi = \mathcal{A}_z(t)\mathcal{A}(t)\Phi$. In addition:*

1. *$\mathcal{A}_z(t)$ generates a uniformly bounded contraction semigroup on $\mathcal{FD}_\otimes^2$ for each t, with $\lim_{z \to \infty} \mathcal{A}_z(t)\Phi = \mathcal{A}(t)\Phi$, for $\Phi \in D$.*
2. *For each n, each set $\tau_1, \cdots, \tau_n \in I$ and each set $a_1, \cdots, a_n$, $a_i \geqslant 0$; $\sum_{i=1}^n a_i \mathcal{A}(\tau_i)$ generates a C_0-semigroup on $\mathcal{FD}_\otimes^2$.*

Assume that $A(t),\ t \in I$ is a closed, weakly continuous and that $D(A(t)) = D(t)$ is dense in $\mathcal{H}$. It follows that this family has a weak HK-integral $Q[a, b] = \int_a^b A(t)dt \in C(\mathcal{H})$ (the closed densely defined linear operators on $\mathcal{H}$). Furthermore, it is not difficult to see that $A_z(t),\ t \in I$, is also weakly continuous and hence the family $\{A_z(t) \mid t \in I\} \subset L(\mathcal{H})$ has a weak HK-integral $Q_z[b, a] = \int_a^b A_z(t)dt \in L(\mathcal{H})$.

The family $\{\mathcal{A}_z(\mathrm{t}) \mid t \in I\}$ has a strong integral $\mathbf{Q}_z[t, a] = \int_a^t \mathcal{A}_z(s)ds$ and the linear operator $\mathbf{Q}_z[t, a]$ generates a uniformly continuous C_0 semigroup, by Fundamental Theorem for Time-Ordered Integrals, [4].

The results up to this point only used the assumption that the family $A(t), t \in I$, is weakly continuous. From now on it is assumed that, for each t, $Q[t, a]$ and $Q^*[t, a]$ are dissipative. This is equivalent to the statement that $Q[t, a]$ generates a C_0-contraction semigroup of bounded linear operators on $\mathcal{H}$, but in practice this is easier to check. (These conditions do not imply that $Q[t, a]$ solves the initial-value theorem on $\mathcal{H}$.)

This assumption is weaker than that of the family $\{A(t)\}$ having a common dense domain. In all counter examples, the family $\{A(t)\}$ is assumed to be strongly continuous, or the conditions imposed implies so (see Fattorini [23], pg. 408).

Theorem 2.8. *With the above assumptions,*

1. *For* $\Phi \in D_0$, $\lim_{z\to\infty} \mathbf{Q}_z[t,s]\Phi = \mathbf{Q}[t,s]\Phi$,
2. $\mathbf{Q}[t,s]$ *generates a* C_0*-contraction semigroup on* $\mathcal{FD}_{\otimes}^2$,
3. $\mathbf{Q}[t,r]\Phi + \mathbf{Q}[r,s]\Phi = \mathbf{Q}[t,s]\Phi$ *(a.s.c.),*
4. $\lim_{h\to 0} [(\mathbf{Q}[t+h,s] - \mathbf{Q}[t,s])/h]\,\Phi = \mathcal{A}(t)\Phi$ *(a.s.c.),*
5. $\lim_{h\to 0} [(\mathbf{Q}[t+h,s] - \mathbf{Q}[t,s])/h]\,\Phi = -\mathcal{A}(s)\Phi$ *(a.s.c.), and*
6. $\lim_{h\to 0} \exp\{\tau\mathbf{Q}[t+h,t]\}\,\Phi\ = \Phi$ *(a.s.c.),*$\tau \geqslant 0$.

The next result is the time-ordered version of the Hille-Yosida Theorem (see Pazy [17], pg. 8). It is assumed that the family $A(t), t \in I$, is closed and densely defined.

Theorem 2.9. *The family* $\mathcal{A}(t), t \in I$*, has a strong HK-integral,* $\mathbf{Q}[t,a]$*, which generates a* C_0*-contraction semigroup on* $\mathcal{FD}_{\otimes}^2$ *if and only if* $\rho(A(t)) \supset (0,\infty)$, $\|R(\lambda, A(t))\| < 1/\lambda$ *for* $\lambda > 0$, $A(t), t \in I$*, satisfies the inequality (7.3) in [4] and has a m-dissipative, closed, densely defined weak HK-integral* $Q[t,a]$ *on* $\mathcal{H}$.

3. Hyperbolic Evolution Equations on $\mathcal{FD}_{\otimes}^2$

Let $B(t),\ t \in I$ be a family of generators of C_0-semigroups $T_t(\tau)$ satisfying:

$$\|T_t(\tau)\|_{\mathcal{H}} \leqslant M(t)e^{-\tau\omega(t)}.$$

Definition 3.1. *We say the family* $B(t),\ t \in I$ *is stable, if for each* t*, the constants* $M(t)$ *and* $\omega(t)$ *(stability constants), are such that:*

1. $\rho\left(B(t)\right) \supset (\omega(t), \infty)$*, for each* $t \in I$.
2. *For* $\lambda_j > \omega(t_j),\ 1 \le j \le k$,

$$\left\|\prod_{j=1}^{k} R\left(\lambda_j, B(t_j)\right)\right\|_{\mathcal{H}} \leqslant \prod_{j=1}^{k} M(t_j)(\lambda_j - \omega(t_j))^{-1}.$$

We note for later that the terms on the left above do not commute.

We assume that:

1. There exist a constant $M = \sup\{M(t_1), M(t_2)\cdots M(t_k)\} < \infty$, where the supremum is taken over all finite subsets of I.
2. The family $\omega(t),\ t \in I$ has a HK-integral $\omega = \int_I \omega(\tau)d\tau$.

If for each $t \in I$, we let $A(t) = B(t) - \omega(t)I$, the family $A(t),\ t \in I$ are generators of C_0 semigroups which satisfy:

$$\|S_t(\tau)\|_{\mathcal{H}} \leqslant M.$$

Remark 3.1. *Following Pazy [17] (in the proof of Theorem 5.2), for each* $t \in I$ *we can re-norm* $\mathcal{H}$ *(equivalently) in our definition of* $\mathcal{H}_{\otimes}^2$ *so that in the* $\mathcal{H}_t$ *norm* $\|S_t(\tau)\|_{\mathcal{H}} \leqslant 1$*, making it a contraction semigroup (at each* t*). This means that the time-ordered family* $\mathcal{A}(t),\ t \in I$ *defined on* $\mathcal{FD}_{\otimes}^2$ *will be generators of* C_0 *contraction semigroups with stability constants* 1 *and* 0.

Thus without loss in generality, for the hyperbolic problem we can assume the following as in [4]:

1. For each $t \in I,\ A(t)$ generates a C_0-contraction semigroup.
2. For each $t \in I,\ A(t)$ is stable with constants $1,\ 0$.
3. The resolvent set $\rho(A(t)) \supset (0, \infty),\ t \in I$, and for every $\tau \in I$ and each finite family $\{t_1, t_2 \ldots, t_k\} \subset I$ we have:
$$\left\| \prod_{j=1}^{k} \exp\{\tau A(t_j)\} \right\| \leqslant 1.$$
4. There exists a Hilbert space $\mathcal{Y}$ densely and continuously embedded in $\mathcal{H}$ such that, for each $t \in I$, $D(A(t)) \supset \mathcal{Y}$ and $A(t) \in L[\mathcal{Y}, \mathcal{H}]$ (i.e., $A(t)$ is bounded as a mapping from $\mathcal{Y} \to \mathcal{H}$), and the function $g(t) = \|A(t)\|_{\mathcal{Y} \to \mathcal{H}}$ is continuous.
5. The space $\mathcal{Y}$ is an invariant subspace for each semigroup $S_t(\tau) = \exp\{\tau A(t)\}$ and $S_t(\tau)$ is a stable C_0-semigroup on $\mathcal{Y}$ with the same stability constants.

The following lemma shows that condition (4) implies that $A(t)$ is strongly continuous, hence, in [4], the equation (7.3) of the Fundamental Theorem for Time-Ordered Integrals is satisfied.

Lemma 3.1. *[4] Suppose conditions (3) and (4) above are satisfied with $\|\varphi\|_{\mathcal{H}} \leqslant \|\varphi\|_{\mathcal{Y}}$. Then the family $A(t),\ t \in I$, is strongly continuous on I.*

We now consider the hyperbolic problem in the time-ordered setting on $\mathcal{FD}^2_{\otimes}$:

$$\begin{aligned} \frac{du(t)}{dt} &= \mathcal{A}(t)u(t) + f(t),\ 0 \leq a \leq s < t \leq b, \\ u(a) &= \Phi. \end{aligned} \tag{3.1}$$

Our approach is to first solve the problem for $\mathcal{A}_z(t)$ and then obtain the solution to (3.1) as a limit. This makes it possible to by-pass the common dense domain implied by use of the subspace $\mathcal{Y}$ in $\mathcal{H}$ (i.e, conditions (4) and (5)). In this case, we have:

$$\begin{aligned} \frac{du_z(t)}{dt} &= \mathcal{A}_z(t)u_z(t) + f(t),\ 0 \leq a \leq s < t \leq b, \\ u_z(a) &= \Phi. \end{aligned} \tag{3.2}$$

We know that by the Fundamental Theorem for Time-Ordered Integrals, [4] $\mathbf{Q}_z[t, a] = \int_a^t \mathcal{A}_z(\tau)d\tau$ generates a uniformly continuous contraction semigroup on $\mathcal{FD}^2_{\otimes}$.

Note that $\mathbf{Q}_z[t, a]$ is bounded whereas $\mathbf{Q}[t, a]$ is unbounded.

Theorem 3.2. *For each $t \in I$, let $\mathbf{Q}_z[t, a]$ be the infinitesimal generator of a C_0-contraction semigroup. The family of generators $\{\mathbf{Q}_z[t, a]\}_{t \in I}$ is stable iff there are constants 0 and 1 such that*

$$\rho(\mathbf{Q}_z[t, a]) \supset (0, \infty), \quad for \quad t \in I,$$

and the following condition is satisfied for $\tau_j \geq 0$, finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ and for fixed t_j

$$|| \prod_{j=1}^{k} exp\{\tau_j \mathbf{Q}_z[t_j, a]\}|| \leq 1.$$

Proof: By the definition of stability, it is sufficient to prove that for a family $\{\mathbf{Q}_z[t,a]\}_{t\in I}$ of infinitesimal generators for which $\rho(\mathbf{Q}_z[t,a]) \supset (0,\infty)$, for $t \in I$,

$$||\prod_{j=1}^{k} exp\{\tau_j \mathbf{Q}_z[t_j,a]\}|| \leq 1,$$

is equavialent to

$$||\prod_{j=1}^{k} R(\lambda_j : \mathbf{Q}_z[t_j,a])|| \leq \prod_{j=1}^{k} \frac{1}{\lambda_j},$$

for $\lambda_j > 0$ and every finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$, $k = 1, 2, \ldots$.

It is known that

$$R(\lambda_j : \mathbf{Q}_z[t_j,a])\Phi = \int_0^\infty e^{-\lambda_j s} e^{\tau_j \mathbf{Q}_z[t_j,a]} \Phi ds,$$

and iterating this a finite number of times we have

$$\prod_{j=1}^{k} R(\lambda_j : \mathbf{Q}_z[t_j,a])\Phi = \int_0^\infty \cdots \int_0^\infty exp\{-\sum_{j=1}^{k} \lambda_j s_j\}$$
$$\cdot \prod_{j=1}^{k} exp\{\tau_j \mathbf{Q}_z[t_j,a]\}\Phi ds_1 \ldots ds_k.$$

Estimation of the last expression gives

$$||\prod_{j=1}^{k} R(\lambda_j : \mathbf{Q}_z[t_j,a])\Phi|| \leq ||\Phi|| \prod_{j=1}^{k} \int_0^\infty e^{-\lambda_j s_j} ds_j = ||\Phi|| \prod_{j=1}^{k} \frac{1}{\lambda_j},$$

and this completes the proof.

3.1. Solution of the Homogeneous Hyperbolic Problem on $\mathcal{FD}^2_\otimes$.

In this section we consider the following homogeneous hyperbolic evolution equation and study its solution:

$$\begin{aligned} \frac{du_z(t)}{dt} &= \mathcal{A}_z(t)u_z(t),\ 0 \leq a \leq s < t \leq b, \\ u_z(a) &= \Phi, \end{aligned} \tag{3.3}$$

where $\Phi \in \mathcal{FD}^2_\otimes$.

For this aim, we first deal with the conditions that the family of operator coefficients has to satisfy. In [17], the author considers homogeneous hyperbolic evolution equation on a Banach space, where the family of operator coefficients satisfies three conditions. In our case, these conditions given in [17] for a Banach space are reduced to the following single condition when considering the time-ordered hyperbolic evolution equations on Feynman-Dyson space $\mathcal{FD}^2_\otimes$ based on Hilbert space:

$(\mathbf{FD-H})$ For each $t \in I$, $\mathbf{Q}_z[t,a]$ is stable with constants 0 and 1 and the resolvent set

$$\rho(\mathbf{Q}_z[t,a]) \supset (0,\infty),\ t \in I,$$

such that for $\tau_j \geq 0$ and finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$

$$||\prod_{j=1}^{k} exp\{\tau_j \mathbf{Q}_z[t_j, a]\}|| \leq 1.$$

In fact, the family $\{\mathbf{Q}_z[t,a]\}_{t\in[0,T]}$ is stable, since any family of infinitesimal generators of C_0-semigroups of contractions is stable, [17].

By the fact that $\{\mathbf{Q}_z[t,a]\}_{t\in[0,T]}$ is defined on all $\mathcal{FD}^2_{\otimes}$, we do not need any additional conditions. This will greatly simplify our approach to the existence and uniqueness for the solution of the homogeneous hyperbolic initial value problem.

Lemma 3.3. *Let us consider the homogeneous hyperbolic evolution equation given in (3.3). Then, $\mathbf{U}_z[t,a] = e^{\mathbf{Q}_z[t,a]}$ is an evolution system,*
where $\mathbf{Q}_z[t,a] = \int_a^t \mathcal{A}_z(\tau)d\tau$.

Proof:

It is obvious that $\mathbf{U}_z[s,s] = e^{\mathbf{Q}_z[s,s]} = I$. Second property

$$\mathbf{U}_z[t,s]\mathbf{U}_z[s,a] = \mathbf{U}_z[t,a],$$

is followed by

$$\mathbf{U}_z[t,s]\mathbf{U}_z[s,a] = e^{\mathbf{Q}_z[t,s]}e^{\mathbf{Q}_z[s,a]}.$$

The mapping $(t,a) \to \mathbf{U}_z[t,a]$ is strongly continuous for $0 \leq a \leq s < t \leq b$.

Theorem 3.4. *, [4] If $a < t < b$,*

(1) $lim_{z\to\infty}\mathbf{U}_z[t,a]\Phi = \mathbf{U}[t,a]\Phi = e^{\mathbf{Q}[t,a]}\Phi$, $\Phi \in \mathcal{FD}^2_{\otimes}$,

(2) $\frac{\partial}{\partial t}\mathbf{U}_z[t,a]\Phi = \mathcal{A}_z(t)\mathbf{U}_z[t,a]\Phi = \mathbf{U}_z[t,a]\mathcal{A}_z(t)\Phi$, $\Phi \in \mathcal{FD}^2_{\otimes}$,

and

(3) $\frac{\partial}{\partial t}\mathbf{U}[t,a]\Phi = \mathcal{A}(t)\mathbf{U}[t,a]\Phi = \mathbf{U}[t,a]\mathcal{A}(t)\Phi$, and $\Phi \in D(\mathbf{Q}_z[b,a]) \supset D_0$.

This theorem allows us to extend the solution of (3.3) which is the solution of the homogeneous part of the hyperbolic evolution equation given by (3.2). Now, we have the following result.

Theorem 3.5. *Let $\mathbf{Q}_z[t,a]$, $0 \leq a \leq t \leq b$, be the infinitesimal generator of a C_0-contraction semigroup $S_t(s)$, $s \geq 0$, on $\mathcal{FD}^2_{\otimes}$. If we consider the family $\{\mathbf{Q}_z[t,a]\}_{t\in[a,b]}$, then there exists a unique evolution system $\mathbf{U}_z[t,a]$, $0 \leq a \leq s \leq t \leq b$, in $\mathcal{FD}^2_{\otimes}$ satisfying*

(1) $||\mathbf{U}_z[t,a]|| \leq 1$, for $0 \leq a \leq s < t \leq b$,

(2) $\frac{\partial}{\partial s}\mathbf{U}_z[t,s]\Phi = -\mathcal{A}_z(s)\mathbf{U}_z[t,s]\Phi$, for $\Phi \in \mathcal{FD}^2_{\otimes}$, $a \leq s < t \leq b$,

where derivatives are in the strong sense in $\mathcal{FD}^2_{\otimes}$.

Proof:

(1) For each $t \in I$, $\mathbf{Q}_z[t,a]$ is stable with constants 0 and 1, and

$$\rho(\mathbf{Q}_z[t,a]) \supset (0,\infty),\ t \in I,$$

such that for $\tau_j \geq 0$ and finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$,

$$||\prod_{j=1}^{k} exp\{\tau_j \mathbf{Q}_z[t_j,a]\}|| \leq 1.$$

It follows that

$$||\mathbf{U}_z[t,a]|| \leq 1, \quad for \ \ 0 \leq a \leq s < t \leq b.$$

(2) For the proof we use the following approach:

$$\mathbf{U}_z[t,s+h] - \mathbf{U}_z[t,s] = e^{\int_{s+h}^{t} \mathcal{A}_z(\xi)d\xi} - e^{\int_{s}^{t} \mathcal{A}_z(\xi)d\xi}$$

$$= e^{(\int_{s}^{t} \mathcal{A}_z(\xi)d\xi - \int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi)} - e^{\int_{s}^{t} \mathcal{A}_z(\xi)d\xi}$$

$$= e^{\int_{s}^{t} \mathcal{A}_z(\xi)d\xi}[e^{-\int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi} - I_\otimes]$$

$$= \mathbf{U}_z[t,s][e^{-\int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi} - I_\otimes].$$

Now we consider the series expansion, [4],

$$e^{w\mathbf{Q}_z[t,a]}\Phi = \left\{ I_\otimes + \sum_{k=1}^{n} \frac{(w\mathbf{Q}_z[t,a])^k}{k!} + \frac{1}{n!}\int_0^w (w-\xi)^n (\mathbf{Q}_z[t,a])^{n+1}\mathbf{U}_z^{\xi}[t,a]d\xi \right\}\Phi,$$

and use it for the expression $\{e^{-\int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi} - I_\otimes\}\Phi$, where $\Phi \in \mathfrak{FD}_\otimes^2$, $w = 1$ and $n = 1$. Hence, we have

$$e^{-\int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi}\Phi = e^{-\mathbf{Q}_z[s+h,s]}\Phi$$

$$= \{I_\otimes - \mathbf{Q}_z[s+h,s] - \int_0^1 (1-\xi)\mathbf{Q}_z[s+h,s]^2 e^{-\int_{s}^{s+h} \mathcal{A}_z(\tau)d\tau} d\xi\}\Phi,$$

$$\begin{aligned} \Rightarrow \{e^{-\int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi} - I_\otimes\}\Phi \quad &= \quad \{-\mathbf{Q}_z[s+h,s]\Phi \\ &- \int_0^1 (1-\xi)\mathbf{Q}_z[s+h,s]^2 e^{-\int_{s}^{s+h} \mathcal{A}_z(\tau)d\tau} d\xi\}\Phi, \end{aligned}$$

$$\begin{aligned} \Rightarrow \{\frac{e^{-\int_{s}^{s+h} \mathcal{A}_z(\xi)d\xi} - I_\otimes}{h}\}\Phi \quad &= \quad \frac{\mathbf{Q}_z[s+h,s]}{h}\Phi \\ &+ \int_0^1 (1-\xi)\frac{\mathbf{Q}_z[s+h,s]^2}{h} e^{-\int_{s}^{s+h} \mathcal{A}_z(\tau)d\tau} d\xi\Phi. \end{aligned}$$

Since

$$\lim_{h\to 0}\frac{\mathbf{Q}_z[s+h,s]}{h}=\mathcal{A}_z(s)\ \ and\ \lim_{h\to 0}\frac{\mathbf{Q}_z[s+h,s]^2}{h}=0,$$

we have:

$$\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi+\mathcal{A}_z(s)\Phi$$
$$=\mathcal{A}_z(s)\Phi-\frac{\mathbf{Q}_z[s+h,s]}{h}\Phi-\int_0^1(1-\xi)\mathbf{Q}_z[s+h,s]e^{-\int_s^{s+h}\mathcal{A}_z(\tau)d\tau}\frac{\mathbf{Q}_z[s+h,s]}{h}d\xi\Phi.$$

Taking limit, we get:

$$\lim_{h\to 0}\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi+\mathcal{A}_z(s)\Phi=0.$$

This gives

$$\lim_{h\to 0}\frac{\mathbf{U}_z[t,s+h]-\mathbf{U}_z[t,s]}{h}\Phi=\mathbf{U}_z[t,s]\lim_{h\to 0}\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi$$
$$=-\mathbf{U}_z[t,s]\mathcal{A}_z(s)\Phi.$$

Indeed,

$$||\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi+\mathcal{A}_z(s)\Phi||\leq||(-\frac{\mathbf{Q}_z[s+h,s]}{h}\Phi+\mathcal{A}_z(s))\Phi||$$
$$+||\int_0^1(1-\xi)\mathbf{Q}_z[s+h,s]\frac{\mathbf{Q}_z[s+h,s]}{h}e^{-\int_s^{s+h}\mathcal{A}_z(\tau)d\tau}d\xi\Phi||.$$

Hence,

$$\begin{aligned}\Rightarrow\ &\lim_{h\to 0}||\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi+\mathcal{A}_z(s)\Phi||\leq\\ \leq\ &\lim_{h\to 0}||(\frac{-\mathbf{Q}_z[s+h,s]}{h})\Phi+\mathcal{A}_z(s)\Phi||\\ +\ &\lim_{h\to 0}||\int_0^1(1-\xi)(-\mathbf{Q}_z[s+h,s])\frac{-\mathbf{Q}_z[s+h,s]}{h}e^{-\int_s^{s+h}\mathcal{A}_z(\tau)d\tau}d\xi\Phi||,\end{aligned}$$

which means

$$0\leq\lim_{h\to 0}||\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi+\mathcal{A}_z(s)\Phi||\leq 0.$$

Thus, we have:

$$\lim_{h\to 0}\frac{e^{-\int_s^{s+h}\mathcal{A}_z(\xi)d\xi}-I_\otimes}{h}\Phi=-\mathcal{A}_z(s)\Phi.$$

On the other hand, since $\mathbf{Q}_z[t,a]$, $0\leq a\leq t\leq b$, is an infinitesimal generator of a C_0-contraction semigroup $S_t(s)$ ($s\geq 0$) on $\mathcal{FD}^2_\otimes$, the evolution system $\mathbf{U}_z[t,a]$ is unique. This completes the proof.

Corollary 3.6. *If* $u(a) \in \mathcal{FD}^2_{\otimes}$, *then*

1. $u_z(t) = \mathbf{U}_z[t,a]u(a)$ *and*

$$\frac{\partial u_z(t)}{\partial t} = \mathcal{A}_z(t)u_z(t), \quad u_z(a) = u(a).$$

2. $u(t) = \lim_{z\to\infty} u_z(t)$ *exists,* $u(t) \in D_0$ *and*

$$\frac{\partial u(t)}{\partial t} = \mathcal{A}(t)u(t), \quad u_z(a) = u(a).$$

3.2. The Time-Ordered Inhomogeneous Hyperbolic Evolution Equations on $\mathcal{FD}$.

We now consider the inhomogeneous hyperbolic problem:

$$\begin{aligned} \frac{du(t)}{dt} &= \mathcal{A}(t)u(t) + f(t), \ 0 \le a \le s \le t \le b, \\ u(a) &= \Phi. \end{aligned}$$

Theorem 3.7. *Let* $\mathbf{Q}_z[t,a]$, $0 \le a < t < b$, *be the infinitesimal generator of the* C_0*-contraction semigroup* $S_t(s)$, $s \ge 0$, *on* $\mathcal{FD}^2_{\otimes}$ *and let* $f \in C^1([t,a] : \mathcal{FD}^2_{\otimes})$, *then for every* $\Phi \in \mathcal{FD}^2_{\otimes}$, *the approximate inhomogeneous initial value problem has a unique solution of the form:*

$$u_z(t) = \mathbf{U_z}[t,a]\Phi + \int_a^t \mathbf{U_z}[t,r]f(r)dr,$$

and the exact inhomogeneous problem has the unique solution:

$$u(t) = \lim_{z\to\infty} u_z(t) = \mathbf{U}[t,a]\Phi + \int_a^t \mathbf{U}[t,r]f(r)dr.$$

Proof:

We first show that $u_z(t)$ satisfies the evolution equation. In fact,

$$\begin{aligned} \frac{d}{dt}u_z(t) &= \frac{d}{dt}(\mathbf{U_z}[t,a]\Phi) + \frac{d}{dt}(\int_a^t \mathbf{U_z}[t,r]f(r)dr) \\ &= \mathcal{A}_z(t)\mathbf{U_z}[t,a]\Phi + \int_a^t \frac{\partial}{\partial t}(\mathbf{U_z}[t,r]f(r))dr + \mathbf{U_z}[t,t]f(t) - 0 \\ &= \mathcal{A}_z(t)\{\mathbf{U_z}[t,a]\Phi + \int_a^t \mathbf{U_z}[t,r]f(r)dr\} + f(t) = \mathcal{A}_z(t)u_z(t) + f(t). \end{aligned}$$

To prove uniqueness, we assume that there is also a $v_z(t)$ solution and we need to prove

$$\frac{\partial}{\partial r}(\mathbf{U_z}[t,r]v_z(r)) = \mathbf{U_z}[t,r]f(r).$$

This implies

$$\begin{aligned} &\frac{\partial}{\partial r}\mathbf{U_z}[t,r]v_z(r) + \mathbf{U_z}[t,r]\frac{\partial}{\partial r}v_z(r) \\ &\qquad = -\mathcal{A}_z(r)\mathbf{U_z}[t,r]v_z(r) + \mathbf{U_z}[t,r][\mathcal{A}_z(r)v_z(r) + f(r)] \\ &\qquad = \mathbf{U_z}[t,r]f(r). \end{aligned}$$

A simple calculation shows

$$\int_a^t \frac{\partial}{\partial r}(\mathbf{U_z}[t,r]v_z(r))dr = \int_a^t \mathbf{U_z}[t,r]f(r)dr,$$

$$\Rightarrow \mathbf{U_z}[t,r]v_z(r)|_a^t = \int_a^t \mathbf{U_z}[t,r]f(r)dr,$$

$$\Rightarrow \mathbf{U_z}[t,r]v_z(t) - \mathbf{U_z}[t,a]v_z(a) = \int_a^t \mathbf{U_z}[t,r]f(r)dr,$$

$$\Rightarrow v_z(t) = \mathbf{U_z}[t,a]\Phi + \int_a^t \mathbf{U_z}[t,r]f(r)dr,$$

giving uniqueness. Since $\mathbf{U_z}[t,s]\Phi \to \mathbf{U}[t,a]\Phi$, it follows that $u_z(t) \to u(t)$, completing our proof.

4. Parabolic Evolution Equations on $\mathcal{FD}_{\otimes}^2$

In this section, we will deal with the solution of the initial value problem in the time-ordered setting,

$$\begin{cases} \frac{du(t)}{dt} + \mathcal{A}(t)u(t) = f(t), \quad 0 \le s < t \le T, \\ u(s) = \Phi, \end{cases} \tag{4.1}$$

which we call the parabolic equation, by taking the term $\mathcal{A}(t)u(t)$ to the left side of the equation, to avoid some of the notational difficulties associated with using fractional powers of $\mathcal{A}(t)$.

Our approach is to first solve the problem for $\mathcal{A}_z(t)$ and then obtain the solution to (3.1) as a limit as in the hyperbolic case. When we consider the homogeneous parabolic initial value problem,

$$\begin{cases} \frac{du_z(t)}{dt} + \mathcal{A}_z(t)u_z(t) = 0, \quad 0 \le s < t \le T, \\ u_z(s) = \Phi, \end{cases} \tag{4.2}$$

we say that the evolution system will be obtained by a different method than the evolution system obtained in the hyperbolic case.

Suppose that $-\mathcal{A}_z(t)$ is the infinitesimal generator of the C_0 semigroup $\mathbf{Q_z}[t,s]$ for every value of t in the interval $[0,T]$ and $\mathbf{U_z}[t,s]$ is an evolutionary system of parabolic problem for $s > 0$ as follows:

$$\mathbf{U_z}[t,s] = e^{-\mathbf{Q_z}[t-s,s]} + \int_s^t e^{-\mathbf{Q_z}[t-\tau,\tau]}R[\tau,s]d\tau, \tag{4.3}$$

then $\mathbf{U_z}[t,s]$ should satisfy equation (4.2). So let us first take the derivative of $\mathbf{U_z}[t,s]$ with respect to t

$$\frac{\partial \mathbf{U_z}[t,s]}{\partial t} = -\mathcal{A}_z(t-s)e^{-\int_s^t \mathcal{A}_z(\tau)d\tau} + \Big[-\int_s^t \mathcal{A}_z(t-\tau)e^{-\int_\tau^{t-\tau} \mathcal{A}_z(\ell)d\ell}R(\tau,s)d\tau\Big] + e^{-\int_t^0 \mathcal{A}_z(\ell)d\ell}.$$

Now we consider equation (4.2) for $\mathbf{U_z}[t,s]$

$$\begin{aligned}\frac{\partial \mathbf{U_z}[t,s]}{\partial t}+\mathcal{A}_z(t)\mathbf{U_z}[t,s] &= [-\mathcal{A}_z(t-s)]e^{-\mathbf{Q_z}[t-s,a]}+\\ &+\int_s^t [-\mathcal{A}_z(t-\tau)]e^{-\mathbf{Q_z}[t-\tau,a]}R(\tau,s)d\tau+\\ &+ e^{-\int_0^a \mathcal{A}_z(\ell)d\ell} R(t,s)+\mathcal{A}_z(t)\Big[e^{-\mathbf{Q_z}[t-s,a]}+\\ &+\int_s^t e^{-\mathbf{Q_z}[t-\tau,a]}R(\tau,s)d\tau\Big]\\ &= 0,\end{aligned}$$

$$\begin{aligned}\Rightarrow\ & \Big[\mathcal{A}_z(t)-\mathcal{A}_z(t-s)\Big]e^{-\mathbf{Q_z}[t-s,a]}+\\ +\ & \int_s^t \Big[\mathcal{A}_z(t)-\mathcal{A}_z(t-\tau)\Big]e^{-\mathbf{Q_z}[t-\tau,a]}R(\tau,s)d\tau+\\ +\ & e^{\int_0^a \mathcal{A}_z(\tau)d\tau} R(t,s)=0.\end{aligned} \tag{4.4}$$

If we denote the factor $\Big[\mathcal{A}_z(t)-\mathcal{A}_z(t-s)\Big]e^{-\mathbf{Q_z}[t-s,a]}$ in this last equation by $R_1(t,s)$, then we write the equation (4.4) as

$$-R_1(t,s)-\int_s^t R_1(t,\tau)R(\tau,s)d\tau+R(t,s)e^{\int_0^a \mathcal{A}_z(\tau)d\tau}=0, \tag{4.5}$$

and from here we get

$$R(t,s)=R_2(t,s)+\int_s^t R_2(t,\tau)R(\tau,s)d\tau, \tag{4.6}$$

where

$$R_2(t,s)=R_1(t,s)e^{-\mathbf{Q_z}[a,0]}=\Big[\mathcal{A}_z(t-s)-\mathcal{A}_z(t)\Big]e^{-\int_0^{t-s}\mathcal{A}_z(\tau)d\tau}. \tag{4.7}$$

Since $\mathbf{U_z}[t,s]$ is an evolution system of (4.2), it follows from (4.4), (4.5) and (4.7) that the integral equation (4.6) giving $R(t,s)$ must be satisfied. In order to express $\mathbf{U_z}[t,s]$ giving by (4.3), it is sufficient to solve the integral equation denoting by $R(t,s)$. For this we will need the following assumptions:

1. For each $t\in I$, $\mathcal{A}_z(t)$ is closed and densely defined, $R(\lambda;\ \mathcal{A}_z(t))$ exists in the sector Δ satisfying
$$\|R\left(\lambda,\mathcal{A}_z(t)\right)\| \leqslant \tfrac{M}{|\lambda|+1} \text{ for } \lambda\in\Delta,\ Re(\lambda)\leq 0,\ t\in I.$$
2. The function $\mathcal{A}_z^{-1}(t)$ is continuously differentiable on I.
3. There are constants $C_1>0$ and $\rho:\ 0<\rho<1$, such that, for each $\lambda\in\Delta$ and every $t\in I$, we have
$$\|D_t R(\lambda;\ \mathcal{A}_z(t))\|\leq C_1/|\lambda|^{1-\rho}.$$

4. The function $D\mathcal{A}_z^{-1}(t)$ is Holder continuous on $\mathcal{H}$ and there are positive constants $C_2,\ \alpha$ such that

$$\left\|D\mathcal{A}_z^{-1}(t) - D\mathcal{A}_z^{-1}(s)\right\| \le C_2 \left|t-s\right|^{\alpha},\ s,\ t \in I.$$

It is clear that the time ordered version of condition (4) will not be true if formulated as a direct translation. For this we need:

Definition 4.1. *[4] An exchange operator $\mathbf{E}[t,t']$ on $L^{\#}[\mathcal{FD}_{\otimes}^2]$ is a linear map defined for pairs t,t' such that:*

1. $\mathbf{E}[t,t'] : L[\mathcal{H}(t)] \to L[\mathcal{H}(t')]$, *(isometric isomorphism),*
2. $\mathbf{E}[s,t']\mathbf{E}[t,s] = \mathbf{E}[t,t']$,
3. $\mathbf{E}[t,t']\mathbf{E}[t',t] = I$,
4. *for* $s \neq t,\ t'$, $\mathbf{E}[t,t']\mathcal{A}_z(s) = \mathcal{A}_z(s)$, *for all* $\mathcal{A}_z(s) \in L[\mathcal{B}_z(s)]$.

The exchange operator acts to exchange the time positions of a pair of operators in a more complicated expression.

Theorem 4.1. *[4] (Existence) There exists an exchange operator for $L^{\#}[\mathcal{FD}_{\otimes}^2]$.*

The time ordered version of Fattorini's conditions become:
(P1) For each $t \in I$, $\mathcal{A}(t)$ is closed and densely defined, $R(\lambda;\mathcal{A}(t))$ exists in the sector Δ satisfying

$$\left\|R\left(\lambda,\mathcal{A}(t)\right)\right\| \leqslant \tfrac{M}{|\lambda|+1} \text{ for } \lambda \in \Delta,\ Re(\lambda) \le 0,\ t \in I.$$

(P2) The function $\mathcal{A}_z^{-1}(t)$ is continuously differentiable on I.
(P3) There are constants $C_1 > 0$ and $\rho:\ 0<\rho<1$, such that, for each $\lambda \in \Delta$ and every $t \in I$, we have

$$\|D_t R(\lambda;\ \mathcal{A}_z(t))\| \le C_1/|\lambda|^{1-\rho}.$$

(P4) The function $D\mathcal{A}_z^{-1}(t)$ is Holder continuous on $\mathcal{FD}_{\otimes}^2$ and there are positive constants $C_2,\ \alpha$ such that

$$\left\|E[\tau,t]D\mathcal{A}_z^{-1}(t) - E[\tau,s]D\mathcal{A}_z^{-1}(s)\right\| \le C_2\left|t-s\right|^{\alpha},\ \tau,s,\ t \in I.$$

It is easy to show that the first condition implies that $\mathcal{A}_z$ generates an analytic contraction semigroup for each $t \in I$.

The four conditions allow Fattorini to prove the following theorem.

Theorem 4.2. *[4]* (Fattorini) *Let the family $A(t),\ t \in I$, have a common dense domain and satisfy assumptions* $(1)-(4)$. *Then the problem*

$$\frac{\partial u(t)}{\partial t} = A(t)u(t), \quad u(a) = u_a,$$

has a unique solution $u(t) = V(t,s)u_a$, for $t,\ s \in I$. Furthermore,

1. *$V(t,s)$ is strongly continuous on I and continuously differentiable (in $\mathcal{H}$ norm) with respect to both s and $t \in I$,*
2. *$V(t,s)\mathcal{H} \subset D(A(t))$,*
3. *$A(t)V(t,s)$ and $V(t,s)A(s)$ are bounded,*
4. *$D_tV(t,s) = A(t)V(t,s),\ D_sV(t,s) = -\overline{V(t,s)A(s)}$, and*
5. *for $t,\ s \in I$,*

$$\|D_tV(t,s)\| \le C/(t-s),\ \ \|D_sV(t,s)\| \le C/(t-s).$$

Fattorini requires seven pages plus five pages of preparatory work to prove this theorem (see [23], pg. 397). The proof of (essentially) the same theorem, requires seventeen pages in Pazy ([17], pg. 149). The following example shows the existence of solutions without a common dense domain (see Fattorini [23], pg. 408).

Example 4.3. *Let the family of operators $A(t)$, $t \in I = [0,1]$, be defined on $\mathcal{H} = L^2(0,1)$ by*

$$A(t)u(r) = -\frac{1}{(t-r)^2}u(r).$$

It is easy to see that each $A(t)$ is selfadjoint and $(A(t)u,u) \le -\|u\|_{\mathcal{H}}^2$ for $u \in D(A(t))$. It follows that the spectrum of $A(t)$, $\sigma(A(t)) \subset (-\infty,-1]$, for $t \in [0,1]$. The first condition is satisfied for any $\delta \in (0,\pi/2)$, while the second condition is clear, and makes the fourth condition obvious. For $\lambda \notin (-\infty,-1]$, we have

$$R(\lambda;\ A(t))u(r) = \frac{(t-r)^2}{\lambda(t-r)^2+1}u(r),$$

so that

$$\|R(\lambda;\ A(t))u(r)\|_{\mathcal{H}}^2 = \int_0^1 \frac{(t-r)^4}{[\lambda(t-r)^2+1]^2}u^2(r)dr \le \frac{1}{|\lambda|^2}\|u\|_{\mathcal{H}}^2.$$

It is now clear that each $A(t)$ generates a contraction semigroup and

$$D_t R(\lambda;\ A(t))u(r) = \frac{2(t-r)}{[\lambda(t-r)^2+1]^2}u(r).$$

From here, an easy estimation shows that for $\lambda \in \Delta$ and $\alpha \in [0,1]$,

$$2\alpha\left|\lambda\alpha^2+1\right|^{-2} \leqslant C|\lambda|^{-1/2},$$

for some constant C, so that

$$\|D_t R(\lambda;\ A(t))\|_{\mathcal{H}} \le \frac{C}{|\lambda|^{1/2}},$$

so that the third condition follows. We now notice that

$$(A(t)-A(s))\,A(\tau)^{-1} = \left[\frac{(\tau-r)^2}{(s-r)} + \frac{(\tau-r)^2}{(t-r)}\right](s-t),$$

so that, for some constants $C>0,\ 0<\beta\le 1$, we have

$$\left\|(A(t)-A(s))\,A(\tau)^{-1}\right\| \leqslant C\,|t-s|^{\beta}\ (a.s)\ \text{ for all } t,\ s,\ \tau \in [0,1].$$

The above theorem would follow if the family had a common dense domain. It is easy to see that $D(A(t)) = \left\{u(r) \in L^2[0,1]\,|r \ne t\right\}$, for each $t \in [0,1]$, so that $\bigcap_{t\in I} D(A(t)) = \{0\}$. Thus the restriction to a common dense domain is an undesirable condition.

The time ordered version of Fattorini's theorem becomes:

Theorem 4.4. *Let the family $A(t)$, $t \in I$ satisfy assumptions $(1)-(4)$. Then the corresponding time-ordered family $\mathcal{A}_z(t)$, $t \in I$ has a HK-integral $\mathbf{Q}_z[t,s]$ on $\mathcal{FD}_{\otimes}^2$, which generates an analytic evolution operator*
$U_z(t,s)$, for $a \le s < t \le b$ and:

$$\frac{\partial u(t)}{\partial t} = \mathcal{A}_z(t)u(t), \quad \lim_{t,s\to a} u(t) = u_a,$$

has a unique solution $u(t) = U_z(t,s)u(s)$, for $t,\ s \in I$. Furthermore,

1. *$U_z(t,s)$ is strongly continuous on I and continuously differentiable with respect to both s and $t \in (a,b)$,*
2. *$U_z(t,s)\mathcal{FD}^2_{\otimes} \subset D(\mathcal{A}_z(t))$.*
3. *$\mathcal{A}_z(t)U_z(t,s)$ and $U_z(t,s)\mathcal{A}_z(s)$ are bounded,*
4. *$D_tU_z(t,s) = \mathcal{A}_z(t)U_z(t,s),\ D_sU_z(t,s) = -\overline{U_z(t,s)\mathcal{A}_z(s)}$, and*
5. *for $t,\ s \in I$,*
$$\|D_tU_z(t,s)\| \le C/(t-s), \quad \|D_sU_z(t,s)\| \le C/(t-s).$$

Proof. The proof of (1) and (2) follows from Theorem 2.8, Theorem 3.4 and Theorem 3.5. Proofs of (3)-(5) are the same as in Fattorini. □

Some consequences of these assumptions are collected in the next lemma.

Lemma 4.5. *Let (1)-(4) be satisfied then*

$$\begin{aligned} \left\| D\mathbf{Q}_z^{-1}[t,a] - D\mathbf{Q}_z^{-1}[s,a] \right\| &\le K|t-s|^{\alpha} \\ & \textit{for} \quad s \in (0,T], t_1, t_2 \in [0,T], \end{aligned} \tag{4.8}$$

$$\begin{aligned} \left\| \mathbf{Q}_z[\tau,a]\left[e^{-\mathbf{Q}_z[s_2,a]} - e^{-\mathbf{Q}_z[s_1,a]}\right] \right\| &\le \frac{C}{s_1 s_2}|s_2 - s_1| \\ & \textit{for} \quad 0 < s_1, s_2 \in (0,T], t, \tau \in [0,T], \end{aligned} \tag{4.9}$$

$$\begin{aligned} \left\| \mathbf{Q}_z[t,a]\left[e^{-\mathbf{Q}_z[\tau_1,a]} - e^{-\mathbf{Q}_z[\tau_2,a]}\right] \right\| &\le \frac{C}{a}|\tau_2 - \tau_1|^{\alpha} \\ & \textit{for} \quad s \in (0,T], t, \tau_1, \tau_2 \in [0,T]. \end{aligned} \tag{4.10}$$

Also, the operator $\mathbf{Q}_z[t,a]e^{-\mathbf{Q}_z[s,a]}$ is bounded on $\mathcal{FD}^2_{\otimes}$, for $s \in (0,T], \tau, t \in [0,T]$ and in the uniform operator topology, this operator is uniformly continuous for $s \in (0,T], \tau, t \in [0,T]$,for every $\varepsilon > 0$.

Proof: Let us prove the first inequality

$$\left\| D\mathbf{Q}_z^{-1}[t,a] - D\mathbf{Q}_z^{-1}[s,a] \right\| \le K|t-s|^{\alpha}.$$

$D\mathbf{Q}_z^{-1}[t,a]$ is as follows:

$$D\mathbf{Q}_z^{-1}[t,a] = \mathbf{Q}_z[t,a]\mathbf{Q}_z^{-1}[\tau,a].$$

Then we get

$$\begin{aligned} \left\| [\mathbf{Q}_z[t_1,a] \right. &- \left. \mathbf{Q}_z[t_2,a]]e^{-\mathbf{Q}_z[\tau,s]} \right\| = \\ &= \left\| [\mathbf{Q}_z[t_1,a] - \mathbf{Q}_z[t_2,a]]\mathbf{Q}_z^{-1}[\tau,a]\mathbf{Q}_z[\tau,a]e^{-\mathbf{Q}_z[\tau,s]} \right\| \\ &= \left\| D\mathbf{Q}_z^{-1}[t_1,a] - D\mathbf{Q}_z^{-1}[t_2,a] \right\| \left\| \mathbf{Q}_z[\tau,a]e^{-\mathbf{Q}_z[\tau,s]} \right\| \\ &\le c_2|t_1 - t_2|^{\alpha} C \\ &\le K|t_1 - t_2|^{\alpha}. \end{aligned}$$

Now to prove (4.9) let $0 < s_1 \le s_2$ and $\Phi \in \mathcal{FD}^2_{\otimes}$. Since $T(t)x - T(s)x = \int_s^t T(\tau)A(x)dx$ for any semigroup, we can write the following equation for our semigroup

$$e^{-\mathbf{Q}_z[s_2,a]}\Phi - e^{-\mathbf{Q}_z[s_1,a]}\Phi = -\int_{s_1}^{s_2} \mathbf{Q}_z[\tau,a]e^{-\mathbf{Q}_z[\sigma,a]}\Phi d\sigma.$$

And then we get

$$\begin{aligned}
\mathbf{Q}_z[\tau,a][e^{-\mathbf{Q}_z[s_2,a]} - e^{-\mathbf{Q}_z[s_1,a]}]\Phi &= \mathbf{Q}_z[\tau,a]\Big[-\int_{s_1}^{s_2} \mathbf{Q}_z[\tau,a]e^{-\mathbf{Q}_z[\sigma,a]}\Phi d\sigma\Big] \\
&= -\int_{s_1}^{s_2} [\mathbf{Q}_z[\tau,a]]^2 e^{-\mathbf{Q}_z[\sigma,a]}\Phi d\sigma \\
&= -\int_{s_1}^{s_2} \Big[\mathbf{Q}_z[\tau,a]]e^{-\mathbf{Q}_z[\frac{\sigma}{2},a]}\Big]^2 \Phi d\sigma.
\end{aligned}$$

and

$$\begin{aligned}
\Big\|\mathbf{Q}_z[t,a][e^{-\mathbf{Q}_z[s_2,a]} - e^{-\mathbf{Q}_z[s_1,a]}]\Phi\Big\| &\leq \Big\|\mathbf{Q}_z[t,a]\mathbf{Q}_z^{-1}[\tau,a]\Big\| \\
&\quad \times \Big\|\mathbf{Q}_z[\tau,a][e^{-\mathbf{Q}_z[s_2,a]} - e^{-\mathbf{Q}_z[s_1,a]}]\Phi\Big\| \\
&\leq \Big\|\mathbf{Q}_z[t,a]\mathbf{Q}_z^{-1}[\tau,a]\Big\| \\
&\quad \times \Big\|-\int_{s_1}^{s_2} \Big[\mathbf{Q}_z[\tau,a]]e^{-\mathbf{Q}_z(\frac{\sigma}{2},a)}\Big]^2 \Phi d\sigma\Big\| \\
&\leq C\Big\|\Phi\Big\| \int_{s_1}^{s_2} \frac{1}{\sigma^2} d\sigma \\
&= \frac{C\Big\|\phi\Big\|}{s_1 s_2}|s_2 - s_1|.
\end{aligned}$$

Thus, we obtain

$$\Big\|\mathbf{Q}_z(\tau,a)\Big[e^{-\mathbf{Q}_z[s_2,a]} - e^{-\mathbf{Q}_z[s_1,a]}\Big]\Big\| \leq \frac{C}{s_1 s_2}|s_2 - s_1|.$$

To prove (4.10) note that from (P1) it follows that

$$\Big\|R(\lambda;\mathbf{Q}_z[s,t-s]\Big\| \leq \frac{1}{|\lambda|}, \qquad \lambda \in \Sigma, t \in I,$$

and therefore we have

$$\begin{aligned}
&\Big\|\mathbf{Q}_z[t,a]\Big[R(\lambda;\mathbf{Q}_z[\tau_1,a]) - R(\lambda;\mathbf{Q}_z[\tau_2,a])\Big]\Big\| = \\
= &\Big\|\mathbf{Q}_z[t,a]R(\lambda;\mathbf{Q}_z[\tau_1,a])\Big[\mathbf{Q}_z[\tau_1,a] - \mathbf{Q}_z[\tau_2,a]\Big]R(\lambda;\mathbf{Q}_z[\tau_2,a])\Big\| \\
\leq &\Big\|\mathbf{Q}_z[t,a]\mathbf{Q}_z^{-1}[\tau_1,a]\Big\| \Big\|\mathbf{Q}_z[\tau_1,a]R(\lambda;\mathbf{Q}_z[\tau_1,a])\Big\| \\
&\times\Big\|\Big[\mathbf{Q}_z[\tau_1,a] - \mathbf{Q}_z[\tau_2,a]\Big]\mathbf{Q}_z^{-1}[\tau_2,a]\Big\| \Big\|\mathbf{Q}_z[\tau_2,a]R(\lambda;\mathbf{Q}_z[\tau_2,a])\Big\| \\
\leq &K|\tau_1 - \tau_2|^\alpha.
\end{aligned}$$

As a result, we get the desired inequality as follows, with Γ being a smooth path

$$\mathbf{Q}_z[t,a]e^{-\mathbf{Q}_z[\tau_1,a]}\Phi - \mathbf{Q}_z[t,a]e^{-\mathbf{Q}_z[\tau_2,a]}\Phi = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{-\lambda a}\mathbf{Q}_z[t,a] \times \Big[R(\lambda;\mathbf{Q}_z[\tau_1,a]) - R(\lambda;\mathbf{Q}_z[\tau_2,a])\Big]\Phi d\lambda,$$

$$\implies \Big\|\mathbf{Q}_z[t,a]e^{-\mathbf{Q}_z[\tau_1,a]}\Phi - \mathbf{Q}_z[t,a]e^{-\mathbf{Q}_z[\tau_2,a]}\Phi\Big\| \leq K|\tau_1-\tau-2|^{\alpha}\Big\|\Phi\Big\|\int\limits_{\Gamma} e^{-\lambda a}d\lambda \leq \frac{K}{a}|\tau_1-\tau_2|^{\alpha}\Big\|\Phi\Big\|,$$

$$\implies \Big\|\mathbf{Q}_z[t,a]\Big[e^{-\mathbf{Q}_z(\tau_1,a)} - e^{-\mathbf{Q}_z(\tau_2,a)}\Big]\Big\| \leq \frac{C}{a}|\tau_2-\tau_1|^{\alpha}. \qquad \square$$

We have not discussed classical, mild, strong or weak solutions; the inhomogeneous problem or the question of asymptotic behavior of solutions, because these topics receive no additional benefit from lifting them to the $\mathcal{FD}_{\otimes}^2$ setting. However, lifting a problem to the $\mathcal{FD}_{\otimes}^2$ setting can have benefit even if the operators are not time dependent. For a detailed discussion, we refer Section 7.5 in [4].

5. Conclusion

In this paper, we have reviewed the Gill-Zachary implementation of Feynman's time ordered operator calculus and showed how it simplifies the study of the hyperbolic and parabolic evolution equations. In closing, we highlight some of the advantages of extending operators to $\mathcal{FD}_{\otimes}^2$ setting:

1. This allows operators acting at different times commute.
2. This lifts the restriction of a common dense domain.
3. This allows weaker continuity conditions on the time dependence.
4. The Yosida approximation can be used to reduce problems to the bounded case.
5. The HK-integral allows terms with non-absolutely convergent integrals.
6. Advantages are possible even if the operators are not explicitly time dependent (see [4]).

Acknowledgments

All our respects and appreciations to the editor and reviewers for their evaluation and valuable remarks. The first, third, fourth, sixth and seventh authors were supported by Research Fund of the Yildiz Technical University. Project Number:FBA-2023-5765".

References

1. R. P. Feynman, *The theory of positrons*, Phys. Rev. 76, 749-759, (1949).
2. R. P. Feynman, *Space-time approach to quantum electrodynamics*, Phys. Rev. 76, 769-789, (1949).
3. T. L. Gill and W. W. Zachary, *Foundations for Relativistic Quantum Theory I: Feynman's Operator Calculus and the Dyson Conjectures*, J. Math. Phys. 43, 69-93, (2002).
4. T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman Operator Calculus*, Springer, New York, (2016).
5. F. J. Dyson, *The radiation theories of Tomonaga, Schwinger, and Feynman*, Physical Review 75, 486-502, (1949).
6. I. Fujiwara, *Operator calculus of quantized operator*, Prog. Theor. Phys. 7, 433-448, (1952).
7. W. L. Miranker and B. Weiss, *The Feynman operator calculus*, SIAM Rev. 8, 224-232, (1966).
8. E. Nelson, *Operants: a functional calculus for non-commuting operators*, in "Functional Analysis and Related Fields", Springer, Berlin, (1970).
9. H. Araki, *Expansional in Banach algebras*, Ann. scient. Ec. Norm. Sup. Math. 4, 67-84, (1973).

10. V. P. Maslov, *Operational Methods*, English translation 1976 (revised from the Russian edition), Moscow, (1973).

11. G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus*, Oxford U. Press, New York, (2000).

12. T. L. Gill and G. A. De Parga, *Foundations for QED, Feynman operator calculus, Dyson conjectures, and Einstein's dual theory*, J. Phys.: Conf. Ser. 2482(1), 012015, (2023).

13. A. Constantin, *The Construction of an Evolution System in the Hyperbolic Case and Applications*, Math. Nachr. 224, 49-73, (2001).

14. G. Da Prato and M. Iannelli, *On a method for studying abstract evolution equations in the Hyperbolic case*, Comm. in Partial Diff. Eqs. 1, 585-608, (1976).

15. T. Kato, *Integration of the equation of evolution in a Banach space*, J. Math. Soc. Japan 5, 208-234, (1953).

16. K. Kobayasi, *On a theorem for linear evolution equations of hyperbolic type*, J. Math. Soc. Japan 31(4), 647-654, (1979).

17. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, (Appl. Math. Sci., vol. 44), Springer-Verlag, New York, (1983).

18. L. Tebou, *Stabilization of some coupled hyperbolic/parabolic equations*, Discrete and Continuous Dynamical Systems - B 14(4), 1601-1620, (2010).

19. S. Pankavich, P. Radu, *Nonlinear instability of solutions in parabolic and hyperbolic diffusion*, Evolution Equations and Control Theory 2(2), 403-422, (2013).

20. X. Dai, *Hyperbolic and parabolic equations with several opposite-sign source at critical initial energy level in heat and vibrating systems*, Nonlinear Analysis 195, 111752, (2020).

21. A. Agresti and M. Veraar, *Nonlinear parabolic stochastic evolution equations in critical spaces Part I. Stochastic maximal regularity and local existence*, Nonlinearity 35(8), 4100, (2022).

22. T. L. Gill and W. W. Zachary, *Feynman operator calculus: the constructive theory*, Expo. Math. 29, 165–203, (2011).

23. H. O. Fattorini, *The Cauchy Problem*, Addison-Wesley, London, (1983).

Erdal Gül,
Department of Mathematics,
Yıldız Technical University,
Türkiye.
E-mail address: gul@yildiz.edu.tr

and

Tepper L. Gill,
Department of Mathematics,
Howard University,
USA.
E-mail address: tgill@howard.edu

and

Özlem Bakşi,
Department of Mathematics,
Yıldız Technical University,
Türkiye.
E-mail address: baksi@yildiz.edu.tr

and

Seda Kızılbudak Çalışkan,
Department of Mathematics,
Yıldız Technical University,
Türkiye.
E-mail address: skizilb@yildiz.edu.tr

and

Ayşe Kara Hansen,
Department of Mathematics,
Aarhus University,
Denmark.
E-mail address: aysekara66@gmail.com

and

Serpil Karayel,
Department of Mathematics,
Yıldız Technical University,
Türkiye.
E-mail address: serpil@yildiz.edu.tr

and

Yonca Sezer,
Department of Mathematics,
Yıldız Technical University,
Türkiye.
E-mail address: ysezer@yildiz.edu.tr