Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.70502

Almost periodic solutions for a class of neutral integro-differential equations *

Marko Kostić and Daniel Velinov †

ABSTRACT: In this paper, we investigate the existence and uniqueness of almost periodic mild solutions for a class of neutral integro-differential equations in Banach spaces. We essentially apply the results from the fixed point theory. At the end of paper, we present some illustrative examples to show the effectiveness of the obtained findings.

Key Words: almost periodic solutions, mild solutions, existence and uniqueness, neutral differential equations, neutral integro-differential equations.

Contents

1. Introduction and preliminaries

The initial analysis of the class of almost periodic functions can be traced back to H. Bohr (1924–1926). For comprehensive insights into almost periodic functions and their applications, the readers are directed to the research monographs such as [2], [5], [9], [14]–[16], [17] and [31]. The significance of almost periodic functions is paramount in the qualitative analysis of solutions to (nonlinear) integro-differential equations within the domain of Banach spaces.

Functional differential equations, specifically referred to as neutral differential equations, are encountered in a variety of phenomena, particularly in the analysis of oscillatory systems and the modeling of various physical problems. For further details and in-depth exploration, interested readers are encouraged to refer to [12], [22], and the references provided therein. The problem of existence and uniqueness of almost periodic solutions of integro-differential equations and neutral integro-differential equations is very popular since they play a crucial role in modeling dynamic systems with delayed interactions, reflecting scenarios encountered in various scientific and engineering areas.

The exploration of generalized periodicity in the solutions of diverse classes of neutral differential equations was initially prompted by considerations of periodic behavior in their solutions. Key contributions to this line of inquiry can be found in works such as [4], [13], [19], [20], [21], [23], [24], [28], and [29]. Researchers E. Ait Dads and K. Ezzinbi [8], as well as A. Fink and J. Gatica [11], extended this exploration by examining almost periodic solutions for specific classes of nonlinear neutral integral equations. Subsequent investigations by S. Abbas and D. Bahuguna [1] and X. Chen and F. Lin [6] further expanded the scope, examining almost periodicity in more general neutral functional differential equations within Banach spaces. Notably, the pursuit of almost periodic solutions for a class of nonlinear integro-differential equations with neutral delay is presented in [32]. Additionally, in a related but distinct conceptualization of almost periodicity, studies on positive pseudo almost periodicity [7], [10], and investigations into (μ, ν)-pseudo S-asymptotic ω-periodicity [27] for solutions of various classes of neutral differential equations have been conducted.

In the work by M. Ayachi [3], a set of sufficient conditions is established to ensure the existence

* This research is partially supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia and Bilateral project between MANU and SANU.
† Corresponding author

Submitted November 23, 2023. Published June 23, 2024
2010 *Mathematics Subject Classification:* 34C27, 34C25, 34K40, 12H20.

and global exponential stability of measure-pseudo almost periodic solutions within a specific class of bi-directional associative memory neural networks. Similarly, in the study conducted by T. Liang, Y. Q. Yang, Y. Liu and L. Li, [18], specific sufficient conditions are provided for the existence and global exponential stability of almost periodic solutions in Cohen–Grossberg neural networks on time scales. These references, along with the cited references therein, underscore the broader significance of almost periodic solutions, transcending theoretical considerations and finding practical applications in diverse fields such as neuroscience, physics, biology, and engineering. The inclusion of illustrative examples in the paper serves to accentuate the practical relevance of almost periodic solutions in addressing real-world problems. Moreover, the paper motivates further exploration of these mathematical concepts within the scientific community, building upon recent results in [3], [18], and [26].

This paper presents a novel contribution by delineating inherently natural conditions under which the specified class of neutral integro-differential equations, defined in a Banach space, exhibits (unique) almost periodic mild solutions. The versatile nature of the considered class, coupled with tailored accommodations and constraints, renders it applicable to modeling scenarios across diverse scientific domains. From applications in neuroscience and dynamical systems to addressing classical engineering problems, this class of equations emerges as a valuable and broadly applicable tool, amplifying its significance in various scientific contexts.

The paper is structured as follows. Following a concise overview of recent literature, we delve into the presentation of results and the underlying motivation driving this research. Subsequently, we expound upon basic notation and auxiliary results pertaining to almost periodic functions and neutral integro-differential equations within the context of Banach spaces. The second section of the paper houses the main contributions, encompassing the articulation of necessary conditions for the uniqueness of almost periodic mild solutions within the considered class of neutral integro-differential equations. Additionally, we delineate the necessary conditions for the existence of almost periodic solutions. The section concludes with illustrative examples, underscoring the significance of the results and extending previously established findings. The paper ends with a conclusion that underscores the attained results and outlines future directions for investigation.

1.1. Preliminaries

Let X be a complex Banach space, equipped with the norm $\|\cdot\|$ and $I=[0,\infty)$. If $f:\mathbb{R}\to X$ is a continuous function and $\epsilon>0$, then a number $\tau>0$ is said to be an ϵ-period for $f(\cdot)$ if

$$\|f(t+\tau)-f(t)\|\leq\epsilon,\quad t\in\mathbb{R}.$$

The set consisting of all ϵ-periods for $f(\cdot)$ will be denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic if for each $\epsilon>0$ the set $\vartheta(f,\epsilon)$ is relatively dense in $[0,\infty)$ meaning that there exists $l>0$ such that any subinterval of $[0,\infty)$ of length l meets $\vartheta(f,\epsilon)$. By $AP(\mathbb{R}:X)$ will be denoted the Banach space of all almost periodic functions $f:\mathbb{R}\to X$. Then $AP(\mathbb{R}:X)$ is a Banach space equipped with the supremum norm given by

$$\|u\|_{AP(\mathbb{R}:X)}:=\sup_{t\in\mathbb{R}}\|u(t)\|.$$

A continuous function $f:\mathbb{R}\times X\to X$ is said to be almost periodic in t uniformly for $u\in X$ if for each $\varepsilon>0$ and for each compact subset K of X, the set of all real numbers τ such that

$$\|f(t+\tau,u)-f(t,u)\|\leq\epsilon,\quad t\in\mathbb{R},\ u\in K$$

is relatively dense in $[0,\infty)$. If a continuous function $f:\mathbb{R}\times X\times Y\to X$ is given, then the number τ is said to be ϵ-period for $f(\cdot,\cdot,\cdot)$ if

$$\|f(t+\tau,u,v)-f(t,u,v)\|\leq\epsilon,\quad t\in\mathbb{R},\ u\in X,\ v\in Y.$$

The set consisting of all ϵ-periods for $f(\cdot,\cdot,\cdot)$ will be denoted by $\vartheta_{X,Y}(f,\epsilon)$. The continuous function $f:\mathbb{R}\times X\times Y\to X$ is said to be almost periodic in t uniformly for $(u,v)\in X\times Y$ if for each $\varepsilon>0$ and for each compact subset E of $X\times Y$ such that the set $\vartheta_{X,Y}(f,\epsilon)$ is relatively dense in $[0,\infty)$.

Here we are going to investigate the existence and uniqueness of almost periodic solutions of neutral integro-differential equation

$$u'(t) = Au(t) + f\big(t, u_t, Fu(t)\big), \quad t \in \mathbb{R}, \tag{1.1}$$

and the integral equation

$$Fu(t) = \int_{-\infty}^{t} k(t-s)g\big(s, u_s\big)\, ds,$$

where A is linear operator on the Banach space X, $f : \mathbb{R} \times \mathcal{C} \to X$ and $g : \mathbb{R} \times \mathcal{C} \times X \to X$ are bounded functions on bounded sets, $k \in L^1(I, I)$ is continuous, nonincreasing function and $u_r(t) = u(t + r)$, for $r \in [-m, 0]$, $m \geq 0$ is a fixed constant, where $\mathcal{C}$ is the space of continuous functions from $[-m, 0]$ to X equipped with the supremum norm.

In this paper, we will consider the case when the operator A is the infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) on X:

(A) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$, such that there exists constant $C, \sigma > 0$ such that $\|T(t)\| \leq Ce^{\sigma t}$, for $t \in [0, \infty)$.

Additionally, in certain statements, we will going to consider the following assumptions:

(C1) The function $k(t) \in L^1(\mathbb{R} : \mathbb{C})$ is continuous and nonincreasing;

(C2) $g \in AP(\mathbb{R} \times \mathcal{C} : X)$;

(C3) $f \in AP(\mathbb{R} \times \mathcal{C} \times X : X)$;

(C4) There exists a positive constant L_g such that $\|g(t, \phi) - g(t, \psi)\| \leq L_g \|\phi - \psi\|_{\mathcal{C}}$;

(C5) There exists a positive constant L_f such that $\|f(t, \phi_1, \psi_1) - f(t, \phi_2, \psi_2)\| \leq L_f\big(\|\phi_1 - \phi_2\|_{\mathcal{C}} + \|\psi_1 - \psi_2\|\big)$;

(C6) There exists a continuous and nondecreasing function $L_g : \mathbb{R} \to \mathbb{R}$ such that for each $\nu > 0$, and $\phi,\ \psi \in \mathcal{C}$ such that $\|\phi\|_{\mathcal{C}} \leq \nu$ and $\|\psi\|_{\mathcal{C}} \leq \nu$, we have

$$\|g(t, \phi) - g(t, \psi)\| \leq L_g(\nu)\|\phi - \psi\|_{\mathcal{C}},$$

$t \in \mathbb{R}$, where $L_g(0) = 0$;

(C7) There exists a continuous and nondecreasing function $L_f : \mathbb{R} \to \mathbb{R}$ such that for each $\nu > 0$, and $\phi_i \in \mathcal{C}$, $\psi_i \in X$ such that $\|\phi_i\|_{\mathcal{C}} \leq \nu$ and $\|\psi_i\| \leq \nu$, $i = 1, 2$ we have

$$\|f(t, \phi_1, \psi_1) - f(t, \phi_2, \psi_2)\| \leq L_f(\nu)\big(\|\phi_1 - \phi_2\|_{\mathcal{C}} + \|\psi_1 - \psi_2\|\big),$$

$t \in \mathbb{R}$, where $L_f(0) = 0$;

(C8) Let $f \in C(\mathbb{R} \times \mathcal{C} \times X : X)$. There exist a bounded measurable function $a_f : \mathbb{R} \to X$ and a constant b_f such that $\|f(t, u, v)\| \leq \|a_f(t)\| + b_f(\|u\|_{\mathcal{C}} + \|v\|)$, and $\sup_{t\in\mathbb{R}} \|a_f(t)\| = a_f^*$;

(C9) Let $g \in C(\mathbb{R} \times \mathcal{C} : X)$. There exist a bounded measurable function $a_g : \mathbb{R} \to X$ and a constant b_g such that $\|g(t, u)\| \leq \|a_g(t)\| + b_g\|u\|_{\mathcal{C}}$, and $\sup_{t\in\mathbb{R}} \|a_g(t)\| = a_g^*$.

A mild solution of (1.1) is given by

$$u(t) = \int_{-\infty}^{t} T(t-s)f\big(s, u_s, Fu(s)\big)\, ds,$$

for $t \in \mathbb{R}$.

2. Existence and uniqueness of almost periodic mild solutions

In this section, we will elucidate the conditions that guarantee the existence of unique almost periodic mild solutions for the problem under consideration described by (1.1).

At the beginning, note that if $u \in AP(\mathbb{R} : X)$ and $g \in AP(\mathbb{R} \times \mathcal{C} : X)$ then $u_t \in AP(\mathbb{R} : X)$ and $h(\cdot) = g(\cdot, u) \in AP(\mathbb{R} : X)$.

We start with the following auxiliary results.

Lemma 2.1 *Let (C1)–(C2) and (C4) hold. If $u \in AP(\mathbb{R} : X)$, then $Fu \in AP(\mathbb{R} : X)$.*

Proof: Since (C2) holds, it is clear that $j(\cdot) = g(\cdot, u) \in AP(\mathbb{R} : X)$ and $\|j\|_\infty \leq \infty$. Then for each $\varepsilon_1 \in \vartheta_X(\phi, \cdot)$ and each compact subset K of X holds

$$\|j(t - s + \omega) - j(t - s)\| \leq \varepsilon_1.$$

Now,

$$\|Fu\| \leq \int_{-\infty}^{t} k(t-s)\|j\|\, ds \leq \|k\|_{L^1} \cdot \|j\|_\infty,$$

so Fu is continuous. Next,

$$\|Fu(t+\omega) - Fu(t)\| = \left\| \int_{-\infty}^{t+\omega} k(t+\omega-s) g(s, u_s)\, ds - \int_{-\infty}^{t} k(t-s) g(s, u_s)\, ds \right\|$$

$$\leq \int_{0}^{+\infty} k(s)\|j(t+\omega-s) - j(t-s)\|\, ds \leq \|k\|_{L^1}\varepsilon_1 \leq \varepsilon.$$

Hence, $Fu \in AP(\mathbb{R} : X)$. □

Lemma 2.2 *Let (A), (C1)–(C3) and (C5) hold. If $u \in AP(\mathbb{R} : X)$, then*

$$(\mathcal{S}u)(t) = \int_{-\infty}^{t} T(t-s) f(s, u_s, Fu(s))\, ds \in AP(\mathbb{R} : X).$$

Proof: Let $u \in AP(\mathbb{R} : X)$. By Lemma 2.1, $Fu \in AP(\mathbb{R} : X)$, $\phi(\cdot) = f(\cdot, u, Fu(\cdot)) \in AP(\mathbb{R} : X)$. Then for each $\varepsilon_1 \in \vartheta_{X,Y}(\phi, \cdot)$ and each compact subset E of $X \times Y$ we have

$$\|\phi(t - s + \omega) - \phi(t - s)\| \leq \varepsilon_1.$$

Now, we have

$$\|\mathcal{S}u(t+\omega) - \mathcal{S}u(t)\|_X = \left\| \int_{0}^{+\infty} T(s)\Big(\phi(t-s+\omega) - \phi(t-s)\Big)\, ds \right\|$$

$$\leq \int_{0}^{+\infty} Ce^{-\sigma s}\|\phi(t-s+\omega) - \phi(t-s)\|_X\, ds \leq \varepsilon_1 \int_{0}^{+\infty} Ce^{-\sigma s}\, ds = \frac{C\varepsilon_1}{\sigma} \leq \varepsilon.$$

Hence, $\mathcal{S}u \in AP(\mathbb{R} : X)$. □

Theorem 2.1 *Let (A) and (C1)–(C5) hold. If $\rho < 1$, where $\rho = \frac{CL_f}{\sigma}(1 + \|k\|_{L^1} L_g)$, then (1.1) has a unique almost periodic mild solution.*

Proof: We define the operator $\mathcal{S} : AP(\mathbb{R} : X) \to AP(\mathbb{R} : X)$ by

$$(\mathcal{S}u)(t) := \int_{-\infty}^{t} T(t-s) f\big(s, u_s, Fu(s)\big)\, ds, \quad t \in \mathbb{R}.$$

By Lemma 2.2, the operator $\mathcal{S}$ is well-defined. Now, let $u,\ v \in AP(\mathbb{R} : X)$. Then we obtain

$$\begin{aligned}
\|\mathcal{S}u - \mathcal{S}v\|_X &\le \int_{-\infty}^{t} \|T(t-s)(f(s, u_s, Fu(s)) - f(s, v_s, Fv(s)))\|\, ds \\
&\le C \int_{-\infty}^{t} e^{-\sigma(t-s)} \|f(s, u_s, Fu(s)) - f(s, v_s, Fv(s))\|\, ds \\
&\le CL_f \int_{-\infty}^{t} e^{-\sigma(t-s)} \big(\|u_s - v_s\|_{\mathcal{C}} + \|Fu(s) - Fv(s)\|\big)\, ds \\
&\le CL_f \int_{-\infty}^{t} e^{-\sigma(t-s)} \Big(\|u_s - v_s\|_{\mathcal{C}} + \int_{-\infty}^{t} k(t-s)\|g(s, u_s) - g(s, v_s)\|\Big)\, ds \\
&\le CL_f \int_{-\infty}^{t} e^{-\sigma(t-s)} \Big(\|u - v\|_\infty \big(1 + \|k\|_{L^1} L_g\big)\Big)\, ds \\
&\le \frac{CL_f}{\sigma} \big(1 + \|k\|_{L^1} L_g\big) \|u - v\|_\infty .
\end{aligned}$$

Hence, by the Banach contraction mapping principle, $\mathcal{S}$ has a unique fixed point in $AP(\mathbb{R} : X)$, so (1.1) has a unique mild solution in $AP(\mathbb{R} : X)$. □

Theorem 2.2 *Let (A), (C1)–(C3) and (C6)–(C7) hold. If there is $\nu > 0$ such that $\rho < 1$, where*

$$\rho = \frac{C}{\sigma}\Big(L_f(\nu)\big(1 + L_g(\nu)\|k\|_{L^1}\big) + \frac{1}{\nu} L_f(\nu)\|k\|_{L^1} \cdot \sup_{t\in\mathbb{R}} \|g(t,0)\| + \frac{1}{\nu} \sup_{t\in\mathbb{R}} \|f(t,0,0)\|\Big),$$

then (1.1) has a unique almost periodic mild solution, with $\|u\|_\infty \le \lambda$.

Proof: Note that f is bounded, so $f(\cdot, 0)$ is also a bounded function in $\mathbb{R}$. We define the operator $\mathcal{R} : AP(\mathbb{R} : X) \to AP(\mathbb{R} : X)$ by

$$(\mathcal{R}u)(t) := \int_{-\infty}^{t} T(t-s) f\big(s, u_s, Fu(s)\big)\, ds, \quad t \in \mathbb{R}.$$

Put $B_\nu := \{u \in AP(\mathbb{R} : X) \, : \, \|u\|_\infty \le \nu\}$. For $u \in B_\nu$, we have

$$\begin{aligned}
\|\mathcal{R}u(t)\| &\le C \int_{-\infty}^{t} e^{-\sigma(t-s)} \big(\|f(s, u_s, Fu(s)) - f(s,0,0) + f(s,0,0)\|\big)\, ds \\
&\le C \int_{-\infty}^{t} e^{-\sigma(t-s)} L_f(\nu) \big(\|u_s\|_{\mathcal{C}} + \|Fu(s)\|\big)\, ds + C \int_{-\infty}^{t} e^{-\sigma(t-s)} \|f(s,0,0)\|\, ds \\
&\le \frac{C\nu L_f(\nu)}{\sigma} \big(1 + L_g(\nu)\|k\|_{L^1}\big) + \frac{CL_f(\nu)}{\sigma} \|k\|_{L^1} \cdot \sup_{t\in\mathbb{R}} \|g(t,0)\| \\
&\quad + \frac{C}{\sigma} \sup_{t\in\mathbb{R}} \|f(t,0,0)\| \le \nu,
\end{aligned}$$

so $\mathcal{R}u \in B_\nu$.

For $u, v \in B_\nu$, we obtain

$$\|\mathcal{R}u - \mathcal{R}v\| \leq CL_f(\nu) \int_{-\infty}^{t} e^{-\sigma(t-s)} \Big(\|u_s - v_s\|_{\mathcal{C}} + \|Fu(s) - Fv(s)\| \Big)\, ds$$
$$\leq \frac{CL_f(\nu)}{\sigma} \big(1 + L_g(\nu)\|k\|_{L^1}\big)\|u - v\|_\infty.$$

Hence,

$$\|\mathcal{R}u - \mathcal{R}v\|_\infty \leq \frac{CL_f(\nu)}{\sigma} \big(1 + L_g(\nu)\|k\|_{L^1}\big)\|u - v\|_\infty.$$

By the condition $\rho < 1$, using Banach contraction mapping principle, the equation (1.1) has a unique mild almost periodic solution. □

Theorem 2.3 *Let (A), (C1)–(C3) and (C8)–(C9) hold. Then the equation (1.1) has at least one solution.*

Proof: We define the closed ball B_r as

$$B_r = \{u \in AP(\mathbb{R} : X) \, : \, \|u\|_\infty < r\},$$

where $r \geq \frac{C(a_f^* + b_f\|k\|_{L^1} a_g^*)}{1 - C(b_f + \|k\|_{L^1} b_g)}$.

Let the operator $\mathcal{G} : AP(\mathbb{R} : X) \to AP(\mathbb{R} : X)$ be defined by

$$(\mathcal{G}u)(t) := \int_{-\infty}^{t} T(t-s) f\big(s, u_s, Fu(s)\big)\, ds, \quad t \in \mathbb{R}.$$

Now, by applying (C8)–(C9), we obtain

$$\|(\mathcal{G}u)(t)\| = \left\| \int_{-\infty}^{t} T(t-s) f(s, u_s, Fu(s))\, ds \right\|$$
$$\leq \int_{-\infty}^{t} \|T(t-s)\| \cdot \|f(s, u_s, Fu(s))\|\, ds$$
$$\leq C \int_{-\infty}^{t} e^{-\sigma(t-s)} \big(\|a_f(t)\| + b_f\big(\|u\|_\infty + \|Fu(s)\|\big)\big)\, ds$$
$$= C \int_{-\infty}^{t} e^{-\sigma(t-s)} \left(a_f^* + b_f \left(\|u\|_\infty + \int_{-\infty}^{t} k(t-\theta)\|g(\theta, u_\theta)\|\, d\theta \right) \right) ds$$
$$\leq C \int_{-\infty}^{t} e^{-\sigma(t-s)} \big(a_f^* + b_f\big(\|u\|_\infty + \|k\|_{L^1}\big(\|a_g(t)\| + b_g\|u\|_\infty\big)\big)\big)\, ds$$
$$\leq C\big(a_f^* + b_f\|k\|_{L^1} a_g^* + \big(b_f + \|k\|_{L^1} b_g\big)\|u\|_\infty\big) \leq r,$$

so $\mathcal{G} : B_r \to B_r$ and $\{\mathcal{G}u\}$ is uniformly continuous.

Now, we are going to prove that $\mathcal{G}$ is continuous. Let (u_n) be a sequence in B_r, such that $u_n \to u$, when $n \to \infty$. Then

$$(\mathcal{G}u_n)(t) = \int_{-\infty}^{t} T(t-s) f(s, (u_n)_s, Fu_n(s))\, ds,$$

where $(u_n)_s(t) = u_n(t+s)$.

Now, using Lebesgue dominated convergence theorem, and having on mind the continuity of the function f and g, we obtain

$$\lim_{n\to\infty}(\mathcal{G}u_n)(t) = \lim_{n\to\infty}\int_{-\infty}^{t} T(t-s)f(s,(u_n)_s, Fu_n(s))\,ds$$

$$= \int_{-\infty}^{t} T(t-s)f\big(s, \lim_{n\to\infty}(u_n)_s, \lim_{n\to\infty} Fu_n(s)\big)\,ds$$

$$= \int_{-\infty}^{t} T(t-s)f(s,u_s,Fu(s))\,ds = (\mathcal{G}u)(t).$$

Note that, in the upper equality, we used that

$$\lim_{n\to\infty} Fu_n(s) = \lim_{n\to\infty}\int_{-\infty}^{t} k(t-s)g(s,(u_n)_s)\,ds$$

$$= \int_{-\infty}^{t} k(t-s)g(s, \lim_{n\to\infty}(u_n)_s)\,ds = \int_{-\infty}^{t} k(t-s)g(s,u_s)\,ds = Fu(s).$$

Next, we prove that $\{\mathcal{G}u\}$ is equicontinuous and the operator $\mathcal{G}$ is relatively compact. Let $u \in B_r$, and $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ and $|t_1 - t_2| < \delta$ for some δ. We have

$$\|(\mathcal{G}u)(t_2) - (\mathcal{G}u)(t_1)\| = \Bigg\|\int_{-\infty}^{t_2} T(t_2-s)f(s,u_s,Fu(s))\,ds$$

$$- \int_{-\infty}^{t_1} T(t_1-s)f(s,u_s,Fu(s))\,ds\Bigg\| = \Bigg\|\int_{-\infty}^{t_1} T(t_2-s)f(s,u_s,Fu(s))\,ds$$

$$+ \int_{t_1}^{t_2} T(t_2-s)f(s,u_s,Fu(s))\,ds - \int_{-\infty}^{t_1} T(t_1-s)f(s,u_s,Fu(s))\,ds\Bigg\|$$

$$= \Bigg\|\int_{-\infty}^{t_1}\Big(T(t_2-s) - T(t_1-s)\Big)f(s,u_s,Fu(s))\,ds\Bigg\| + \Bigg\|\int_{t_1}^{t_2} T(t_2-s)f(s,u_s,Fu(s))\,ds\Bigg\|$$

$$\le \int_{-\infty}^{t_1} \|T(t_1-s)\big(T(t_2-t_1) - I\big)f(s,u_s,Fu(s))\|\,ds$$

$$+ \int_{t_1}^{t_2} \|T(t_2-s)f(s,u_s,Fu(s))\|\,ds.$$

Hence, $\|(\mathcal{G}u)(t_2) - (\mathcal{G}u)(t_1)\| \to 0$, when $t_1 \to t_2$. Now, by using Schauder's fixed point theorem, we obtain existence of at least one solution of (1.1). □

The following example is a generalization of certain results in [18] (see also [3], [26] and [30]):

Example 2.1 *Let us consider the Cohen–Grossberg neural network with delays given by the system*

$$u_i'(t) = -a_i u_i(t) + \sum_{j=1}^{n} b_{ij}(t) f_j\big(u_j(t)\big) + \sum_{j=1}^{n} c_{ij} g_j\big(u_j(t-\tau_{ij})\big) \tag{2.1}$$

$$+\sum_{j=1}^{n} d_{ij} \int\limits_{-\infty}^{t} k_{ij}(t-s) v_j\big(u_j(s)\big)\, ds + I_i(t),$$

for $i = 1, 2, ..., n$. In the context of this neural network model, n represents the number of units, $u_i(t)$ signifies the state of the ith unit at time t, $a_i > 0$ denotes the rate at which the ith unit resets its potential to the resting state in isolation, when detached from both the network and external inputs. The parameter $b_{ij}(t)$ indicates the strength of influence from the jth unit on the ith unit at time t, while $c_{ij}(t)$ represents the strength of the jth unit in the ith unit at time $t - \tau_{ij}$, where τ_{ij} corresponds to the transmission delay along the axon from the jth unit to the ith unit at time t. The terms f_j, g_j, and v_j refer to the measured response or activation in response to incoming potentials for the jth unit, and $I_i(t)$ characterizes the varying external input signals directed to the ith unit at time t.

Let $b_{ij}(t), c_{ij}(t), d_{ij}(t), I_i(t) \in AP(\mathbb{R} : \mathbb{R})$, $k_{ij}(t) = e^{-t}$, $i, j = 1, 2, ..., n$. Additionally, let the following holds: There exist positive constants $L_{f_j}, L_{g_j}, L_{v_j}$ such that

$$|f_j(u) - f_j(v)| \le L_{f_j}|u - v|, \qquad |g_j(u) - g_j(v)| \le L_{g_j}|u - v|,$$

$$|v_j(u) - v_j(v)| \le L_{v_j}|u - v|,$$

for all $u, v \in \mathbb{R}$.

Let $A = diag(-a_1, -a_2, ..., -a_n)$ and X be the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. The semigroup generated by A is given by $T(t) = e^{tA} = diag(e^{-ta_1}, e^{-ta_2}, ..., e^{-ta_n})$ and $\|T(t)\| \le Ce^{-\sigma t}$, where $C = 1$ and $\sigma = \min_{1\le i\le n} a_i$. Hence, (A) holds. Let

$$f(t, u(t-\tau), Fu(t)) = \Bigg(\sum_{j=1}^{n} b_{1j}(t) f_j(u_j(t)) + \sum_{j=1}^{n} c_{1j}(t) g_j(u_j(t-\tau_{1j}))$$

$$+\sum_{j=1}^{n} d_{1j}(t) \int\limits_{-\infty}^{t} k_{1j}(y-s) v_j(u_j(s))\, ds + I_1(t), ...,$$

$$\sum_{j=1}^{n} b_{nj}(t) f_j(u_j(t)) + \sum_{j=1}^{n} c_{nj}(t) g_j(u_j(t-\tau_{nj}))$$

$$+\sum_{j=1}^{n} d_{nj}(t) \int\limits_{-\infty}^{t} k_{nj}(y-s) v_j(u_j(s))\, ds + I_n(t)\Bigg)^T.$$

Note that (C4)–(C5) are fulfilled. Hence, by using Theorem 2.1, for $\rho = L_f(1 + L_g) < \sigma$, where

$$L_f = \max_{t\in\mathbb{R}} \Bigg(\sum_{i=1}^{n}\sum_{j=1}^{n} L_{f_j} b_{ij}(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} L_{g_j} c_{ij}(t)\Bigg) + 1,$$

$$L_g = \max_{t\in\mathbb{R}} \Bigg(\sum_{i=1}^{n}\sum_{j=1}^{n} L_{v_j} d_{ij}(t)\Bigg),$$

the Cohen–Grossberg neural network (2.1) has a unique almost periodic solution.

Example 2.2 *Let we consider the following neutral integro-differential equation*

$$u'(t) = -au(t) + \frac{1}{2}\sin u(t-\tau) + \cos\frac{1}{3}\int_{-\infty}^{t} e^{-(t-s)}\cos(s-\tau)\,ds, \tag{2.2}$$

for $a \in \mathbb{R}$. We put $f(t,u,v) = \frac{1}{2}\sin u + \cos v$, $Fu = \int_{-\infty}^{t} k(t-s)g(s,u_s)\,ds$, $k(t) = e^{-t}$ and $g(t,u) = \frac{1}{3}\cos u$. The semigroup generated by A is given by $T(t) = e^{-at}$, so $\|T(t)\| \le Ce^{-\sigma t}$, where $C = 1$ and $\sigma = -a$, so (A) is satisfied. Note that the functions $f(t,u,v)$ and $g(t,u)$ are almost periodic functions. Hence, the conditions (C1)–(C3) are fulfilled. Moreover, we have

$$|f(t,u,v)| \le \frac{1}{2} + (|u| + |v|) \quad \textit{and} \quad |g(t,u)| \le \frac{1}{3} + |u|,$$

so $a_f^* = \frac{1}{2}$, $b_f = 1$, $a_g^* = \frac{1}{3}$ and $b_g = 1$. We conclude that (C8)–(C9) hold.

Now, by using Theorem 2.3, the equation (2.2) has at least one almost periodic solution.

3. Conclusion

In the course of this investigation, we have established the criteria for the existence and uniqueness of almost periodic mild solutions within a specified class of neutral integro-differential equations, formulated within the framework of a Banach space. Our findings extend the current body of knowledge in the field of almost periodic solutions for specific categories of neutral differential equations. The provided illustrative examples underscore the significance of our results and their potential applicability in domains such as neural networks and engineering.

For future research endeavors, our focus will center on delineating the prerequisites governing the existence and uniqueness of almost periodic mild solutions in more general classes of neutral integro-differential equations. This will encompass scenarios where the operator A in (1.1) possesses a non-dense domain. Additionally, we aim to investigate the presence of almost periodic mild solutions in classes of nonautonomous neutral integro-differential equations.

References

1. Abbas, S., Bahuguna, D., *Almost periodic solutions of neutral functional differential equations*, Comput. Math. with Appl., 55, 2593–2601, (2008).
2. Amerio, L., Prouse G., *Almost Periodic Functions and Functional Equations*, Van Nostrand–Reinhold, New York, (1971).
3. Ayachi, M., *Measure-pseudo almost periodic dynamical behaviors for BAM neural networks with D operator and hybrid time–varying delays*, Neurocomputing, 486, 160–173, (2022).
4. Bellour, A., Ait Dads, E., *Periodic solutions for nonlinear neutral delay integro-differential equations*, Electron. J. Differ. Equ., 100, 1–9, (2015).
5. Besicovich, A. S., *Almost Periodic Functions*, Dover Publications, New York, (1958).
6. Chen, X., Lin, F., *Almost periodic solutions of neutral functional differential equations*, Nonlinear Anal. Real World Appl., 11(2), 1182–1189, (2010).
7. Ait Dads, E., Cieutat, P., Lhachimi, L., *Positive pseudo almost periodic solutions for some nonlinear infnite delay integral equations*, Math. Comput. Modelling, 49, 721–739, (2009).
8. Ait Dads, E., Ezzinbi, K., *Almost periodic solution for some neutral nonlinear integral equation*, Nonlinear Anal., 28, 1479–1489, (1997).
9. Diagana, T., *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer-Verlag, New York, (2013).
10. Diagana, T., Hernández, E., Rabello, M., *Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay*, Math. Comput. Model., 45(9-10), 1241–1252, (2007).
11. Fink, A., Gatica, J., *Positive almost periodic solutions of some delay integral equations*, J. Difference Equ., 83, 166–178, (1990).
12. Hale, J. K., Lunel, V. M., *Introduction to Functional Differential Equations*, in: Appl. Math. Sci., Vol. 99, Springer–Verlag, New York, (1993).

13. Islam, M. N., Raffoul, Y. N., *Periodic solutions of neutral nonlinear system of differential equations with functional delay*, J. Math. Anal. Appl., 331, 1175–1186, (2007).

14. Kostić, M., *Almost Periodic and Almost Automorphic Type Solutions of Abstract Volterra Integro-Differential Equations*, W. de Gruyter, Berlin, (2019).

15. Kostić, M., *Selected Topics in Almost Periodicity*, W. de Gruyter, Berlin, (2022).

16. Kostić, M., *Metrical Almost Periodicity and Applications to Integro-Differential Equations*, W. de Gruyter, Berlin, (2023).

17. Levitan, B. M., Zhikov, V., *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Moscow, (1978).

18. Liang, T., Yang, Y. Q., Liu, Y., Li, L., *Existence and global exponential stability of almost periodic solutions to Cohen-Grossberg neural networks with distributed delays on time scales*, Neurocomputing, 123, 207–215, (2014).

19. Nussbanum, R., *A periodicity threshold theorem for some nonlinear integral equations*, SIAM J. Math. Anal., 9, 356–376, (1978).

20. Park, J. Y., Kwumn, Y. C., Jeong, J. M., *Existence of periodic solutions for delay evolution integrodifferential equations*, J. Math. and Comput. Model., 65, 597–603, (2004).

21. Raffoul, Y., *Positive periodic solutions in neutral nonlinear differential equations*, Electron. J. Qual. Theory Differ. Equ., 16, 1–10, (2007).

22. Rubanik, V. P., *Oscillations of Quasilinear Systems with Retardation*, Nauka, Moscow, (1969).

23. Sattayatham, P., Tangmanee, S., Wei, W., *On periodic solutions of nonlinear evolutions equations in Banach spaces*, J. Math. Anal., 276(1), 98–108, (2002).

24. Sforza, D., *Existence in the Large for a Semilinear Integrodifferential Equation with Infinite Delay*, J. Differ. Equ., 120, 289–303, (1995).

25. Smith, H. L., *On periodic solutions of a delay integral equation modelling epidemics*, J. Math. Biol., 4, 69–80, (1977).

26. Song, Q. K., Cao, J. D., *Stability analysis of Cohen–Grossberg neural networks with both time–varying and continuously distributed delays*, J. Comput. Appl. Math., 197(1), 188–203, (2006).

27. Xia, Z., Wang, D., Wen, C.-F., Yao, J.-C., *Pseudo asymptotically periodic mild solutions of semilinear functional integro-differential equations in Banach spaces*, Math. Meth. Appl. Sci., 2017, 1–23, (2017).

28. Xiang, X., Ahmed, N. U., *Existence of periodic solutions of semilinear evolution equations with time lags*, Nonlinear Anal., 18, 1063–1070, (1992).

29. Wang, J. R., Xiang, X., Wei, W., *Periodic solutions of a class of integrodifferential impulsive periodic systems with time-varying generating operators on Banach space*, Electron. J. Qual. Theory Differ. Equ., 4, 1–17, (2009).

30. Zhang, J. Y., Suda, Y., Iwasa, T., *Absolutely exponential stability of a class of neural networks with unbounded delay*, Neural Netw., 17(3), 391–397, (2004).

31. Zaidman, S., *Almost-Periodic Functions in Abstract Spaces*, Pitman Research Notes in Math, Vol. 126, Pitman, Boston, (1985).

32. Zou, Q.-F., Ding, H.-S., *Almost periodic solutions for a nonlinear integro-differential equation with neutral delay*, J. Nonlinear Sci. Appl., 9, 4500–4508, (2016).

Marko Kostić,
Faculty of Technical Sciences,
University of Novi Sad,
Serbia.
E-mail address: marco.s@verat.net

and

Daniel Velinov,
Faculty of Civil Engineering,
Ss. Cyril and Methodius University in Skopje,
N. Macedonia.
E-mail address: velinovd@gf.ukim.edu.mk