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Existence of weak solutions to the fractional p-Laplacian problems of Kirchhoff-type via topological degree

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ABSTRACT: In this work, we show the existence result of weak solutions for a class of Kirchhoff-type problems with Dirichlet-type boundary conditions involving p-Laplacian operator. Under some appropriate conditions, the existence of weak solutions is obtained by employing the Berkovits degree theory.

Key Words: Kirchhoff, weak solutions, fractional p-Laplacian, elliptic problems, topological degree, Berkovits topological degree.

Contents

1. Introduction

In this paper, we are interested in the following Kirchhoff problem

$$\begin{cases} M\left(\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)(-\Delta)_p^s(u) = f(x,u) \quad \text{ in } \Omega, \\ u=0 \quad \text{ in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain of $\mathbb{R}^n$, $Q=(\mathbb{R}^n\times\mathbb{R}^n)\backslash(\mathbb{R}^n\backslash\Omega\times\mathbb{R}^n\backslash\Omega)$, $M:\mathbb{R}^+\to\mathbb{R}^+$, f is a given function that satisfies some conditions given later. The fractional p-Laplacian operator $(-\Delta)_p^s$ is defined as follows

$$(-\Delta)_p^s u(x) = P.V\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}dy, \quad x\in\mathbb{R}^n,$$

where $x\in\mathbb{R}^n$ and $P.V$, which stands for "in the principal value sense," is a frequently used abbreviation.

The study of fractional Laplacian and non-local operators has received a lot of interest in recent years. This kind of operator arises in various applications such as population dynamics, continuum mechanics, image process, phase transition phenomena, Lévy processes, and game theory [2,7,8,9,17]. For this reason, it is particularly important to study equations where such non-local operators are involved. When the fractional p-Laplacian is concerned, we refer the interested reader to [3,16,18,23,24] for the parabolic and elliptical problems involving fractional p-Laplacian.

The stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

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is presented by Kirchhoff [14] and is related to problem (1.1). The parameters ρ, ρ_0, h, E and L in (1.2) are the constants, that extend the classical D'Alambert wave equation by taking into account the effects of the variations in the length of the strings during the vibrations. In detail, the nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ is dependent on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ of the Kinetic energy $\left|\frac{\partial u}{\partial x}\right|^2$ on $[0, L]$ and is no longer a pointwise identity.

When it comes to the problem (1.1), [25] investigated the existence results using the Mountain Pass Theorem. Based on the fountain theorem, the existence results to the problem same to (1.1) is discussed in [27].

Several authors have studied problem (1.1) with $M \equiv 1$. For example, the existence of weak solutions to the fractional p-Laplacian system using the theory of Young measures is proven in [3]. The authors in [19] proved the existence of nonnegative solutions using Leray-Schauder's nonlinear alternative.

The case $M \equiv 1$ and $p = 2$ is introduced in [20] and [21], also the authors in [13] obtained the existence of solution by computing the Morse theory and critical groups. Based on variational methods, the existence of a weak solution is proved by [12]. We refer also to [5,6] for more information.

In the theory of nonlinear partial differential equations, the method of topological degree has recently become an increasingly important tool to discuss the existence of solutions. The theory used is developed by [4] for operators of generalized monotone type. Authors in [15] studied the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces, where the topological degree theory was first established. We suggest to the readers to consult [1,10] which has been applied to some elliptical problems.

Inspired by the above work, we use the method of topological degree to investigate the existence of a weak solution to the problem (1.1). To the best of our knowledge, problem (1.1) has never been studied by the topological degree theory.

This article is organized into four sections. In section 2, we give some background information on fractional Sobolev spaces and a review of the Berkovits degree theory. In the last section, we state the hypothesis and technical Lemmas to prove Theorem 4.1, and we finish by proving the main result.

2. Mathematical preliminaries

2.1. Fractional Sobolev spaces

In this part, we recall some notations and definitions, and some of the results that will be applied to this work.

Let $0 < s < 1$, $1 < p < \infty$ be real numbers and we define p_s^* the fractional critical exponent giving by:

$$p_s^* = \begin{cases} \infty \text{ if } ps \geq n, \\ np/(n-ps) \text{ if } ps < n. \end{cases}$$

Let Ω an open set in $\mathbb{R}^n$ and $Q = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathbb{R}^n \setminus \Omega \times \mathbb{R}^n \setminus \Omega)$. It is obvious that $\Omega \times \Omega$ is strictly contained in Q. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx\right)^{\frac{1}{p}}.$$

And the closed linear subspace

$$W_0 = \{u \in W : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

In W_0, we may also use the norm

$$\|u\|_{W_0} = \left(\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx\right)^{\frac{1}{p}}.$$

It is known that $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex reflexive Banach space (see [25]), its dual is indecated by $\left(W_0^*, \|\cdot\|_{W_0^*}\right)$.

Lemma 2.1 *[11] The following embedding $W_0 \hookrightarrow L^{\theta}(\Omega)$ is compact for all $1 \leq \theta < p_s^*$, and continuous for all $1 \leq \theta \leq p_s^*$.*

2.2. Topological degree theory

Let X be a real separable reflexive Banach space and X^* be its dual space, and let Ω be a nonempty subset of X. We recall that $\rightharpoonup$ represents the weak convergence and $\langle \cdot, \cdot \rangle$ is continuous dual pairing. Let Y be another real Banach space. Now, we introduce some topological degree results and properties.

Definition 2.1 *Let $J : \Omega \subset X \to Y$ be an operator. We recall that a mapping J is:*
1)- Bounded, if it takes any bounded set into a bounded set.
2)- Demicontinuous, if for any sequence $(u_k) \subset \Omega, u_k \to u$ implies $J(u_k) \rightharpoonup J(u)$.
3)- Compact, if it is continuous and the image of any bounded set is relatively compact.
4)- (S_+) type, if for any sequence $(u_k) \subset \Omega$ with $u_k \rightharpoonup u$ and $\limsup_{k\to\infty} \langle Ju_k, u_k - u\rangle \leq 0$, we have $u_k \to u$.
5)- Quasimonotone, if $u_k \rightharpoonup u$ implies $\limsup_{k\to\infty} \langle Ju_k, u_k - u\rangle \geq 0$.
6)- $(S_+)_T$ type, with $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$, if for any sequence $(u_k) \subset \Omega$ with $u_k \rightharpoonup u$, $\quad y_n := Tu_k \rightharpoonup y$ and $\limsup_{k\to\infty} \langle Ju_k, y_n - y\rangle \leq 0$, we have $u_k \to u$.

Remark 2.1 *[26]*
We can see that any compact map in a set is quasi-monotone in that set. Moreover, any demi-continuous map of type (S_+) in a set is quasimonotone in that set.

In the sequel, $\mathcal{U}$ be the collection of all bounded open set in X. We consider the following classes of operators:

$$\begin{aligned}
&\mathcal{J}_1(\Omega) := \{J : \Omega \to X^* \mid J \text{ is bounded, demicontinuous and of type } (S_+)\},\\
&\mathcal{J}_{T,B}(\Omega) := \{J : \Omega \to X \mid J \text{ is bounded, demicontinuous and satifies}\\
&\ \text{condition } (S_+)_T\},\\
&\mathcal{J}_T(\Omega) := \{J : \Omega \to X \mid J \text{ is demicontinuous and satifies condition } (S_+)_T\},\\
&\mathcal{J}_B(X) := \left\{J \in \mathcal{J}_{T,B}(\bar{N}) \mid N \in \mathcal{U}, T \in \mathcal{J}_1(\overline{N})\right\}.
\end{aligned}$$

Lemma 2.2 *[4] Let $S : D_S \subset X^* \to X$ be demicontinuous and $T \in \mathcal{J}_1(\bar{N})$ be continuous such that $T(\bar{N}) \subset D_S$, where N is a bounded open set in X. The following assertions are true:*
1. If S is quasimonotone, then $I + SoT \in \mathcal{J}_T(\bar{N})$, where I stands for the the identity operator.
2. If S is of type (S_+), then $SoT \in \mathcal{J}_T(\bar{N})$.

Definition 2.2 *Let N is to be a bounded open subset of X, let $J, S \in \mathcal{J}_T(\bar{N})$ where $T \in \mathcal{J}_1(\bar{N})$ a continuous operator. We define an affine homotopy $\Phi : [0,1] \times \bar{N} \to X$ by*

$$\Phi(t, u) := (1-t)Ju + tSu \quad \text{for} \quad (t, u) \in [0,1] \times \bar{N}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.2 *[4] The affine homotopy in definition 2.2 satisfies condition $(S_+)_T$.*

Next, we give the Berkovits topological degree for a class of demicontinuous operators satisfying condition $(S_+)_T$. For more information see [4].

Theorem 2.1 *Let*

$$M_T = \left\{(J, N, g) \mid N \in \mathcal{U},\, T \in \mathcal{J}_1(\bar{N}),\, J \in \mathcal{J}_{T,B}(\bar{N}),\, g \notin J(\partial N)\right\}.$$

There exists a unique degree function $d : M_T \longrightarrow \mathbb{Z}$ which satisfies the following properties :
1. (Normalization) For any $g \in N$, we have $d(I, N, g) = 1$.
2. (Additivity) Let $J \in \mathcal{J}_{T,B}(\bar{N})$. If N_1 and N_2 are two disjoint open subsets of N such that $g \notin J\left(\bar{N} \setminus (N_1 \cup N_2)\right)$, then we have

$$d(J, N, g) = d(J, N_1, g) + d(J, N_2, g).$$

3. (Homotopy invariance) If $\Phi : [0,1] \times \bar{N} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $g : [0,1] \to X$ is a continuous path in X such that $g(t) \notin \Phi(t, \partial N)$ for all $t \in [0,1]$, then the value of $d(\Phi(t,\cdot), N, g(t))$ is constant for all $t \in [0,1]$.
4. (Existence) If $d(J; N; g) \neq 0$, then the equation $Ju = g$ has a solution in N.

3. Essential Assumptions and technical lemmas

In this part, we will define a weak solution for the problem (1.1). We start by assuming the following hypothesis:
(A_1): The function f is a Carathéodory function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ satisfying:

$$\text{there exists } C_1 > 0, \quad 0 \leq \gamma < p-1 \quad \text{such that} \quad |f(x,\zeta)| \leq \mathcal{E}(x) + C_1 |\zeta|^{\gamma},$$

for all $\zeta \in \mathbb{R}$ and almost every $x \in \Omega$, where $\mathcal{E} \in L^{p'}(\Omega)$, with $\mathcal{E} \geq 0$ almost everywhere in Ω.
(A_2): M is a countinous and non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and satisfies

$$m_0 s^{\beta-1} \leq M(s) \leq m_1 s^{\beta-1},$$

for all $s > 0$ and m_0, m_1 are real numbers such that $0 < m_0 \leq m_1$ and $\beta \geqslant 1$.

Definition 3.1 *We say a function $u \in W_0$ is a weak solution of the problem (1.1) if*

$$M\left(||u||_{W_0}^p\right) \iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}(v(x)-v(y))dxdy = \int_\Omega f(x,u)v(x)dx$$

for any $v \in W_0$.

Let denote $L : W_0 \to W_0^*$ defined by

$$\langle L(u), v\rangle = M\left(||u||_{W_0}^p\right) \iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}(v(x)-v(y))dxdy \tag{3.1}$$

for all $u, v \in W_0$.
Take into account the following functional

$$\Psi_M(u) = \frac{1}{p}\mathcal{M}\left(||u||_{W_0}^p\right) \quad \text{for all } u \in W_0, \tag{3.2}$$

where $\mathcal{M}$ be the primitive of the function M, defined by

$$\begin{array}{rcl} \mathcal{M} : [0,+\infty[& \to & [0,+\infty[\\ t & \mapsto & \mathcal{M}(t) = \int_0^t M(\xi)d\xi. \end{array}$$

Lemma 3.1 *Suppose that (A_1) and (A_2) holds, then*
1)- L is bounded,
2)- L is coercive,
3)- L is continuous operator,
4)- L is strictly monotone,
5)- L is of type (S_+).

Proof: 1) By the Hölder inequality and (A_2), we get

$$\begin{aligned}|\langle L(u),v\rangle| &= \left| M\left(||u||^p_{W_0}\right)\iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}(v(x)-v(y))dxdy\right| \\ &\leq m_1\left(||u||^p_{W_0}\right)^{\beta-1}||u||^{p-1}_{W_0}||v||_{W_0} \\ &\leq C_2||v||_{W_0},\end{aligned}$$

then L is bounded.
2)- We prove that L is a ceorcive operator. As $\beta \geqslant 1$, we get

$$\begin{aligned}\langle L(u),u\rangle &= M\left(||u||^p_{W_0}\right)\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy \\ &\geqslant m_0\left(||u||^p_{W_0}\right)^{\beta-1}||u||^p_{W_0} \\ &\geqslant m_0||u||^p_{W_0},\end{aligned}$$

we conclude that

$$\lim_{||u||_{W_0}\to\infty}\frac{\langle L(u),u\rangle}{||u||_{W_0}}=\infty.$$

Next, we prove that the operator L is continuous.
According to (3.2), Ψ_M is continuously Gateaux-differentiable in W_0 and

$$\langle \Psi'_M(u),v\rangle = \langle L(u),v\rangle \text{ for all } u,\ v\in W_0,$$

then L is continous.
To prove that L is strictly monotone, we define a functional $\mathcal{L}: W_0\to\mathbb{R}$ as

$$\mathcal{L}(u)=\frac{1}{p}\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy,$$

by virtue of [25, Lemma 3.3], we get $\mathcal{L}\in C^1(W_0)$ and

$$\langle \mathcal{L}'(u),v\rangle = \iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}(v(x)-v(y))dxdy.$$

Taking into the Simon inequalities (see [22]), for all $x,y\in\mathbb{R}^n$,

$$\begin{aligned}&|x-y|^p \leq c_p\left(|x|^{p-2}x-|y|^{p-2}y\right)(x-y) \text{ for } p\geq 2\\ &|x-y|^p \leq C_p\left[\left(|x|^{p-2}x-|y|^{p-2}y\right)(x-y)\right]^{\frac{p}{2}}\left(|x|^p+|y|^p\right)^{\frac{2-p}{2}} \text{ for } 1<p<2,\end{aligned} \tag{3.3}$$

where $c_p=\left(\frac{1}{2}\right)^{-p}$ and $C_p=\frac{1}{p-1}$.
We obtain that $\mathcal{L}'$ is strictly monotone. According to [26, Proposition 25.10], $\mathcal{L}$ is strictly convex. Moreover, as M is nondecreasing, then $\mathcal{M}$ is convex. This proves that Ψ_M is strictly convex, since $\Psi'_M=L$ in W_0^*, as a result L is strictly monotone in W_0.
5) - Let $(u_k)_k$ be a sequence in W_0 such that

$$\begin{cases} u_k\rightharpoonup u & \text{in } W_0\\ \limsup\limits_{k\to\infty}\langle L(u_k)-L(u),u_k-u\rangle\leq 0.\end{cases} \tag{3.4}$$

We will prove that $u_k\to u$ in W_0.
For each $u\in W_0$, we take into account the functional $\mathcal{P}(u): W_0\to W_0^*$ by

$$\langle \mathcal{P}(u),v\rangle = \iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}(v(x)-v(y))dxdy,$$

for all $v \in W_0$. Then, $\mathcal{P}(u)$ is a continuous linear functional in W_0 and

$$|\langle \mathcal{P}(u), v\rangle| \leq \|u\|_{W_0}^{p-1}\|v\|_{W_0} \quad \text{for all } v \in W_0.$$

Let $u_k \rightharpoonup u$ in W_0 and let $\{u_k\}_k$ satisfy (3.4). Then, the weak convergence of $\{u_k\}_k$ in W_0 implies that

$$\lim_{k\to\infty} \langle \mathcal{P}(u), u_k - u\rangle = 0,$$

and $(u_k)_k$ is bounded in W_0. The continuity of M implies that $\left\{M\left(||u_k||_{W_0}^p\right)\right\}_k$ is bounded, then

$$\lim_{k\to\infty} M\left(||u_k||_{W_0}^p\right)\langle \mathcal{P}(u), u_k - u\rangle = 0. \tag{3.5}$$

Similarly,

$$\lim_{k\to\infty} M\left(||u||_{W_0}^p\right)\langle \mathcal{P}(u), u_k - u\rangle = 0. \tag{3.6}$$

It follows from (3.4) that

$$\lim_{k\to\infty} \left\langle M\left(||u_k||_{W_0}^p\right)\mathcal{P}(u_k) - M\left(||u||_{W_0}^p\right)\mathcal{P}(u), u_k - u\right\rangle = 0.$$

Hence, by (3.5) and (3.6)

$$\lim_{k\to\infty} M\left(||u_k||_{W_0}^p\right)\langle \mathcal{P}(u_k) - \mathcal{P}(u), u_k - u\rangle = 0.$$

Moreover, assumption (A_2) implies that

$$\lim_{k\to\infty} \langle \mathcal{P}(u_k) - \mathcal{P}(u), u_k - u\rangle = 0.$$

If $p \geqslant 2$, then

$$||u_k - u||_{W_0}^p \leq C_3\langle \mathcal{P}(u_k) - \mathcal{P}(u), u_k - u\rangle,$$

if $1 < p < 2$, then

$$||u_k - u||_{W_0}^p \leq C\left(\langle \mathcal{P}(u_k) - \mathcal{P}(u), u_k - u\rangle\right)^{\frac{p}{2}}\left(||u_k||_{W_0}^p + ||u||_{W_0}^p\right)^{\frac{2-p}{2}},$$

therefore, $u_k \to u$ strongly in W_0. □

Lemma 3.2 *The operator $S : W_0 \to W_0^*$ setting by*

$$\langle Su, v\rangle = \int_\Omega f(x,u)v(x)dx, \quad \forall u, v \in W_0$$

is compact.

Proof: Let $\psi : W_0 \to L^{p'}(\Omega)$ be the operator defined by

$$\psi u(x) := -f(x,u) \text{ for } u \in W_0,\ x \in \Omega.$$

We have

$$|x-y|^p \leq 2^{p-1}\left(|x|^p + |y|^p\right), \quad 1 < p, \tag{3.7}$$

and since $1 < \frac{p}{p-1}$, for each $u \in W_0$, it follows (without loss of generality, we may assume that $\gamma = p-1$) that

$$\begin{aligned}
||\psi u||_{p'}^{p'} &= \int_\Omega |f(x,u)|^{p'}dx \\
&\leq \int_\Omega |\mathcal{E}(x) + c|u|^{p-1}|^{p'}dx \\
&\leq C_4\left(||\mathcal{E}||_{p'}^{p'} + C_5||u||_p^p\right) \\
&\leq C_4\left(||\mathcal{E}||_{p'}^{p'} + C_6||u||_{W_0}^p\right)
\end{aligned}$$

where the continuous embedding $W_0 \hookrightarrow L^p$ was employed. Therefore, ψ is bounded on W_0.
For the continuity of ψ. Let $u_k \to u$ in W_0, then $u_k \to u$ in $L^p(\Omega)$, thus there exists measurable function $\lambda \in L^p(\Omega)$ and a subsequence still denoted by (u_k) such that

$$\begin{aligned} u_k(x) &\to u(x) \\ u_k(x) &\le \lambda(x), \end{aligned}$$

for almost every $x \in \Omega$. Thanks to (A_1), we obtain

$$\begin{aligned} f(x, u_k(x)) \to f(x, u_k(x)) \quad \text{a.e} \quad x \in \Omega, \\ |f(x, u_k(x))| \le \mathcal{E}(x) + C_2|\lambda(x)|^{p-1}. \end{aligned} \tag{3.8}$$

Since $\mathcal{E}(x) + C_2|\lambda(x)|^{p-1} \in L^{p'}(\Omega)$ and from (3.8), we get

$$\int_\Omega |f(x, u_k) - f(x, u)|^{p'}\, dx \to 0.$$

We conclude that by applying the dominated convergence theorem

$$\psi u \to \psi u_k \quad \text{in} \quad L^{p'}(\Omega).$$

Then ψ is continuous.
Since the embedding $I : W_0 \to L^p(\Omega)$ is compact, then the adjoint operator $I^* : L^{p'}(\Omega) \to W_0^*$ is also compact. Consequently, the composition $I^* o\psi : W_0 \to W_0^*$ is compact. We find that $S = I^* o\psi$ is compact. □

4. Main results

In this section, we transform the problem (1.1) into a problem determined by a Hammerstein equation. Using the Berkovits topological degree theory introduced in the above section, we prove the existence of weak solutions for our problem.

Theorem 4.1 *Suppose that (A_1) and (A_2) holds, then problem (1.1) has a weak solution u in W_0.*

Proof: Let $u \in W_0$, u is a weak solutions of the problem (1.1) if and only if

$$Lu = -Su, \tag{4.1}$$

L and S are two operators defined respectively in (3.1) and Lemma 3.2.

According to Lemma 3.1, the operator L is bounded, continuous, coercive, strictly monotone, and of (S_+) type. Then, the inverse operator $T := L^{-1} : W_0^* \to W_0$ exists according to the Minty-Browder Theorem (see [26, Theorem 26]). Moreover, it is bounded, continuous, and of the type (S_+).
In addition, from Lemma 3.2 the operator S is bounded, quasimonotone, and continuous. Consequently, equation (4.1) is equivalent to

$$u = Tv \quad \text{and} \quad v + SoTv = 0. \tag{4.2}$$

We will use the degree theory discussed in the section above to solve (4.2). So, we state first that the set

$$B := \left\{ v \in W_0^* \mid \quad v + tSoTv = 0 \quad \text{for some} \quad t \in [0, 1] \right\}$$

is bounded in W_0^*.
In fact Let $v \in B$ and take $u := Tv$, as $\beta \geqslant 1$, we obtain

$$\begin{aligned} m_0 ||Tv||_{W_0}^p &= m_0 ||u||_{W_0}^p \\ &\leq m_0 \left(||u||_{W_0}^p\right)^\beta \\ &= m_0 \left(||u||_{W_0}^p\right)^{\beta-1} ||u||_{W_0}^p \\ &\leq M\left(||u||_{W_0}^p\right) \iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \\ &= \langle Lu, u\rangle \\ &= \langle v, Tv\rangle \\ &= -t\langle SoTv, Tv\rangle \\ &\leq t \int_\Omega f(x,u) u dx \\ &\leq t\left(C||\mathcal{E}||_{p'} ||u||_{W_0} + C_7^{\gamma+1} ||u||_{W_0}^{\gamma+1}\right). \end{aligned}$$

We get the boundedness of $\{Tv | v \in B\}$. Since the operator S is bounded and from (4.2), it follows that the set B is bounded in W_0^*. Then, there exists a positive constant R such that

$$\|v\|_{W_0^*} < R \text{ for all } v \in B.$$

As a result,

$$v + tSoTv \neq 0 \quad \text{for all} \quad v \in \partial B_R(0) \quad \text{and all} \quad t \in [0,1].$$

It follows from Lemma 2.2 that

$$I + SoT \in \mathcal{J}_T\left(\overline{B_R(0)}\right) \quad \text{and} \quad I = LoT \in \mathcal{J}_T\left(\overline{B_R(0)}\right).$$

Since the operators I, S and T are bounded, $I + SoT \in \mathcal{J}_{T,B}\left(\overline{B_R(0)}\right)$ and $I \in \mathcal{J}_{T,B}\left(\overline{B_R(0)}\right)$.
Consider a homotopy $\Phi : [0,1] \times \overline{B_R(0)} \to W_0^*$ given by

$$\Phi(t,v) := v + tSoTv \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the properties 1) and 3) of the degree d stated in Theorem 2.1, we have

$$d\left(I + SoT, B_R(0), 0\right) = d\left(I, B_R(0), 0\right) = 1,$$

and as a result, there is a point $v \in B_R(0)$ where

$$v + SoTv = 0.$$

As a conclusion $u = Tv$ is a weak solution of problem (1.1). This ends the proof. □

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