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Geometry of η-Ricci Yamabe Soliton on Nearly Sasakian Manifold

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ABSTRACT: The present paper is devoted to study η-Ricci-Yamabe soliton on nearly-Sasakian manifolds. We examine Ricci-semisymmetricity and Einstein- semisymmetric on nearly-sasakian manifold to obtain condition for shrinking or steady or expanding. Further, we analyze the 3-dimensional nearly-Sasakian manifolds admitting η-Ricci-Yamabe soliton satisfying certain geometric conditions.

Key Words: Nearly-Sasakian manifold, η-Ricci-Yamabe soliton, Einstein tensor, η-Einstein manifold, Projective curvature tensor.

Contents

1. INTRODUCTION

A Ricci soliton is nothing but a generalised concept of the Einstein metric. In a Riemannian manifold $(\Omega, \mathfrak{g})$ of dimension n, the metric $\mathfrak{g}$ is called a Ricci soliton if

$$(\mathfrak{L}_{\Upsilon}\mathfrak{g})(\mathcal{A}_1,\mathcal{A}_2) + 2S(\mathcal{A}_1,\mathcal{A}_2) + 2\lambda_1\mathfrak{g}(\mathcal{A}_1,\mathcal{A}_2) = 0, \tag{1.1}$$

where $\mathfrak{L}_{\Upsilon}\mathfrak{g}$ is the Lie derivative of the Riemannian metric $\mathfrak{g}$ along the vector field Υ, S is the Ricci tensor, λ_1 is a constant and $\mathcal{A}_1, \mathcal{A}_2$ are arbitrary vector fields on $T\Omega$. The Ricci soliton is said to be expanding if $\lambda_1 = positive$; shrinking if $\lambda_1 = negative$; and steady if $\lambda_1 = 0$. A Ricci soliton with $\Upsilon = 0$ reduces to an Einstein equation. Cho and Kimura(2009) studied an η-Ricci soliton $(\mathfrak{g}, \Upsilon, \lambda_1, \lambda_2)$ as the generalization of the Ricci soliton $(\mathfrak{g}, \Upsilon, \lambda_1)$ [4]. They have explained the extension of Ricci soliton as a notion of η-Ricci soliton [4]. Hamilton(1988) initiated the concepts – Ricci flow and Yamabe flow [5]. Some related works can be found in [1-3,6-17]. The η-Ricci Yamabe soliton (η-RYS) concept was defined by Siddiqi and Akyol [18] as follows:

$$(\mathfrak{L}_{\Upsilon}\mathfrak{g})(\mathcal{A},\mathcal{B}) + 2\rho S(\mathcal{A},\mathcal{B}) + (2\lambda_1 - qr)\mathfrak{g}(\mathcal{A},\mathcal{B}) + 2\lambda_2\eta(\mathcal{A})\eta(\mathcal{B}) = 0, \tag{1.2}$$

where λ_1 and λ_2 are constants.

In this paper, Ricci tensor for nearly Sasakian η-RYS is attained, and Ricci-semisymmetricity and Einstein-semisymmetricity conditions are studied. Nearly-Sasakian η-RYS satisfying the situation $S \cdot R =$

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0 is discussed. η-RYS on a 3-dimensional nearly-Sasakian manifold satisfying the situation $Q \cdot R = 0$ is discussed. Further, η-RYS on a 3-dimensional nearly-Sasakian manifold satisfying the situation $R(\mathcal{A}, \varsigma) \cdot P - P(\mathcal{A}, \varsigma) \cdot R = 0$ is discussed.

2. BASICS

Let Ω denotes a smooth manifold of dimension n. An almost contact structure on Ω is given by $(\Theta, \varsigma, \eta)$, where Θ is a one-one tensor field, ς is a vector field, and η is a 1-form on Ω satisfying the relations

$$\Theta^2\mathcal{A} = -\mathcal{A} + \eta(\mathcal{A})\varsigma, \ \ \eta(\varsigma) = 1, \ \ \Theta(\varsigma) = 0, \ \ \eta(\varsigma\mathcal{A}) = 0, \tag{2.1}$$

$$\mathfrak{g}(\Theta\mathcal{A}, \Theta\mathcal{B}) = \mathfrak{g}(\mathcal{A}, \mathcal{B}) - \eta(\mathcal{A})\eta(\mathcal{B}), \tag{2.2}$$

for all $\mathcal{A}, \mathcal{B}$ in the Lie algebra $\chi(\Omega)$ of all vector fields on Ω. An almost contact manifold provided with a Riemannian metric is called as an almost contact metric manifold(ACMM) and it is expressed by $\Omega(\Theta, \varsigma, \eta, \mathfrak{g})$. In addition, on a Nearly-Sasakian manifold(NSM), below mentioned identities are preserved:

$$R(\mathcal{A}_1, \mathcal{A}_2)\varsigma = \eta(\mathcal{A}_2)\mathcal{A}_1 - \eta(\mathcal{A}_1)\mathcal{A}_2, \tag{2.3}$$

$$R(\varsigma, \mathcal{A}_1)\mathcal{A}_2 = -R(\mathcal{A}_1, \varsigma)\mathcal{A}_2 = \mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2)\varsigma - \eta(\mathcal{A}_2)\mathcal{A}_1, \tag{2.4}$$

$$S(\mathcal{A}_1, \varsigma) = (n-1)\eta(\mathcal{A}_1), \tag{2.5}$$

$$Q\varsigma = (n-1)\varsigma, \tag{2.6}$$

$$S(\Theta\mathcal{A}_1, \Theta\mathcal{A}_2) = S(\mathcal{A}_1, \mathcal{A}_2) - (n-1)\eta(\mathcal{A}_1)\eta(\mathcal{A}_2), \tag{2.7}$$

for all vector fields $\mathcal{A}_1$,$\mathcal{A}_2$ on Ω. The following definitions are needed:

Definition 2.1 *Let Ω be an n-dimensional NSM. We say that Ω is Ricci semisymmetric if $R \cdot S = 0$.*

Definition 2.2 *Let Ω be an n-dimensional NSM. We say that Ω is Einstein semisymmetric if $R \cdot E = 0$, where E is the Einstein tensor given by*

$$E(\mathcal{A}_1, \mathcal{A}_2) = S(\mathcal{A}_1, \mathcal{A}_2) - \left(\frac{r}{n}\right)\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2), \tag{2.8}$$

where S is the Ricci tensor and r is a scalar curvature of the manifold Ω.

Definition 2.3 *We say that a manifold Ω is an η-Einstein manifold if its Ricci tensor is given by*

$$S(\mathcal{A}_1, \mathcal{A}_2) = c_1\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2) + c_2\eta(\mathcal{A}_1)\eta(\mathcal{A}_2),$$

for all vector fields $\mathcal{A}_1$, $\mathcal{A}_2$ on Ω, where c_1 and c_2 are smooth functions on Ω.

3. η-RYS ON NSM

Let Ω be a NSM. Let $(\mathfrak{g}, \varsigma, \lambda_1, \lambda_2)$ be an η-RYS on Ω. Then by (1.2),

$$2\rho S(\mathcal{A}, \mathcal{B}) = -(\mathfrak{L}_v\mathfrak{g})(\mathcal{A}, \mathcal{B}) - (2\lambda_1 - qr)\mathfrak{g}(\mathcal{A}, \mathcal{B}) - 2\lambda_2\eta(\mathcal{A})\eta(\mathcal{B}). \tag{3.1}$$

On a nearly-Sasakian manifold, from a nearly-Sasakian structure $\nabla_{\mathcal{A}}\varsigma = -\Theta\mathcal{A}$, we have

$$(\mathfrak{L}_v\mathfrak{g})(\mathcal{A}, \mathcal{B}) = 0. \tag{3.2}$$

Using (3.2) in (3.1), we obtain

$$S(\mathcal{A}, \mathcal{B}) = \frac{(qr - 2\lambda_1)}{2\rho}\mathfrak{g}(\mathcal{A}, \mathcal{B}) - \frac{\lambda_2}{\rho}\eta(\mathcal{A})\eta(\mathcal{B}), \tag{3.3}$$

$$Q\mathcal{A} = \frac{(qr - 2\lambda_1)}{2\rho}\mathcal{A} - \frac{\lambda_2}{\rho}\eta(\mathcal{A})\varsigma. \tag{3.4}$$

Putting $\mathcal{B} = \varsigma$ in (3.3) and (3.4), we obtain

$$S(\mathcal{A}, \varsigma) = \frac{(qr - 2\lambda_1 - 2\lambda_2)}{2\rho}\eta(\mathcal{A}), \tag{3.5}$$

$$Q\varsigma = \frac{(qr - 2\lambda_1 - 2\lambda_2)}{2\rho}\varsigma. \tag{3.6}$$

Comparing (3.5) with (2.5), we obtain

$$n = 1 + \frac{(qr - 2\lambda_1 - 2\lambda_2)}{2\rho}.$$

Consequently, we obtain the following outcome.

Theorem 3.1 *For an η-RYS on a NSM Ω, the Ricci tensor is in the form (3.3) and* $\lambda_1 + \lambda_2 = \rho(1-n) + \frac{qr}{2}$.

4. η-RYS ON RICCI-SEMISYMMETRIC NSM

Consider a Ricci-semisymmetric NSM Ω. By Definition 2.1,

$$S(R(\varsigma, \mathcal{A}_1)\mathcal{A}_2, \mathcal{A}_3) + S(\mathcal{A}_2, R(\varsigma, \mathcal{A}_1)\mathcal{A}_3) = 0. \tag{4.1}$$

Suppose Ricci-semisymmetric NSM admits an η-RYS. Then by Eq.(3.3), Eq.(4.1) becomes

$$\begin{aligned}&\left[\frac{qr - 2\lambda_1}{2\rho}\right]\mathfrak{g}(R(\varsigma, \mathcal{A}_1)\mathcal{A}_2, \mathcal{A}_3) - \frac{\lambda_2}{\rho}\eta(R(\varsigma, \mathcal{A}_1)\mathcal{A}_2)\eta(\mathcal{A}_3)\\ &+\left[\frac{qr - 2\lambda_1}{2\rho}\right]\mathfrak{g}(\mathcal{A}_2, R(\varsigma, \mathcal{A}_1)\mathcal{A}_3) - \frac{\lambda_2}{\rho}\eta(R(\varsigma, \mathcal{A}_1)\mathcal{A}_3)\eta(\mathcal{A}_2) = 0.\end{aligned} \tag{4.2}$$

Using (2.4), we have

$$\begin{aligned}&\left[\frac{qr - 2\lambda_1}{2\rho}\right]\mathfrak{g}(\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2)\varsigma - \eta(\mathcal{A}_2)\mathcal{A}_1, \mathcal{A}_3) - \frac{\lambda_2}{\rho}\eta(\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2)\varsigma - \eta(\mathcal{A}_2)\mathcal{A}_1)\eta(\mathcal{A}_3) +\\ &\left[\frac{qr - 2\lambda_1}{2\rho}\right]\mathfrak{g}(\mathcal{A}_2, \mathfrak{g}(\mathcal{A}_1, \mathcal{A}_3)\varsigma - \eta(\mathcal{A}_3)\mathcal{A}_1) - \frac{\lambda_2}{\rho}\eta(\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_3)\varsigma - \eta(\mathcal{A}_3)\mathcal{A}_1)\eta(\mathcal{A}_2) = 0\end{aligned} \tag{4.3}$$

which implies

$$\frac{\lambda_2}{\rho}[\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2)\eta(\mathcal{A}_3) + \eta(\mathcal{A}_2)\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_3) - 2\eta(\mathcal{A}_1)\eta(\mathcal{A}_2)\eta(\mathcal{A}_3)] = 0. \tag{4.4}$$

Putting $\mathcal{A}_3 = \varsigma$ in (4.4), yields

$$\frac{\lambda_2}{\rho}[\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2) - \eta(\mathcal{A}_1)\eta(\mathcal{A}_2)] = 0. \tag{4.5}$$

Replace $\mathcal{A}_1$ by $\Theta\mathcal{A}_1$ and $\mathcal{A}_2$ by $\Theta\mathcal{A}_2$, we obtain

$$\frac{\lambda_2}{\rho}\mathfrak{g}(\Theta\mathcal{A}_1, \Theta\mathcal{A}_2) = 0. \tag{4.6}$$

Hence $\lambda_2 = 0$. Therefore, by Theorem 3.1, we have $\lambda_1 = \rho(1-n) + \frac{qr}{2}$. Thus, we have

Theorem 4.1 *A nearly-Sasakian η-RYS cannot be Ricci-semisymmetric and* $\lambda_1 = \rho(1-n) + \frac{qr}{2}$.

5. NEARLY-SASAKIAN η-RYS SATISFYING $S \cdot R = 0$

Consider a nearly-Sasakian η-RYS satisfying

$$(S(\mathcal{A}_1, \mathcal{A}_2) \cdot R)(\mathcal{U}, \Upsilon)\mathcal{A}_3 = 0, \tag{5.1}$$

for all $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{U}$ and Υ on Ω. Equation (5.1) implies that

$$\begin{aligned}(\mathcal{A}_1 \Lambda_s \mathcal{A}_2)R(\mathcal{U}, \Upsilon)\mathcal{A}_3 \,+\, R((\mathcal{A}_1 \Lambda_s \mathcal{A}_2)\mathcal{U}, \Upsilon)\mathcal{A}_3 \,+\, R(\mathcal{U}, (\mathcal{A}_1 \Lambda_s \mathcal{A}_2)\Upsilon)\mathcal{A}_3 \\ + R(\mathcal{U}, \Upsilon)(\mathcal{A}_1 \Lambda_s \mathcal{A}_2)\mathcal{A}_3 = 0,\end{aligned} \tag{5.2}$$

where $(\mathcal{A}_1 \Lambda_s \mathcal{A}_2)\mathcal{A}_3 = S(\mathcal{A}_2, \mathcal{A}_3)\mathcal{A}_1 - S(\mathcal{A}_1, \mathcal{A}_3)\mathcal{A}_2$. Using (5.2) in (5.1) and taking $\mathcal{A}_1 = \varsigma$, we obtain

$$\begin{aligned}S(\mathcal{A}_2, R(\mathcal{U}, \Upsilon)\mathcal{A}_3)\varsigma - S(\varsigma, R(\mathcal{U}, \Upsilon)\mathcal{A}_3)\mathcal{A}_2 + S(\mathcal{A}_2, \mathcal{U})R(\varsigma, \Upsilon)\mathcal{A}_3 - S(\varsigma, \mathcal{U}), R(\mathcal{A}_2, \Upsilon)\mathcal{A}_3 \\ + S(\mathcal{A}_2, \Upsilon)R(\mathcal{U}, \varsigma)\mathcal{A}_3 - S(\varsigma, \Upsilon)R(\mathcal{U}, \mathcal{A}_2)\mathcal{A}_3 + S(\mathcal{A}_2, \mathcal{A}_3)R(\mathcal{U}, \Upsilon)\varsigma \\ - S(\varsigma, \mathcal{A}_3)R(\mathcal{U}, \Upsilon)\mathcal{A}_2 = 0.\end{aligned} \tag{5.3}$$

Taking inner product with ς, using (3.3) and putting $\Upsilon = \mathcal{A}_3 = \varsigma$, in (5.3), we obtain

$$\begin{aligned}&\left[\frac{qr - 2\lambda_1}{2\rho}\right] [(\mathfrak{g}(\mathcal{A}_2, R(\mathcal{U}, \varsigma)\varsigma) - \eta(\mathcal{A}_2)\mathfrak{g}(\varsigma, R(\mathcal{U}, \varsigma)\varsigma) - \eta(\mathcal{U})\,\eta(R(\mathcal{A}_2, \varsigma)\varsigma) \\ &+ \eta(\mathcal{A}_2)\,\eta(R(\mathcal{U}, \varsigma)\varsigma) - \eta(R(\mathcal{U}, \mathcal{A}_2)\varsigma) + \eta(\mathcal{A}_2)\,\eta(R(\mathcal{U}, \varsigma)\varsigma) - \eta(R(\mathcal{U}, \varsigma)\mathcal{A}_2))] \\ &+ \frac{\lambda_2}{\rho}\{\eta(\mathcal{U})\eta(R(\mathcal{A}_2, \varsigma)\varsigma) - \eta(\mathcal{A}_2)\eta(R(\mathcal{U}, \varsigma)\varsigma + \eta(R(\mathcal{U}, \mathcal{A}_2)\varsigma) - \eta(\mathcal{A}_2)\eta(R(\mathcal{U}, \varsigma)\varsigma) \\ &+ \eta(R(\mathcal{U}, \varsigma)\mathcal{A}_2))\} = 0.\end{aligned} \tag{5.4}$$

Using (2.3) and (2.4) in (5.4) we obtain

$$\left[\frac{2\lambda_1 - qr + \lambda_2}{\rho}\right]\mathfrak{g}(\mathcal{A}_2, \mathcal{U}) - \eta(\mathcal{U})\eta(\mathcal{A}_2) = 0,$$
$$\lambda_1 = \frac{qr - \lambda_2}{2}.$$

Then by (3.3), we have

$$S(\mathcal{A}_1, \mathcal{A}_2) = \frac{\lambda_2}{\rho}\left[\frac{1}{2}\mathfrak{g}(\mathcal{A}_1, \mathcal{A}_2) - \eta(\mathcal{A}_1)\eta(\mathcal{A}_2)\right]. \tag{5.5}$$

Substituting $\mathcal{A}_2 = \varsigma$ in (5.5), we obtain

$$S(\mathcal{A}_1, \varsigma) = -\left[\frac{\lambda_2}{\rho}\right]\eta(\mathcal{A}_1). \tag{5.6}$$

Comparing this with (2.5) results

$$\lambda_2 = \rho(1 - n), \qquad \lambda_1 = \frac{(qr - \rho(1 - n))}{2}.$$

Hence, we have

Theorem 5.1 *If a nearly-Sasakian η-RYS satisfies the condition $S \cdot R = 0$, then $\lambda_2 = \rho(1 - n)$, $\lambda_1 = \frac{qr - \rho(1-n)}{2}$ and the underlying manifold is η-Einstein.*

6. η-RYS ON EINSTEIN-SEMISYMMETRIC NSMs

Consider an Einstein-semisymmetric NSM Ω. By Definition 2.2,

$$E(R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3,\mathcal{W})+E(\mathcal{A}_3,R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{W})=0. \tag{6.1}$$

Substituting $\mathcal{A}_1=\varsigma$ and using (2.8), we get

$$S(R(\varsigma,\mathcal{A}_2)\mathcal{A}_3,\mathcal{W})+S(\mathcal{A}_3,R(\varsigma,\mathcal{A}_2)\mathcal{W})-\left(\frac{r}{n}\right)[\mathfrak{g}(R(\varsigma,\mathcal{A}_2)\mathcal{A}_3,\mathcal{W})+\mathfrak{g}(\mathcal{A}_3,R(\varsigma,\mathcal{A}_2)\mathcal{W})]=0. \tag{6.2}$$

If the Einstein-semisymmetric NSM admits an η-RYS, then by (3.3) we get

$$\begin{aligned}&\left[\frac{qr-2\lambda_1}{2\rho}\right]\mathfrak{g}(R(\varsigma,\mathcal{A}_2)\mathcal{A}_3,\mathcal{W})-\frac{\lambda_2}{\rho}\eta(R(\varsigma,\mathcal{A}_2)\mathcal{A}_3)\eta(\mathcal{W})-\left[\frac{r}{n}\right]\mathfrak{g}(R(\varsigma,\mathcal{A}_2)\mathcal{A}_3,\mathcal{W})\\&\quad+\left[\frac{qr-2\lambda_1}{2\rho}\right]\mathfrak{g}(\mathcal{A}_3,R(\varsigma,\mathcal{A}_2)\mathcal{W})-\frac{\lambda_2}{\rho}\eta(R(\varsigma,\mathcal{A}_2)\mathcal{W})\eta(\mathcal{A}_3)\\&\quad-\frac{r}{n}\mathfrak{g}(\mathcal{A}_3,R(\varsigma,\mathcal{A}_2)\mathcal{W})=0.\end{aligned} \tag{6.3}$$

Now by (2.4) and taking $\mathcal{W}=\varsigma$, we obtain

$$-\frac{\lambda_2}{\rho}[\mathfrak{g}(\mathcal{A}_2,\mathcal{A}_3)\eta(\mathcal{W})-2\eta(\mathcal{A}_2)\eta(\mathcal{A}_3)\eta(\mathcal{W})+\eta(\mathcal{A}_3)\mathfrak{g}(\mathcal{A}_2,\mathcal{W})]=0, \tag{6.4}$$

which implies $\lambda_2=0$. Then, by theorem (3.1), we have $\lambda_1=\rho(1-n)+\frac{qr}{2}$. Thus, we have

Theorem 6.1 *A nearly-Sasakian η-RYS cannot be Einstein-semisymmetric and* $\lambda_1=\rho(1-n)+\frac{qr}{2}$.

7. η-RYS ON 3-DIMENSIONAL NSM SATISFYING $Q\cdot R=0$

Let Ω be a 3-dimensional NSM admitting η-RYS that satisfies $Q\cdot R=0$. Then

$$Q(R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3-R(Q\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3-R(\mathcal{A}_1,Q\mathcal{A}_2)\mathcal{A}_3-R(\mathcal{A}_1,\mathcal{A}_2)Q\mathcal{A}_3=0, \tag{7.1}$$

for all $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ on Ω. In view of (3.4), Eq. (7.1) takes the form

$$\begin{aligned}&\frac{\lambda_2}{\rho}[\eta(\mathcal{A}_1)R(\varsigma,\mathcal{A}_2)\mathcal{A}_3-\eta(R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3)\varsigma+\eta(\mathcal{A}_2)R(\mathcal{A}_1,\varsigma)\mathcal{A}_3+\eta(\mathcal{A}_3)R(\mathcal{A}_1,\mathcal{A}_2)\varsigma]\\&\qquad-\frac{(qr-2\lambda_1)}{2\rho}[2R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3]=0.\end{aligned}$$

Taking inner product with ς and using the equations (2.3) and (2.4), above equation reduces to

$$\eta(R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3)\left[\frac{2\lambda_1-\lambda_2-qr}{\rho}\right]+\frac{\lambda_2}{\rho}[\eta(\mathcal{A}_1)\mathfrak{g}(\mathcal{A}_2,\mathcal{A}_3)-\eta(\mathcal{A}_2)\mathfrak{g}(\mathcal{A}_1,\mathcal{A}_3)]=0. \tag{7.2}$$

Taking $\mathcal{A}_1=\varsigma$ in (7.2), we get

$$\frac{(2\lambda_1-qr)}{\rho}[\mathfrak{g}(\mathcal{A}_2,\mathcal{A}_3)-\eta(\mathcal{A}_2)\eta(\mathcal{A}_3)]=0.$$

Upon simplification above equation yields

$$\lambda_1=\frac{qr}{2}.$$

Thus, we have:

Theorem 7.1 *If a 3-dimensional NSM admitting η-RYS satisfies $Q\cdot R=0$, then the Ricci soliton is*

1. *expanding,* *for* $\frac{qr}{2}>0$,
2. *shrinking,* *for* $\frac{qr}{2}<0$.

8. η-RYS ON 3-DIMENSIONAL NSM SATISFYING $R(\mathcal{A},\varsigma)\cdot P - P(\mathcal{A},\varsigma)\cdot R = 0$

Let M be a 3-dimensional NSM admitting η-RYS that satisfies

$$R(\mathcal{A},\varsigma)\cdot P - P(\mathcal{A},\varsigma)\cdot R = 0 \tag{8.1}$$

We require the following definition:

Definition 8.1 *The projective curvature tensor P in a 3-dimensional NSM is defined by*

$$P(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3 = R(\mathcal{A}_1,\mathcal{A}_2)\mathcal{A}_3 - \frac{1}{n}[S(\mathcal{A}_2,\mathcal{A}_3)\mathcal{A}_1 - S(\mathcal{A}_1,\mathcal{A}_3)\mathcal{A}_2]. \tag{8.2}$$

The following results are due to (7.1):

$$P(\mathcal{A}_1,\varsigma) = \left(\frac{2n\rho - qr + 2\lambda_1 + 2\lambda_2}{2n\rho}\right)\eta(\mathcal{A}_2)\mathcal{A}_1 - g(\mathcal{A}_1,\mathcal{A}_2)\varsigma + \frac{1}{n}S(\mathcal{A}_1,\mathcal{A}_2)\varsigma, \tag{8.3}$$

$$P(\mathcal{A}_1,\mathcal{A}_2)\varsigma = \left(\frac{2n\rho - qr + 2\lambda_1 + 2\lambda_2}{2n\rho}\right)(\eta(\mathcal{A}_2)\mathcal{A}_1 - \eta(\mathcal{A}_1)\mathcal{A}_2), \tag{8.4}$$

$$P(\mathcal{A}_1,\varsigma)\varsigma = \left(\frac{2n\rho - qr + 2\lambda_1 + 2\lambda_2}{2n\rho}\right)\mathcal{A}_1 - \eta(\mathcal{A}_1)\varsigma. \tag{8.5}$$

From (8.1), we have

$$\begin{aligned}
&R(\mathcal{A},\varsigma)P(\mathcal{U},\Upsilon)\mathcal{W} - P(R(\mathcal{A},\varsigma)\mathcal{U},\Upsilon)\mathcal{W} - P(\mathcal{U},R(\mathcal{A},\varsigma)\Upsilon)\mathcal{W}\\
&-P(\mathcal{U},\Upsilon)R(\mathcal{A},\varsigma)\mathcal{W} - P(\mathcal{A},\varsigma)R(\mathcal{U},\Upsilon)\mathcal{W} + R(P(\mathcal{A},\varsigma)\mathcal{U},\Upsilon)\mathcal{W}\\
&+R(\mathcal{U},P(\mathcal{A},\varsigma)\Upsilon)\mathcal{W} + R(\mathcal{U},\Upsilon)P(\mathcal{A},\varsigma)\mathcal{W} = 0.
\end{aligned}$$

Substituting $\mathcal{U} = \mathcal{W} = \varsigma$, yields

$$\begin{aligned}
R(\mathcal{A},\varsigma)P(\varsigma,\Upsilon)\varsigma - P(R(\mathcal{A},\varsigma)\varsigma,\Upsilon)\varsigma - P(\varsigma,R(\mathcal{A},\varsigma)\Upsilon)\varsigma - P(\varsigma,\Upsilon)R(\mathcal{A},\varsigma)\varsigma&\\
- P(\mathcal{A},\varsigma)R(\varsigma,\Upsilon)\varsigma + R(P(\mathcal{A},\varsigma)\varsigma,\Upsilon)\varsigma&\\
+ R(\varsigma,P(\mathcal{A},\varsigma)\Upsilon)\varsigma + R(\varsigma,\Upsilon)P(\mathcal{A},\varsigma)\varsigma = 0.&
\end{aligned} \tag{8.6}$$

Now, using (2.3),(2.4),(8.3)-(8.5) in (8.6), we obtain

$$\frac{2}{n}S(\Upsilon,\mathcal{A})\varsigma - g(\Upsilon,\mathcal{A})\varsigma + \eta(\mathcal{A})\eta(\Upsilon)\varsigma\left[\frac{n\rho - qr + 2\lambda_1 + 2\lambda_2}{n\rho}\right] = 0.$$

Applying inner product with ς in the aforementioned equation, we achieve

$$S(\Upsilon,\mathcal{A}) = \left(\frac{n}{2}\right)g(\Upsilon,\mathcal{A}) - \eta(\mathcal{A})\eta(\Upsilon)\left(\frac{n\rho - qr + 2\lambda_1 + 2\lambda_2}{2\rho}\right). \tag{8.7}$$

Putting $\mathcal{A} = \varsigma$ in (8.7) and using (2.5), we get $\lambda_1 = (1-n)\rho + \frac{qr}{2} - \lambda_2$.
Hence (8.7) reduces to

$$S(\Upsilon,\mathcal{A}) = \left(\frac{n}{2}\right)\mathfrak{g}(\Upsilon,\mathcal{A}) + \left(\frac{n}{2} - 1\right)\eta(\mathcal{A})\eta(\Upsilon).$$

Hence, we have

Theorem 8.1 *If a NSM of dimension 3 admitting η-RYS satisfies the relation*

$$R(\mathcal{A},\varsigma)\cdot P - P(\mathcal{A},\varsigma)\cdot R = 0,$$

then the Ricci soliton is steady and the manifold is Einstein.

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