Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.70586

Novel Kinds of Generalized Clairaut Equations and Their Singular Solutions

Ginés R. Pérez Teruel

ABSTRACT: The aim of this paper is twofold. On the one hand, we investigate the solutions of classical Clairaut-type equations armed with special function methods for the ODE, PDE and functional counterparts, generalizing some of the results found in the literature. On the other hand, we introduce a new generalization of Clairaut-type equations and investigate the properties of their solutions, with specific focus on the singular solutions. Though the extension of Clairaut's equations we provide can be considered as their linear or minimal extension, interestingly enough, we find that the singular solutions are non-linearly extended, and include extra functional terms that endow these singular solutions with very different properties when compared to those of classical Clairaut's equations. We show that this feature of Generalized Clairaut's equations is exclusive of the singular solutions, since the general solutions do not have such attribute.

Key Words: Clairaut-type equations, special functions, singular solutions.

Contents

Contents

1. Introduction and Basic Notions

One of the most remarkable examples of a nonlinear first-order differential equation is the standard Clairaut's equation (A. Clairaut [1], 1713-1765) defined by

$$y - xy' = \psi(y'). \tag{1.1}$$

Where $y = y(x)$ is an unknown real function of $x \in \mathbb{R}$, $y' = \frac{dy}{dx}$ and $\psi(y')$ is smooth function of its first derivative. Through the decades, Clairaut's equation has found a wide range of applications in several branches of applied mathematics and theoretical physics. Indeed, regarding the physics case, perhaps the most paradigmatic example is given by the usual functional equation in classical mechanics [2] [3], which connects the Hamiltonian with the Lagrangian of a dynamical system via the standard Legendre transformation. This relation resembles a Clairaut-type equation when the dependence on the functions that conform the variables of these functionals satisfy some conditions. Another more recent example is given by the effective action with composite fields in quantum field theory (QFT). Indeed, to see how

Submitted November 30, 2023. Published April 20, 2025
2010 *Mathematics Subject Classification:* 34A34, 35A99.

Clairaut- type equations (CTEs) arise in the context of QFT, suppose we have a model given in terms of the non-degenerate action $S[\phi]$ that depends on the scalar field $\phi = \phi(x)$. Then, defining $J = J(x)$ and $J\phi = \int dx J(x)\phi(x)$, where J is usual source to ϕ, the theory is described by the generating functional of the Green function, $Z[J]$

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi]+J\phi)} = e^{iW[J]}. \tag{1.2}$$

The effective action $\Gamma = \Gamma[\Phi]$ can be introduced by the Legendre transformation of $W[J]$ as

$$\Gamma[\Phi] = W[J] - J\Phi, \tag{1.3}$$

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x), \quad \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = -J(x). \tag{1.4}$$

After eliminating the source J from the last equations, it is straightforward to find a functional equation with an identical structure to the CTE

$$\Gamma - \frac{\delta\Gamma}{\delta\Phi}\Phi = W\Big[-\frac{\delta\Gamma}{\delta\Phi(x)}\Big]. \tag{1.5}$$

For a detailed discussion about these particular applications of Clairaut's equation in physics, the reader is referred to Refs. [2,3,4,5].
In order to solve Clairaut's equation, it is convenient to introduce a new variable z such that $z = \frac{dy}{dx}$. Differentiation of Eq. (1.1) with respect to x, yields

$$y''\Big(\psi'(z) + x\Big) = 0 \tag{1.6}$$

The general solution corresponds to the family of linear functions provided by

$$y_g(x) = Cx + \psi(C), \tag{1.7}$$

where $C \in \mathbb{R}$. More interesting is the singular solution, the name is due to the fact that is only one curve, which geometrically turns out to be the envelope of the family of straight lines of the general solution. This special solution arises by solving the differential equation

$$\psi'(z) + x = 0, \tag{1.8}$$

where $\psi'(z) = \frac{d\psi}{dz}$. The solution of this equation can be expressed in parametric form

$$\begin{cases} x = -\psi'(z) \\ y = \psi(z) - z\psi'(z) \end{cases} \tag{1.9}$$

No general methods are known to build these special or singular solutions of CTEs. A useful strategy employed recently in the literature [4,5,6,7,8,9,10], exploits the fact that, by virtue of Eq.(1.8), one can find $z = \phi(x)$, where $\phi(x)$ is a smooth function. If such smooth function exists, then it is possible to write directly the singular solution as a functional equation of the form

$$y_s(x) = x\phi(x) + \psi(\phi(x)). \tag{1.10}$$

On the other hand, the remarkable property of Clairaut's equation, namely, the fact that the derivative can be substituted by an arbitrary constant, attracted the attention of a number of researchers, who attempted to find more general equations with identical feature. After some initial (and now considered flawed) attempts, Raffy [11] is credited as the first who successfully provided a rigorous contribution to the issue of generalizing the class of Clairaut's equation. In particular, he proved that any differential equation with the structure

$$y = F(f(y')) + F(x - f(y')), \tag{1.11}$$

where F' and f are inverse functions, has the general solution

$$y = F(C) + F(x - C), \tag{1.12}$$

Therefore, it is sufficient to exchange the derivative by an arbitrary constant. Trough the years, higher order Clairaut's equations have been studied in the literature [12,13,14,15]. One of the most natural examples (cf. [16]) is the equation

$$p = xq + f(q), \tag{1.13}$$

where $p = dy/dx$, $q = d^2y/dx^2$, being f a smooth function of its argument. This equation can be easily solved and the general solution is given by

$$y_g(x) = \frac{1}{2}rx^2 + f(r)x + s \tag{1.14}$$

for $r, s \in \mathbb{R}$. Other interesting variations and generalizations of Clairaut's equations can be found in the work of Mitrinovic and Keckic [13].

The paper is organized as follows. Section 2 deals with a new family of singular solutions of CTEs and their PDE and functional counterparts, in terms of a special function such as the Lambert W function. In section 3, we generalize the structure of Clairaut's equation and provide a study of its singular solutions. Finally, in section 4, we discuss the results of the work.

2. Singular Solutions of Clairaut-type Equations with Special Functions Methods

Recently, several singular solutions of Clairaut-type equations were investigated for a variety of non-linear ansatzes for the function $\psi(z)$, like the important case $\psi(z) = \alpha \ln(1 + \beta z)$ [4,5,6,7]. Other non-algebraic choices for $\psi(z)$ can be addressed by means of special functions methods. Let us consider the transcendental function

$$\psi(z) = -\alpha z e^{\beta z}, \tag{2.1}$$

where α, β, $\in \mathbb{R}$. Substitution in Eq.(1.8) yields

$$\alpha e^{\beta z}\Big(1 + \beta z\Big) = x. \tag{2.2}$$

It is possible to solve this transcendental equation, and therefore to express z as a function of x, in the following closed form

$$z = \beta^{-1}\Big(W\Big(\frac{e}{\alpha}x\Big) - 1\Big)\ , \quad \beta \neq 0, \tag{2.3}$$

where W is the Lambert function of the argument $\frac{e}{\alpha}x$, e is obviously the base of the natural logarithm, and we have used the property, $W(xe^x) = x$, of the Lambert function. The $W(x)$ function is real for $x \geq -1/e$. When $-1/e \leq x < 0$, there are two possible real values of $W(x)$. The branch satisfying $-1 \leq W(x)$ is denoted as W_0 or just $W(x)$ when there is no possibility for confusion. The branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$. W_0 is usually called the *principal branch* of the $W(x)$ function. The interested reader can find in Ref. [17] a complete and systematic summary of the properties of this important special function. Physical applications of the Lambert W function include the relation among current,voltage and resistance in a diode [18], or the trajectories of a ballistic projectile affected by air resistance [19], among many others.

By virtue of the result given by Eq.(2.3), the singular solution of the Clairaut equation will therefore take the explicit expression

$$y_s(x) = \beta^{-1}\Big(W\Big(\frac{e}{\alpha}x\Big) - 1\Big)\Big(x - e^{W(\frac{e}{\alpha}x)-1}\Big)\ , \quad \beta \neq 0. \tag{2.4}$$

On the other hand, of the most interesting CTEs for physical applications is the PDE analogue of Eq.(1.1), that is of the form

$$y-\sum_{i=1}^{n}x^iz_i=\psi(z),\quad z_i=\frac{\partial y}{\partial x^i},\tag{2.5}$$

where $y=y(x)$ is an unknown function of the real variables $x=\{x^i\}$, and $\psi=\psi(z)$ a real prescribed function that depends on $z=\{z^i\}$, $i=1,2,...,n$. Differentiation with respect to $\{x^j\}$ provides

$$\sum_{i=1}^{n}\frac{\partial z_i}{\partial x^j}\Big(\frac{\partial\psi}{\partial z_i}+x^i\Big)=0,\quad j=1,2,...,n.\tag{2.6}$$

In the case of a vanishing Hessian matrix, $H_{ij}=0$, where

$$H_{ij}=\frac{\partial z_i}{\partial x^j}=\frac{\partial^2 y}{\partial x^i\partial x^j},\tag{2.7}$$

the general solution is the linear family given by

$$y_g(x)=\sum_{i=1}^{n}x^ic_i+\psi(c),\quad c=\{c_1,c_2,...,c_n\}=const.\tag{2.8}$$

Regarding the singular solution, it arises as the solution of the following PDE

$$\frac{\partial\psi}{\partial z_i}+x^i=0,\quad i=1,2,...,n.\tag{2.9}$$

If we can find the derivatives z_i as

$$z_i=\phi_i(x),\quad i=1,2,...,n,\tag{2.10}$$

the singular solution would be written as the functional equation

$$y_s(x)=\sum_{i=1}^{n}x^i\phi_i(x)+\psi(\phi(x)).\tag{2.11}$$

A number of particular solutions have been obtained elsewhere [4,5,6,7], corresponding to different choices of the ansatz

$$\psi(z)=-G(Z),\quad Z=\sum_{i=1}^{n}a^iz_i,\tag{2.12}$$

where $G(Z)$ is any smooth function, being $\{a^i\}$ a set of n real constants. Let us consider the non-algebraic function

$$G(Z)=\Big(\sum_{i=1}^{n}a^iz_i\Big)\exp\Big(\sum_{i=1}^{n}a^iz_i\Big)=Ze^Z,\quad \{a^i\}=const.\tag{2.13}$$

Then, with that substitution, Eq.(2.9) yields

$$a^ie^Z\Big(1+Z\Big)=x^i,\quad i=1,2,...,n.\tag{2.14}$$

Let c_j be a set of constant with the property

$$\sum_{j=1}^{n}a^jc_j=1.\tag{2.15}$$

Multiplying the last equation by c_i and summing the outcome, we find the result

$$e^Z\Big(1+Z\Big) = \sum_{i=1}^{n} c_i x^i. \tag{2.16}$$

The following property of the Lambert W function,

$$W\Big(\Big(Z+1\Big)e^{Z+1}\Big) = Z+1, \tag{2.17}$$

allows us to express the quantity Z in terms of the $\{x^i\}$ as

$$Z = W\Big(e\sum_{i=1}^{n} c_i x^i\Big) - 1. \tag{2.18}$$

Notice that if we demand injectivity of the Lambert W function (only one branch of the two possible real branches $W_0(x)$ and $W_{-1}(x)$) then one should impose the condition, $\sum_{i=1}^{n} c_i x^i \geq 0$, in the affine hyperplane. Let us introduce now the compact notation

$$X = e\sum_{i=1}^{n} c_i x^i. \tag{2.19}$$

In order to find the singular solution of the Clairaut equation we only have to express the $z_i x^i$ in terms of the $c_i x^i$. This can be done by parts: First, we multiply Eq.(2.14) by z^j, then take into account that $Z(X) = W(X) - 1$. With all these results, we finally obtain

$$\sum_{j=1}^{n} z_j x^j = Z\Big(1+Z\Big)e^Z = \Big(W(X)-1\Big)W(X)e^{W(X)-1}. \tag{2.20}$$

By virtue of the outcome above, the singular solution of Clairaut's partial differential equation takes the final compact expression

$$\begin{aligned} y_s(x) &= \sum_{i=1}^{n} x^j z_j - G(Z) = Z(1+Z)e^Z - Ze^Z = Z^2 e^Z \\ &= \Big(W(X)-1\Big)^2 e^{W(X)-1}. \end{aligned} \tag{2.21}$$

2.1. Transcendental Singular Solutions of Functional Clairaut-Type Equations

The results obtained for the PDE case can be extended in a natural way to functional Clairaut-type equations. Let $\Gamma = \Gamma[F]$ be a functional of fields $F^m = F^m(x), m = 1, 2, ..., N$, these $F^m(x)$ are integrable functions of real variables $x = \{x^i\} \in \mathbb{R}^n$. Functional Clairaut-type equations have the following structure [5,4,6]

$$\Gamma - \frac{\delta\Gamma}{\delta F^m}F^m = \Psi\Big[\frac{\delta\Gamma}{\delta F}\Big], \tag{2.22}$$

where $\Psi = \Psi[Z]$ is a given real-valued functional of real variables $Z_m = Z_m(x)$, $m = 1, 2, ..., N.$, and it is employed the following notation

$$\frac{\delta\Gamma}{\delta F^m}F^m = \int d^n x \frac{\delta\Gamma}{\delta F^m(x)}F^m(x). \tag{2.23}$$

The property that defines the functional derivatives is given by

$$\frac{\delta F^m(x)}{\delta F^j(x')} = \delta^m_j \delta(x - x'). \tag{2.24}$$

The next step is to introduce the fields $Z_m(x)$ as

$$Z_m(x) = \frac{\delta\Gamma}{\delta F^m(x)}, \tag{2.25}$$

with such an ansatz, the Clairaut-type functional equation becomes

$$\Gamma - F^m Z_m = \Psi[Z]. \tag{2.26}$$

The functional differentiation of the above with respect to the fields $F^j(x)$ yields

$$\int dx dx' \frac{\delta Z_m(x')}{\delta F^j(x)} \Big(\frac{\delta \Psi[Z]}{\delta Z_m(x')} + F^m(x') \Big) = 0, \quad j = 1, 2, ..., N. \tag{2.27}$$

The last integral functional equation always admits a solution for $Z_m(x) = C_m = const$, $m = 1, 2, ..., N$, which generates the general solution with a functional $\Gamma[F]$ given by

$$\Gamma[F] = F^m C_m + \Psi[C]. \tag{2.28}$$

On the other hand, regarding the singular solution, it arises in turn by solving

$$\frac{\delta \Psi[Z]}{\delta Z_m(x')} + F^m(x') = 0, \quad m = 1, 2, ..., N. \tag{2.29}$$

If this system admits a solution of the type, $Z_m = \phi_m(F(x))$, then the functional $\Gamma[F] = F^m \phi_m(F) + \Psi[\phi(F)]$, will be the singular solution of Clairaut functional-type equation. One of the most interesting cases for physical applications (as in QFT for composite operators) [4,5] is the following choice of the functional $\Psi[Z]$

$$\Psi[Z] = -G(z), \quad z = A^j Z_j = \int dx A^j(x) Z_j(x). \tag{2.30}$$

The case, $G(z) = \alpha \ln(1 + z)$, $\alpha \in \mathbf{R}$, has already been studied by Lavrov et al [4,5,6], although we will come back to it in a future section of this work. Let us consider now the case

$$G(z) = \alpha \Big(\int dx A^j(x) Z_j(x) \Big) e^{\int dx A^j(x) Z_j(x)} = \alpha z e^z \tag{2.31}$$

Taking into account the ansatz $\Psi[Z] = -G(z)$, Eq.(2.29) implies

$$\alpha e^z A^m(x') \Big(1 + z \Big) = F^m(x') \tag{2.32}$$

Let $B_m(x)$, $m = 1, 2, ..., N$, be a set of functions such that

$$A^m B_m = \int dx' A^m(x') B_m(x') = 1 \tag{2.33}$$

By virtue of this property, we can multiply Eq.(2.32) by $B_m(x)$ to obtain

$$\alpha e^{z+1} \Big(1 + z \Big) = e \int dx' B_m(x') F^m(x') = e F^m B_m \tag{2.34}$$

That transcendental functional equation can be solved by means of the Lambert W function

$$z = W \Big(\frac{e}{\alpha} F^m B_m \Big) - 1. \tag{2.35}$$

With the aid of the same algorithm employed for the partial differential case, it is not difficult to find the following singular solution

$$\Gamma[F] = \Big(W \Big(\frac{e}{\alpha} F^m B_m \Big) - 1 \Big)^2 e^{W \left(\frac{e}{\alpha} F^m B_m \right) - 1}. \tag{2.36}$$

It is worth noting that we have not exploited all the possibilities that the Lambert W function offers to obtain new exact singular solutions of Clairaut-type equations. Indeed, for other transcendental choices of the function $G(z)$, as for example the case, $G(z) = \alpha z^z$, or $G(z) = \alpha z \ln(1+z)$, similar results can be obtained.The details are left to the reader.

3. Generalized Clairaut-type Equations

Let us begin this section by considering the generalization of CTEs to a broader class that can be regarded as their natural extension. Suppose that $A(x)$, $B(x)$, $C(x)$ are real smooth functions. Then, the nonlinear ODE

$$\psi\Big(y', y\Big) + A(x)y' + B(x)y = C(x), \tag{3.1}$$

includes not only the Clairaut equation as a particular case, but also other famous nonlinear ODEs. Indeed, when $\psi\Big(y', y\Big) = \psi(y') = (y')^2$, $A(x) = Ax$, $B(x) = B$, $C(x) = -Cx^2$, $A, B, C \in \mathbb{R}$, Eq.(3.1) becomes

$$(y')^2 + Axy' + By + Cx^2 = 0. \tag{3.2}$$

This differential equation is known in the literature as Chrystal's equation [23,24,25,26,27,29]. In the last years, a wide variety of applications of this equation have been found in several scientific fields like fusion physics [28], poroacoustic wave phenomena [27], and optics [30]. Of course, standard Clairaut's equation is recovered from the homogeneous case when: $\psi(y', y) \equiv \psi(y')$, $B(x) = -1$, $A(x) = x$, $C(x) = 0$. Differentiating 3.1 with respect to x, we obtain

$$\Big(\frac{\partial \psi}{\partial y'} + A(x)\Big)y'' + \Big(A'(x) + B(x) + \frac{\partial \psi}{\partial y}\Big)y' + B'(x)y = C'(x). \tag{3.3}$$

In general, it is not possible to solve 3.1 or 3.3 with respect to y. To this purpose, we need to constrain or specify the explicit form of the functional ψ and also the smooth functions A, B, C should fulfill some conditions (such is the case of Chrsytal's equation and Clairaut's equation). In the next subsections, we proceed to discuss different exact solutions of 3.1, depending on the assumed relation among A, B and C.

3.1. Case I. $A'(x) = B(x)$

For this specific choice, Eq. (3.3) assumes the form

$$\Big(\frac{\partial \psi}{\partial y'} + A(x)\Big)y'' + \Big(2A'(x) + \frac{\partial \psi}{\partial y}\Big)y' + A''(x)y = C'(x). \tag{3.4}$$

To proceed further, we need to know the exact structure of the function $\psi(y', y)$. As an example of a prescribed function $\psi(y', y)$ that provides an exactly integrable equation, let us consider the ansatz

$$\psi(y', y) = y^n y' \ , \qquad n \in \mathbb{Z}, \tag{3.5}$$

Inserting this into 3.4, we obtain

$$\Big(y^n + A(x)\Big)y'' + \Big(2A'(x) + ny^{n-1}y'\Big)y' + A''(x)y = C'(x). \tag{3.6}$$

This nonlinear second order differential equation turns out to be integrable, namely, it can be recast in terms of

$$\frac{d}{dx}\Big(\Big\{\frac{y^{n+1}}{n+1} + A(x)y\Big\}' - C(x)\Big) = 0 \ , \qquad n \neq -1. \tag{3.7}$$

From which one can get an algebraic equation for the unknown function $y(x)$ as

$$\frac{y^{n+1}}{n+1} + A(x)y = \int C(x)dx + C_1 x + C_2 \ , \qquad n \neq -1, \tag{3.8}$$

where C_1, C_2 are integration constants. Consistency with the first-order equation 3.1, constrains one of these constants to be $C_1 = 0$. For the particular case $n = 1$, a closed form of the solution can be provided as follows

$$y(x) = -A(x) \pm \sqrt{A^2(x) + 2\Big(\int C(x)dx + C_2\Big)}. \tag{3.9}$$

In general, if $\psi(y', y)$ is a total derivative, i.e.

$$\psi(y', y) = \frac{d\phi(y)}{dx}, \tag{3.10}$$

the differential equation (3.3) will lead to a completely integrable equation. Therefore, the solution can be expressed by solving the implicit integral equation

$$\phi(y) + A(x)y = \int C(x)dx + C_1\ , \quad (C_1 = constant). \tag{3.11}$$

Special function methods can be successfully employed in some cases. For instance, let us consider

$$\psi(y', y) = e^y y' = \frac{d\phi(y)}{dx} = (e^y)' \tag{3.12}$$

The implicit integral 3.11 equation for this example is given by

$$e^{y(x)} + A(x)y(x) = \int C(x)dx + C_1 \tag{3.13}$$

A closed and compact expression can be obtained for the solution of this equation, regardless of the specific form of the functions A,C. The solution is (see the Appendix for details)

$$y(x) = \frac{\int C(x)dx + C_1}{A(x)} - W\Big(\frac{e^{\frac{\int C(x)dx + C_1}{A(x)}}}{A(x)}\Big) \tag{3.14}$$

For $A \neq 0$. As an example, we have plotted the trends of the family of solutions corresponding to the particular case $A(x) = x^2$, $C(x) = \cos x$.

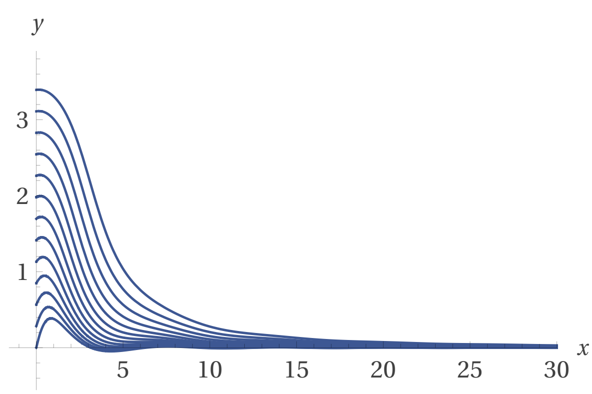

Figure 1: Trends of the solutions of implicit integral equation 3.11 for $A(x) = x^2$, $C(x) = cosx$.

3.2. Case II. $A'(x) = -B(x)$

It would be interesting to investigate in more detail the case $A'(x) = -B(x)$, because Clairaut's ODE belongs to this class. Indeed, notice that CTEs satisfies, $A(x) = x$, $B(x) = const = -1$, therefore it falls into this category. With such assumed relation among the functions $A(x)$ and $B(x)$, Eq. (3.3), acquires the following formal structure

$$\Big(\frac{\partial\psi}{\partial y'}+A(x)\Big)y''+\frac{\partial\psi}{\partial y}y'-A''(x)y=C'(x). \tag{3.15}$$

This specific case allows us to establish a more direct connection with the CTEs. The exact correspondence is achieved by further imposing the conditions: $C(x)=const$, $A''(x)=0$, $\frac{\partial\psi}{\partial y}=0$, or equivalently: $A(x)=ax+b$, $a,\, b\in\mathbb{R}$, $\frac{\partial\psi}{\partial y'}=\frac{d\psi}{dy'}$. Inserting these equalities into the above equation, we obtain

$$\Big(\frac{d\psi}{dy'}+A(x)\Big)y''=0. \tag{3.16}$$

Since $A(x)$ is a linear function, the formal analogies among this equation and Clairaut's equation 1.6 are evident. However, its worth studying its properties with rigor, because despite the formal similarities, there also exists subtleties and significant differences regarding the singular solutions. In fact, the extended version includes an additional functional term that is absent in the standard case, and that can account for different regularity properties of the solutions. Therefore, a more careful inspection of the structure of the singular solutions of this extended version of Clairaut's equation is in order now.

3.3. Extended Ordinary Clairaut-Type Equations

Let us investigate the case II in more detail. Consider the first-order equation

$$\psi\Big(\frac{dy}{dx}\Big)+\Big(ax+b\Big)\frac{dy}{dx}-ay=K, \tag{3.17}$$

where a, b, K are real constants with the only constraint that $a\neq 0$. Standard Clairaut's equation is recovered when $a=1$, $b=0$, $K=0$. Defining a new variable $x'\in\mathbb{R}$, such that $x'=ax+b$, Eq.(3.17) reduces to

$$a^{-1}\psi\Big(a\frac{dy}{dx'}\Big)+x'\frac{dy}{dx'}-y=a^{-1}K, \tag{3.18}$$

which has an structure that resembles Clairaut-type equations. Nevertheless, the exact identification is only possible when

$$\psi\Big(a\frac{dy}{dx'}\Big)=a\psi\Big(\frac{dy}{dx'}\Big), \tag{3.19}$$

namely, when ψ is a linear function of its argument, and this won't be the general case. Then, although at first glance it seems only a matter of convenience to work with Eq.(3.17) or with Eq.(3.18), in what follows we will employ Eq.(3.17) to see more directly the similarities and differences with respect to standard CTEs. We expect only significant departures regarding the singular solutions, since the general solutions are linear functions that will only be rescalated by a linear reparametrization of the argument. We will illustrate later on the fact that singular solutions may present important differences with a particular example for the function $\psi(y')=\alpha\ln(1+\beta y')$, and we will see that the singular solution of the standard Clairaut equation has a different behavior (and even different regularity properties) with respect to the singular solution of the linear extended Clairaut equation.
For now, let us set $\frac{dy}{dx}=z$, and differentiating Eq.(3.17) with respect to x provides the factorization

$$\frac{dz}{dx}\Big(\psi'(z)+ax+b\Big)=0. \tag{3.20}$$

The general solution of the extended Clairaut's equation is a linear family with a more general structure than the usual one

$$y_g(x)=Cx+a^{-1}\Big(\psi(C)+bC-K\Big)\ , \qquad a\neq 0. \tag{3.21}$$

Obviously, for the specific values, $a=1$, $b=0$, $k=0$, the solution reduces to $y_g(x)=Cx+\psi(C)$, which is the family of straight lines solution of standard Clairaut's equation. Some comments are in order now.

It is worth noting that, contrary to the standard general solutions of the Clairaut equation represented by straight lines that do not cross the origin of coordinates (notice that the requirement, $\psi(C) = 0$, is trivial because it would imply, $C = 0$ and therefore $y = 0$, for most of the relevant choices of $\psi(y')$ in physical applications, as is the case given by, $\psi(y') = \alpha \ln(1+\beta y')$, the extended family can include some straight lines that cross the origin, these particular cases are determined by the values of C that satisfy the condition $\psi(C) + bC - K = 0$, an equation that, depending on the form of ψ, may have more than one solution for C different of zero.
On the other hand, the singular solution of extended CTEs arises by solving the equation

$$\psi'(z) + ax + b = 0. \tag{3.22}$$

This equation admits a parametric solution given by the system

$$\begin{cases} x = \dfrac{-b - \psi'(z)}{a}, \\ y = \dfrac{\psi(z) - z\psi'(z) - K}{a}. \end{cases} \tag{3.23}$$

In some situations, however, it is possible to express from Eq.3.22 z explicitly as a function of x, i.e, $z = \phi(x)$, where $\phi(x)$ may be any regular function. In such cases, the singular solution reads ($a \neq 0$)

$$y_s(x) = \phi(x)x + a^{-1}\Big(\psi(\phi(x)) + b\phi(x) - K\Big)\ , \qquad a \neq 0. \tag{3.24}$$

Hence, the presence of the extra term $b\phi(x)$ represents the main departure with respect to the singular solution of the standard Clairaut equation given by Eq.(1.10), and accounts for most of the different regularity properties of both singular solutions. As an example, let us investigate the choice, $\psi(z) = -G(z) = -\alpha \ln(1 + \beta z)$, α, $\beta \in \mathbb{R}$. Substitution into Eq.(3.22) provides the result

$$-\frac{\alpha\beta}{1+\beta z} + ax + b = 0. \tag{3.25}$$

This equation can be easily solved for z to obtain $z = \phi(x)$ in the form

$$z(x) = \beta^{-1}\Big(\frac{\alpha\beta}{ax+b} - 1\Big)\ , \qquad \beta \neq 0. \tag{3.26}$$

Inserting $z(x) = \phi(x)$ in Eq.(3.24), we obtain the singular solution of the extended Clairaut equation, for $\beta \neq 0$, as

$$y_s(x) = \beta^{-1}\Big(\frac{\alpha\beta}{ax+b} - 1\Big)x + a^{-1}\Big[-\alpha \ln\Big(\frac{\alpha\beta}{ax+b}\Big) + b\beta^{-1}\Big(\frac{\alpha\beta}{ax+b} - 1\Big) - K\Big]. \tag{3.27}$$

In Figure 2, We plot the singular solution given by Eq.(3.27) of the extended Clairaut's equation together with the singular solution of the stantard Clairaut's equation for the same function $\psi(z) = -\alpha \ln(1+\beta z)$. As the reader can check, the singular solution of the extended Clairaut's equation is well defined everywhere for $x \geq 0$, while the singular solution of standard CTE cannot be evaluated at $x = 0$.

Let us investigate now the transcendental functional $\psi(z) = -G(z) = -\alpha z e^{\beta z}$, where α, β are real constants different from zero. For this ansatz, Eq.(3.22) reads

$$ax + b = \alpha e^{\beta z}\Big(1 + \beta z\Big). \tag{3.28}$$

This transcendental equation can be solved by means of the Lambert W function as

$$z = \beta^{-1}\Big(W\Big[\frac{e}{\alpha}(ax+b)\Big] - 1\Big) = \beta^{-1}\Big(W(X) - 1\Big)\ , \qquad \beta \neq 0. \tag{3.29}$$

where we have defined a new variable X such that

$$X = \frac{e}{\alpha}(ax+b). \tag{3.30}$$

Therefore, the singular solution of the extended Clairaut equation has the following explicit form

$$y_s(x) = \beta^{-1}x\Big(W(X) - 1\Big) + a^{-1}\Big[b\beta^{-1}\Big(W(X) - 1\Big) - \alpha\beta^{-1}\Big(W(X) - 1\Big)e^{W(X)-1} - K\Big]. \tag{3.31}$$

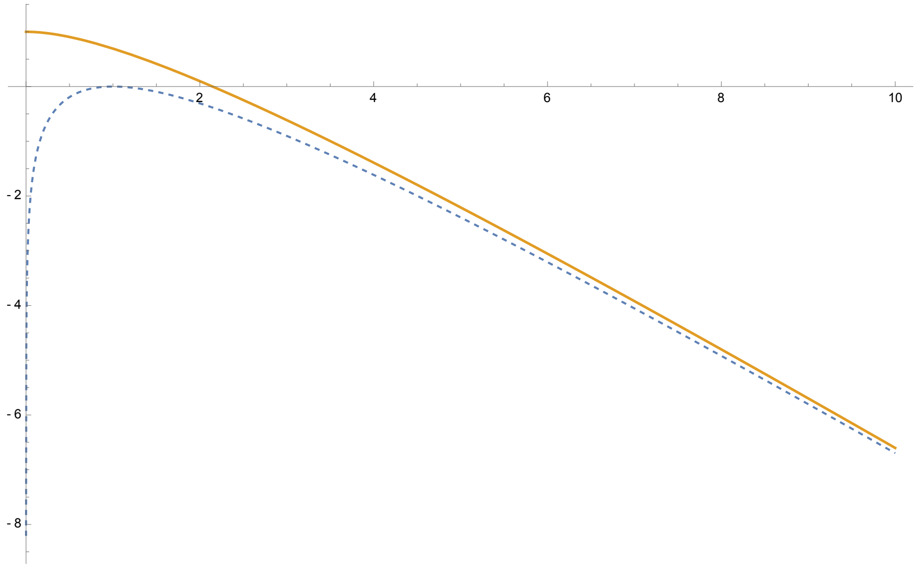

Figure 2: Comparison among the singular solution $y_s(x)$ of extended Clairaut equation (solid), with respect to the singular solution of standard Clairaut equation (dashed) for the ansatz function $\psi(z) = -\alpha \ln(1+\beta z)$. The values of the parameters are $\alpha = \beta = a = b = 1$, and $K = 0$.

3.4. Higher-Order Extended Ordinary Clairaut-Type Equations

The second-order extended Clairaut's generalization of Eq.(1.13) is given by

$$\psi\Big(\frac{d^2y}{dx^2}\Big) + \Big(ax+b\Big)\frac{d^2y}{dx^2} - a\frac{dy}{dx} = K\ , \qquad a \neq 0, \tag{3.32}$$

Differentiation with respect to x yields

$$\frac{d^3y}{dx^3}\Big(\psi'\Big(\frac{d^2y}{dx^2}\Big) + ax + b\Big) = 0. \tag{3.33}$$

Geometrically, the general solution of this equation is a parabolic curve displaced with respect to that of the classical second-order Clairaut's equation, namely

$$y_g(x) = \frac{1}{2}rx^2 + \Big(\frac{\psi(r) + br - K}{a}\Big)x + s\ , \qquad a \neq 0. \tag{3.34}$$

where $r, s \in \mathbb{R}$. More interesting is the singular solution, which can be formed inserting $r = \phi(x)$ into 3.34

$$y_s(x) = \frac{1}{2}\phi(x)x^2 + \Big(\frac{\psi(\phi(x)) + b\phi(x) - K}{a}\Big)x + s\ , \qquad a \neq 0. \tag{3.35}$$

Therefore, the structure of this singular solution is noteworthy, because there is an extra term $C\phi(x)x$, where $C = a/b$, that is not present in the structure of the singular solution of the normal second-order Clairaut's equation 3.32. Although the enlargement of the Clairaut's equation is characterized by a linear extension, interestingly enough, the solution obtained does not follow such a linear extension, because the function $C\phi(x)x$ will not be linear in general, and this is enough to endow the solution of very different properties. So the conclusion is clear: any linear extension of a Clairaut-type equation leads to singular solutions non-linearly extended. The higher-order generalization of 3.32 is straightforward presented by

$$\psi\Big(\frac{d^ny}{dx^n}\Big) + \Big(ax+b\Big)\frac{d^ny}{dx^n} - a\frac{d^{n-1}y}{dx^{n-1}} = K\ , \qquad a \neq 0, \tag{3.36}$$

The same methods that worked for the first-order and second-order versions, are still valid to obtain the general y_g and singular solutions y_s of the n-esime order equation. Similarly to the cases above, one can replace the nth-derivative by an arbitrary constant. From differentiation with respect to x, it follows

$$\frac{d^{n+1}y}{dx^{n+1}}\Big(\psi'\Big(\frac{d^n y}{dx^n}\Big)+ax+b\Big)=0. \tag{3.37}$$

From which one can derive two family of solutions, a general solution y_g given by a n-degree polynomial, and a singular solution y_s with a non-prescribed functional structure (it depends on the specific form of the ψ functional). The details are left to the readers, but one can prove that the singular solution y_s incorporates and additional extra term $C\phi(x)x^{n-1}$ that accounts for very different properties when compared to its classical Clairaut's equation counterpart.

3.5. Extended Clairaut PDE and Functional Clairaut-Type Equations

The PDE analogue of Eq.(3.17) is given by

$$\psi(z)+\sum_{i=1}^{n}\Big(ax^i+b^i\Big)z_i-ay=K, \quad z_i=\frac{\partial y}{\partial x^i}, \tag{3.38}$$

where a, $K\in\mathbb{R}$, and $b=\{b^1,b^2,...,b^n\}$ is a n-tuple constant vector in a real vector space V. Differentiation with respect to x^j yields

$$\sum_{i=1}^{n}\frac{\partial z_i}{\partial x^j}\Big(\frac{\partial\psi}{\partial z_i}+ax^i+b^i\Big)=0, \quad j=1,2,...,n. \tag{3.39}$$

If the Hessian matrix vanishes, i.e, $H_{ij}=\partial_j z_i=0,\forall i,j=1,...,n$ then $z_i=c_i=const$, and the general solution reads

$$y(x)=\sum_{i=1}^{n}x^i c_i+a^{-1}\Big(\psi(C)+\sum_{i=1}^{n}b^i c_i-K\Big). \tag{3.40}$$

Regarding the singular solution, it arises by solving the partial differential equation

$$\frac{\partial\psi}{\partial z_i}+ax^i+b^i=0, \quad i=1,2,...,n, \tag{3.41}$$

which represents a linear (minimal) extension of Eq.(2.9). For the particular choice of the function $\psi(z)$ given by Eq.(2.12), i.e, $\psi(z)=-G(z)$, $Z=\sum n^i z_i$, $\{n^i\}=const$, where $G(Z)$ may be any regular function of its argument, we have

$$-G'(z)n^i+ax^i+b^i=0, \quad i=1,2,...,n, \quad G'(z)=\frac{dG}{dz}. \tag{3.42}$$

The, if c_j, $j=1,2,...,n$ is a set of constants, we can express $G'(Z)$ as

$$G'(Z)=C^{-1}\Big[\sum_{i=1}^{n}(c_i(ax^i+b^i)\Big], \quad C=\sum_{i=1}^{n}n^i c_i. \tag{3.43}$$

Some particular cases are addressed now, involving the singular solutions and their differences with respect to those of the standard Clairaut's PDE. Let us begin by considering the function

$$\psi(z)=\alpha\ln\Big(1-\beta\sum_{i=1}^{n}z_i r^i\Big), \tag{3.44}$$

where α, β are real parameters and $r=\{r^1,r^2,...,r^n\}$ forms a n-tuple constant vector in a vector space V. Therefore, the general solution given by Eq.(3.40) applied to the function of Eq.(3.44) acquires the form

$$y_g(x) = \sum_{i=1}^{n} x^i c_i + a^{-1}\Big[\alpha \ln\Big(1 - \beta\sum_{i=1}^{n} c_i r^i\Big) + \sum_{i=1}^{n} b^i c_i - K\Big]. \tag{3.45}$$

As for the singular solution, we have

$$\frac{-\alpha\beta}{1-\beta(\sum z_j r^j)} r^i + ax^i + b^i = 0, \quad i = 1, 2, ..., n. \tag{3.46}$$

A feature of this system is that the required structure $\sum z_i r^i$ and $\sum (ax^i + b^i) z_i$ as functions of $x = \{x_1, x_2, ..., x_n\}$ can be computed by analytic means. To this purpose, let us introduce the constant n-tuple $d = \{d_1, d_2, ..., d_n\}$ in the dual vector space V^* so that $\sum d_j r^j = 1$. Multiplying Eq.(3.46) by d_i and summing the results

$$\frac{-\alpha\beta}{1-\beta(\sum z_j r^j)} + a\sum_{i=1}^{n} x^i d_i + \sum_{i=1}^{n} b^i d_i = 0,$$
$$\Rightarrow \sum_{i=1}^{n} z_i r^i = \beta^{-1}\Big[1 - \frac{\alpha\beta}{\sum_{i=1}^{n}(ax^i + b^i)d_i)}\Big]. \tag{3.47}$$

Multiplying in turn Eq.(3.46) by z_i and summing the resulting equations we obtain

$$\sum_{i=1}^{n}\Big(ax^i + b^i\Big) z_i = \beta^{-1}\Big(\sum_{i=1}^{n}(ax^i + b^i)d_i\Big) - \alpha, \tag{3.48}$$

where it was used Eq.(3.47). The singular solution found can be explicitly written in the form

$$y_s(x) = \beta^{-1}\Big(\sum_{i=1}^{n}(x^i + \frac{b^i}{a})d_i\Big) - a^{-1}\Big[\alpha + K - \alpha\ln\alpha\beta + \alpha\ln\Big(\sum_{i=1}^{n}(ax^i + b^i)d_i\Big)\Big]. \tag{3.49}$$

This solution is the corresponding generalization of that presented in Ref. [10] for the same choice of $\psi(z)$. For another non-algebraic choices of the $\psi(z)$ function, like that given by Eq.(2.13), we find after some computations that the singular solution of the extended partial differential Clairaut equation has, in terms of the Lambert W function, the following structure

$$y_s(x) = a^{-1}\Big[\Big(W(X) - 1\Big)^2 e^{W(X)-1} - K\Big],$$
$$X = e\sum_{i=1}^{n}(ax^i + b^i)c_i, \quad \sum_{j=1}^{n} a^j c_j = 1. \tag{3.50}$$

Regarding the extended functional Clairaut-type equations, their structure is

$$\Psi\Big[\frac{\delta\Gamma}{\delta F}\Big] + \Big(aF^m + B^m\Big)\frac{\delta\Gamma}{\delta F^m} - a\Gamma = K$$
$$\Rightarrow \Psi[Z] + \Big(aF^m + B^m\Big)Z_m - a\Gamma = K \tag{3.51}$$

where $B^m = B^m(x)$, $m = 1, 2, ..., N$ is the set of given integrable functions of real variables x, a, $K \in \mathbb{R}$, and it was introduced again the notation $Z^m = \frac{\delta\Gamma}{\delta F^m}$. Functional differentiation with respect to $F^j(x)$ yields

$$\int dxdx' \frac{\delta Z_m(x')}{\delta F^j(x)}\Big(\frac{\delta\Psi[Z]}{\delta Z_m(x')} + aF^m(x') + B^m(x')\Big) = 0. \tag{3.52}$$

Let us investigate a functional of the type

$$\Psi[Z] = \alpha \ln\Big(1 - \beta(Z_m A^m)\Big), \tag{3.53}$$

where $A^m = A^m(x)$, $m = 1, 2, ..., N$ are N integrable real valued functions, and α, β real parameters different from zero. Omitting the trivial case of the general solution, that yields a lineal functional Γ, and repeating almost the same arguments made for the extended PDE case, we obtain the singular solution as

$$\Gamma[F] = \beta^{-1}\Big((F^i + \frac{B^i}{a})D_i\Big) - a^{-1}\Big[\alpha + K - \alpha \ln \alpha\beta + \alpha \ln\Big((aF^i + B^i)D_i\Big)\Big], \tag{3.54}$$

where $D_i = D_i(x)$, $i = 1, 2, ..., N$, are field variables that satisfy $D_i A^i = \int dx D_i(x) A^i(x) = 1$.

On the other hand, for a transcendental functional given by Eq.(2.30), the singular solution is another non-algebraic functional given in terms of the Lambert W function that has a similar structure than that obtained for the extended PDE case, i.e, Eq.(3.50), replacing the vectorial magnitudes x^i, b^i, c_i, for the fields $F^i(x)$, $B^i(x)$, $D_i(x)$, where $A^i D_i = \int dx A^i(x) D_i(x) = 1$.
Since the CTEs are important in several branches of theoretical physics, it will be interesting to continue studying its possible extensions. For instance, in future works it will be interesting to investigate more general CTEs of the type

$$\Phi(y - xz) = \psi(z), \quad z = \frac{dy}{dx}. \tag{3.55}$$

And their PDE and functional equation analogues, namely

$$\Phi\Big(y - \sum_{i=1}^{n} x^i z_i\Big) = \psi(z), \quad z_i = \frac{\partial y}{\partial x^i}, \tag{3.56}$$

$$\Phi\Big(\Gamma - \frac{\delta\Gamma}{\delta F^m} F^m\Big) = \Psi\Big[\frac{\delta\Gamma}{\delta F}\Big], \tag{3.57}$$

where Φ is a prescribed function of its argument, for example $\Phi(y - xz) = (y - xz)^2$. It is not difficult to show that such differential equations also admit a singular solution. Indeed, differentiating the ODE given by Eq.(3.55) with respect to x one obtains

$$z'\Big(\Phi'(y - xz)x + \psi'(z)\Big) = 0. \tag{3.58}$$

The general solution is again the solution of $z' = 0$, and obviously represents a family of straight lines. The singular solution arises by setting $\Phi'(y - xz)x + \psi'(z) = 0$ for specific choices of Φ and ψ. In some cases, as for example the ansatzes $\Phi(y - xz) = (y - xz)^2$ and $\psi(z) = z^2 + C$, $C \in \mathbb{R}$, one finds that, unlike the standard CTEs case and their linear extensions, it is not possible to express z as a function of x solely. However, the differential equation obtained for this particular choice of the functions $\Phi(x - yz)$ and $\psi(z)$ is separable and can be easily integrated, but we cannot claim that this will be true in general. Therefore, we cannot expect that the standard strategy proved successful for CTEs to explicitly obtain z as a function of x, i.e, $z = \phi(x)$ will work as well for a more general framework. This and other relevant aspects will be addressed in more detail elsewhere.

4. Concluding Remarks

In this work, we have derived new singular (transcendental) solutions of the CTEs in terms of special functions such as the Lambert W function, and we have shown that new exact singular solutions can be obtained for non-algebraic choices of the function $\psi(z)$. Furthermore, we have studied the minimal (linear) extension of the CTEs, in their ODE, PDE, and functional analogues, respectively. We have found that the non-trivial singular solutions of the CTEs and their linear extensions have different properties, due to the fact that the singular solution of the extended version includes an extra functional term i.e, $y_s = x\phi(x) + a^{-1}\Big(\psi(\phi(x)) + b\phi(x) - K\Big)$, in comparison to, $y_s = x\phi(x) + \psi(\phi(x))$ of the standard ODE case. This additional extra term that depends on $\phi(x)$ accounts for different regularity properties in general, as we have proved for the particular choice $\psi(z) = -\alpha \ln(1 + \beta z)$. We have shown that this fact is also true for the extended second-order Clairaut's equation and its higher-orders. A summary of the findings is the following

- For the ODE case, any linear extension of Clairaut's equation contains singular solutions that are non-linearly extended. This is due to the presence of new non-trivial functional terms that are absent in the standard version.
- Remarkably, this property is exclusive of the singular solutions (general solutions turn out to be linearly extended), and is also true even for second-order Clairaut's equations and beyond.
- For the PDE and functional counterparts, the analysis in more involved and some simplification should be adopted to render the equations tractable. In the cases subjected to study, this property does not seems to hold, and the extensions of the singular solutions are more direct and natural.
- We have broadened the usual view that reduces Clairaut's differential equation to a particular case of Chrystal's equation. With the framework introduced in section 3, it was shown that both differential equations arise as particular cases inside a common ODE structure.

5. Bibliography

References

1. A.C.Clairaut, Solution de plusieurs problems. *Historie de l'Academie Royale de Sciences, Paris* (1734), 196-215.
2. L.D.Landau and E.M.Lifshitz, *Mechanics* (Butterworth-Heinemann, 1976)
3. N.N.Bogoliubov and D.V.Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, 1959)
4. P.M. Lavrov and B.S. Merzlikin, *Phys. Rev. D* **92**, 085038 (2015)
5. O.V.Zyryanova and O.V.Kolesnikov, *Russ. Phys. J.* **59**, 1663 (2016)
6. P.M. Lavrov and B.S.Mezlikin, *PoS.* **CORFU2016**. *D* 029 (2017)
7. J.M.Cornwell, R.Jackiw and E. Tomboulis, *Phys. Rev. D* **10**, 2428 (1974)
8. V.I. Mudruk, O. V. Zyryanova, *Mod. Phys. Lett. A* **34**, 1950258 (2019).
9. O.V.Zyryanova and V.I. Mudruk, *Russ. Phys. J.* **61**, 635 (2018)
10. P.M. Lavrov and B.S.Merzlikin, *Phys. Lett. B* **756**, 188 (2016).
11. L.Raffy: Sur des équations qu'on intégre en les differentiant. Ibid. **23** (1895),50-61.
12. J.D.Keckic, Clairaut's equations of higher order. *Publ.Inst.Math.***12** (1971),63-66
13. D.S.Mitrinovic and J.D.Keckic, Variations and generalisations of Clairaut's equations. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* **716-734** (1981), 11-21.
14. W.H.Witty, Generalization of Clairaut's differential equation. *Amer. Math. Monthly* **59** (1952), 100-102
15. M. Takahashi, On implicit second-order ordinary differential equations: completely integrable and Clairaut type. *J. Dyn.Control Syst.*
16. M.Bhupal, On singular solutions of implicit second-order odinary differential equations. *Hokkaido Math. J.* **32** (2003), 623-641.
17. R.M.Corless, G.H.Gonnet, D.E.G.Hare, D.J.Jeffrey and D.E.Knuth, *Adv. Comp. Math.* **5**, 329-359 (1996)
18. T.C. Banwell and A. Jayakumar, *Electronics. Lett.* **36**, 291-292 (2000)
19. E. Packel and D. Yuen, *College. Math. J* **35**, 337-350 (2004)
20. M.Walker, S.Duplij, Cho-Duan-Ge, *Phys. Rev. D* **91** 064022 (2015)
21. J. Polchinski, *Nucl. Phys. B* **231** (1984) 269
22. P.M.Lavrov, I.L. Shapiro, *JHEP* **2013** (06), 086 (2013)
23. G. Chrystal, On the p-discriminant of a Differential Equation of the First order on Certain Points in the General Theory of Envelopes Connected Therewith, *Trans.Roy.Soc.Edin,* Vol. 38, (1896) pp. 803-824.

24. E.L.Ince,Ordinary Differential Equations, Dover,New York,1956.pp.90-91.

25. H.T.Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962.pp-51-53

26. L.Dresner, Applications of Lie's Theory of Ordinary and Partial Differential Equations, Institute of Physics, Bristol,UK,1999.pp.30,177.

27. P.M. Jordan, A note on Chrystal's equation, *Appl. Math. Comput.* 217 (2) (2010) 933-936

28. R.Buckmire, K.McMuttry, R.E.Mickens, Numerical studies of a nonlinear heat equation with square root reaction term, *Numer.Methods Partial Differ.Equ.* **25** (2009) 598-609.

29. P.M.Jordan, Finite-amplitude acoustic traveling waves in a fluid that saturates a porous medium:acceleration wave formation,*Phys.Lett.A* **355** (2006)

30. M.Tenenbaum,H.Pollard,Ordinary Differential Equations,Dover,New York,1985.pp.759-760

6. Appendix. Explicit Analytic Solution of $e^{y(x)} + A(x)y(x) = \int C(x)dx + C_1$

The purpose of this brief appendix is to show that solving non-algebraic equations (usually viewed as non-tractable analytically) in terms of the Lambert W function is an art in itself. Consider the integral equation

$$e^{y(x)} + A(x)y(x) = \int C(x)dx + C_1, \tag{6.1}$$

where C_1 is an arbitrary constant, and A, B, two smooth functions. It is convenient to isolate the exponential term. Doing so, and dividing by $A(x)$, we can rewrite the equation,for $A \neq 0$, as

$$\frac{e^{y(x)}}{A(x)} = \frac{\int C(x)dx + C_1}{A(x)} - y(x). \tag{6.2}$$

Now the strategy is to take advantage of the property

$$W\Big((x-a)e^{x-a}\Big) = x - a. \tag{6.3}$$

To this end, we multiply both sides of 6.2 for the appropriate exponential terms. We find

$$\frac{e^{\frac{\int C(x)dx+C_1}{A(x)}}}{A(x)} = \Big(\frac{\int C(x)dx + C_1}{A(x)} - y(x)\Big)e^{\frac{\int C(x)dx+C_1}{A(x)} - y(x)}. \tag{6.4}$$

Using the property of the Lambert function above, and rearranging terms, we finally obtain

$$y(x) = \frac{\int C(x)dx + C_1}{A(x)} - W\Big(\frac{e^{\frac{\int C(x)dx+C_1}{A(x)}}}{A(x)}\Big), \tag{6.5}$$

which is the desired result. The more general integral equation,

$$B(x)e^{y(x)} + A(x)y(x) = \int C(x)dx + C_1, \tag{6.6}$$

can be solved exactly in the same fashion. The reader can try its solution as an exercise.

Ginés R. Pérez Teruel,
Consellería de Educación, Universidades y Empleo
Ministerio de Educación y Formación Profesional,
Spain.
E-mail address: gines.landau@gmail.com
E-mail address: gr.perezteruel@edu.gva.es