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A study of skew Lie product involving generalized derivations *

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ABSTRACT: Let $\mathfrak{R}$ be a ring with an involution σ. Notation $\nabla[t_1, t_2]$ denotes the skew Lie product and defined by $t_1t_2 - t_2\sigma(t_1)$. The main objective of this paper is to investigate the commutativity of σ-prime rings with an involution σ of the second kind equipped with skew Lie product involving generalized derivation. Finally, we provide some examples to demonstrate that the conditions assumed in our results are not unnecessary.

Key Words: σ-prime ring, derivation, involution, generalized derivation, skew Lie product.

Contents

1. Introduction

All through this paper, $\mathfrak{R}$ will be used to describe an associative ring, and $\mathfrak{J}_Z$ is the centre of $\mathfrak{R}$. For any $t_1, t_2 \in \mathfrak{R}$, the notation $[t_1, t_2]$ illustrates the commutator $t_1t_2 - t_2t_1$, and $t_1 \circ t_2$ illustrates the anti-commutator $t_1t_2 + t_2t_1$. A ring $\mathfrak{R}$ is called 2-torsion free if $2t_1 = 0 \implies t_1 = 0$ for $t_1 \in R$. We use the basic identities $[t_1t_2, t_3] = t_1[t_2, t_3] + [t_1, t_3]t_2$ and $[t_1, t_2t_3] = [t_1, t_2]t_3 + t_2[t_1, t_3]$ for all $t_1, t_2, t_3 \in \mathfrak{R}$ very frequent. Recall that an involution is an order 2 anti-automorphism. Ring $\mathfrak{R}$ is called σ-prime if $a\mathfrak{R}b = a\mathfrak{R}\sigma(b) = (0)$ or $\sigma(a)\mathfrak{R}b = a\mathfrak{R}b = (0)$ implies $a = 0$ or $b = 0$. Every prime ring is a σ-prime ring but converse is not true in general; for instance let $S = \mathfrak{R}\times\mathfrak{R}^0$, where $\mathfrak{R}^0$ is an opposite ring of prime ring $\mathfrak{R}$. The mapping σ on S as $\sigma(t_1, t_2) = (t_2, t_1)$. Thus, S is a σ-prime ring but S is not a prime ring. We define "an element t_1 in $\mathfrak{R}$ is said to be hermitian if $\sigma(t_1) = (t_1)$ and skew-hermitian if $\sigma(t_1) = -(t_1)$." Where $\mathfrak{J}_H$ denotes the set of hermitian elements and $\mathfrak{J}_S$ denotes the set of skew-hermitian elements of $\mathfrak{R}$. If $char(\mathfrak{R}) \neq 2$ then every $t_1 \in \mathfrak{R}$ can be uniquely expressed as $2t_1 = h + k$ where $h \in \mathfrak{J}_H$ and $k \in \mathfrak{J}_S$. If $\mathfrak{J}_Z \subseteq \mathfrak{J}_H$, then σ is said to be of the first kind otherwise, it is called the second kind and in this case $\mathfrak{J}_S \cap \mathfrak{J}_Z \neq (0)$. Any element $t_1 \in \mathfrak{R}$ is called normal, if its commutes with its image under involution σ, and $\mathfrak{R}$ is called normal if every elements of $\mathfrak{R}$ is normal. See in [8].

A mapping ψ on $\mathfrak{R}$ is termed as derivation if $\psi(t_1 + t_2) = \psi(t_1) + \psi(t_2)$ and $\psi(t_1t_2) = \psi(t_1)t_2 + t_1\psi(t_2)$ for all $t_1, t_2 \in \mathfrak{R}$. Let $b \in \mathfrak{R}$ be a fixed element of $\mathfrak{R}$, then the mapping ψ on $\mathfrak{R}$ defined by $\psi(t_1) = [b, t_1] = bt_1 - t_1b$ for all $t_1 \in \mathfrak{R}$ is called inner derivation induced by b, an additive mapping $D : \mathfrak{R} \to \mathfrak{R}$ is called generalized derivation if there exists a derivation ψ on $\mathfrak{R}$ such that $D(t_1t_2) = D(t_1)t_2 + t_1\psi(t_2)$ for all $t_1, t_2 \in \mathfrak{R}$. A mapping $f : \mathfrak{R} \to \mathfrak{R}$ is called centralizing on $\mathfrak{R}$ if $[f(t_1), t_1] \in \mathfrak{J}_Z$ holds for all $t_1 \in \mathfrak{R}$. In particular, if $[f(t_1), t_1] = 0$ holds for all $t_1 \in \mathfrak{R}$, then it is called commuting. Stimulated by the description of centralizing map, a map f from $\mathfrak{R}$ into itself is called σ-centralizing if $[f(t_1), \sigma(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$ and is called σ-commuting if $[f(t_1), \sigma(t_1)] = 0$ for all $t_1 \in \mathfrak{R}$. The narrative of centralizing and commuting maps dates back to 1955, when Divinsky proved that if a simple artinian ring has commuting non-trivial automorphisms, then it is commutative. After few years, Posner [16] established that the presence of a nonzero centralizing derivation on a prime ring implies commutativity of rings. The study of centralizing (resp. commuting) derivations and various generalizations of concept of a centralizing (resp. commuting) maps are the main concepts emerging directly from Posner's result, with many applications in various areas. Recently, a number of algebraists

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demonstrated the commutativity theorem for prime and semi-prime rings with or without involution, accepting identities on automorphism, derivations, left centralizers and generalized derivations (for example) [2,4,7,10,11,12].

In 2014, Ali and Dar, [2] starts the study of σ-centralizing derivation in prime rings with an involution and proved σ-version of classical results of Posner [16], and they proved that "Let $\mathfrak{R}$ be a prime ring with involution σ such that $char(\mathfrak{R}) \neq 2$. Let ψ be a nonzero derivation of $\mathfrak{R}$ such that $[\psi(t_1), \sigma(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$ and $\psi(\mathfrak{J}_S \cap \mathfrak{J}_Z) \neq \{0\}$. Then $\mathfrak{R}$ is commutative". Further, this result was extended by Najjar et al. [13] for the second kind involution instead of condition $\psi(\mathfrak{J}_S \cap \mathfrak{J}_Z) \neq \{0\}$. Recently Alahmadi et al. [4] generalized the above result for generalized derivation and they prove that "Let $\mathfrak{R}$ be a prime ring with an involution σ of the second kind such that $char(\mathfrak{R}) \neq 2$, if $\mathfrak{R}$ admits a nonzero generalized derivation F associated with a derivation d such that $[F(t), \sigma(t)] \in \mathfrak{J}_Z$ for all $t \in \mathfrak{R}$ then $\mathfrak{R}$ is commutative". In this direction a lots of work have been done in the recent years (See for reference [5,6,17] where further references can be found).

The main target of our paper is to investigate generalized derivation involving skew Lie product on σ prime rings with involution, further we identify the structure of σ-prime rings that satisfy some central identities. In fact our results are generalization of some results proved in [3]. At the last we provide some examples to demonstrate that the conditions assumed in our results are not unnecessary. To prove our main results, we need some lemmas as well as some facts, so we start with the proof of these lemmas and facts.

2. Main Results

Lemma 2.1 *Let $\mathfrak{R}$ be a σ-prime rings with involution σ, for any $a \in \mathfrak{R}$ and $z \in \mathfrak{J}_Z$, if $az \in \mathfrak{J}_Z$ and $a\sigma(z) \in \mathfrak{J}_Z$ then $a \in \mathfrak{J}_Z$ or $z = 0$.*

Proof: Since, $az \in \mathfrak{J}_Z$ and $a\sigma(z) \in \mathfrak{J}_Z$, $0 = [az, r] = [a\sigma(z), r]$ for all $r \in \mathfrak{R}$, implies $0 = z[a, r] = \sigma(z)[a, r]$ further implies $(0) = z\mathfrak{R}[a, r] = \sigma(z)\mathfrak{R}[a, r]$, by the definition of σ-prime rings we have either $z = 0$ or $a \in \mathfrak{J}_Z$ □

Lemma 2.2 *Let $\mathfrak{R}$ be a σ-prime rings with involution σ, for any $a \in \mathfrak{R}$ and $z \in \mathfrak{J}_Z$, if $az \in \mathfrak{J}_Z$ and $\sigma(a)z \in \mathfrak{J}_Z$ then $a \in \mathfrak{J}_Z$ or $z = 0$.*

Proof: Since, $az \in \mathfrak{J}_Z$ and $\sigma(a)z \in \mathfrak{J}_Z$, $0 = [az, r] = [\sigma(a)z, r]$ for all $r \in \mathfrak{R}$, implies $0 = z[a, r] = z[\sigma(a), r]$ implies $(0) = z\mathfrak{R}[a, r] = z\mathfrak{R}[\sigma(a), r]$ further implies $(0) = z\mathfrak{R}[a, r] = z\mathfrak{R}\sigma([a, r])$, by the definition of σ-prime rings we have either $z = 0$ or $a \in \mathfrak{J}_Z$ □

Lemma 2.3 *Let $\mathfrak{R}$ be a σ-prime ring of $char(\mathfrak{R}) \neq 2$, then $\mathfrak{R}$ is 2-torsion free.*

Proof: Let, $u \in \mathfrak{R}$ and $2u = 0$ suggests, $2u(vw) = 0$ for all $v, w \in \mathfrak{R}$ and $u\mathfrak{R}(2w) = (0)$ for all $w \in \mathfrak{R}$. Since, $char(\mathfrak{R}) \neq 2$ and $\mathfrak{R} \neq (0)$ then there exist $0 \neq p \in \mathfrak{R}$ such that $2p \neq 0$, forces $u\mathfrak{R}(2p) = (0) = u\mathfrak{R}\sigma(2p)$, by the definition of σ-prime rings we have, either $u = 0$ or $2p = 0$ second case is not possible by the assumption and first case implies $\mathfrak{R}$ is 2-torsion free. □

Lemma 2.4 *In σ-prime ring, $\mathfrak{J}_Z \cap \mathfrak{J}_H$ and $\mathfrak{J}_Z \cap \mathfrak{J}_S$ are free from zero-divisor.*

Proof: Let $a \in R$ and $b \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, such that $ab = 0$, implies $abu = 0$ for all $u \in \mathfrak{R}$ provide us $a\mathfrak{R}b = (0) = a\mathfrak{R}\sigma(b)$. So by the definition of σ-prime ring, we have either $a = 0$ or $b = 0$. □

Lemma 2.5 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind. Let ψ be the derivation on $\mathfrak{R}$, if $\psi(h) = 0$ for all $h \in \mathfrak{J}_H \cap \mathfrak{J}_Z$, then $\psi(z) = 0$ for all $z \in \mathfrak{J}_Z$.*

Proof: By the given condition we have $\psi(h) = 0$, where $h \in \mathfrak{J}_H \cap \mathfrak{J}_Z$, then $\psi(k^2) = 0$ for $k \in \mathfrak{J}_S \cap \mathfrak{J}_Z$ implies $k\ \psi(k) = 0$ by Lemma 2.4 we have either $k = 0$ or $\psi(k) = 0$ first case is not possible because σ is of the second kind involution. We have $\psi(k) = 0$ for $k \in \mathfrak{J}_S \cap \mathfrak{J}_Z$, for all $z \in \mathfrak{J}_Z$ we have for 2-torsion free rings $2z = h + k$, finally we have $\psi(2z) = \psi(h) + \psi(k) = 0$ implies $\psi(z) = 0$ for all $z \in \mathfrak{J}_Z$. □

Lemma 2.6 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind. If $t_1^2 \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof: $t_1^2 \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, after linearizing we get, $t_1t_2 + t_2t_1 \in \mathfrak{J}_Z$ for all $t_1, t_2 \in \mathfrak{R}$. Since σ is of the second kind, there exist $0 \neq c \in \mathfrak{J}_Z \cap \mathfrak{J}_S$. Replacing t_2 by c, we have $t_1c \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, since $\mathfrak{R}$ is 2-torsion free. $[t_1c, r] = 0$ for all $r \in \mathfrak{R}$, implies $[t_1, r]c = 0$. Now by using Lemma 2.4, we get $[t_1, r] = 0$ for all $t_1, r \in \mathfrak{R}$, implies $\mathfrak{R}$ is commutative. □

Fact 2.1 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind, if $\nabla[t_1, \sigma(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof: By the given condition

$$\nabla\ [t_1, \sigma(t_1)] \in \mathfrak{J}_Z \qquad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.1}$$

Linearizing above equation we have

$$\nabla\ [t_1, \sigma(t_2)] + \nabla\ [t_2, \sigma(t_1)] \in \mathfrak{J}_Z \qquad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.2}$$

Taking t_2k in place of t_2 where; $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, and using

$$(-\nabla\ [t_1, \sigma(t_2)] + t_2\sigma(t_1) + \sigma(t_1)\sigma(t_2)])\ k\ \in \mathfrak{J}_Z \qquad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.3}$$

The last relation further implies

$$[-\nabla\ [t_1, \sigma(t_2)] + t_2\sigma(t_1) + \sigma(t_1)\sigma(t_2), r]\ k\ = 0 \qquad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.4}$$

By Lemma 2.4, we have either $k = 0$ or $[-\nabla\ [t_1, \sigma(t_2)] + t_2\sigma(t_1) + \sigma(t_1)\sigma(t_2), r] = 0$ for all $t_1, t_2 \in \mathfrak{R}$. The first case is not possible because σ is of the second kind involution. Later case implies

$$(-\nabla\ [t_1, \sigma(t_2)] + t_2\sigma(t_1) + \sigma(t_1)\sigma(t_2))\ \in \mathfrak{J}_Z \qquad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.5}$$

By combing (2.2) and (2.5), we have

$$2t_2\sigma(t_1)\ \in \mathfrak{J}_Z \qquad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.6}$$

The last relation further implies

$$t_2\sigma(t_1)\ \in \mathfrak{J}_Z \qquad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.7}$$

Replacing $\sigma(t_1)$ by t_1 and t_2 by k, where $0 \neq k \in \mathfrak{J}_Z \cap \mathfrak{J}_S$ and using $char(\mathfrak{R}) \neq 2$, we have

$$t_1k\ \in \mathfrak{J}_Z \quad \text{and} \quad t_1\sigma(k)\ \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.8}$$

By Lemma 2.1, we have

$$t_1 \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.9}$$

Last relation implies commutativity of $\mathfrak{R}$. □

Fact 2.2 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime rings with involution σ which is of the second kind, if $\mathfrak{R}$ is normal then $\mathfrak{R}$ is commutative.*

Proof: Since, $\mathfrak{R}$ is normal, i. e., $hk = kh$ where $h \in \mathfrak{J}_H$ *and* $k \in \mathfrak{J}_S$ respectively. Take any $t_1 \in \mathfrak{R}$, then $t_1 - \sigma(t_1) \in \mathfrak{J}_S$.

$$h(t_1 - \sigma(t_1)) = (t_1 - \sigma(t_1))h, \quad \text{for all } t_1 \in \mathfrak{R} \text{ and } \ h \in \mathfrak{J}_H. \tag{2.10}$$

Take $s \in \mathfrak{J}_S \cap \mathfrak{J}_Z$, then $s(t_1 + \sigma(t_1)) \in \mathfrak{J}_S$ for all $t_1 \in \mathfrak{R}$, so by normality of $\mathfrak{R}$, we have $hs(t_1 + \sigma(t_1)) = s(t_1 + \sigma(t_1))h$ for all $t_1 \in \mathfrak{R}$ and $h \in \mathfrak{J}_H$.

$$s\{h(t_1 + \sigma(t_1)) - (t_1 + \sigma(t_1))h\} = 0, \quad \text{for all } \ t_1 \in \mathfrak{R}. \tag{2.11}$$

So, by Lemma 2.4, we have either $s = 0$ or $h(t_1 + \sigma(t_1)) = (t_1 + \sigma(t_1))h$. First case is not possible, since σ is of the second kind and latter case together with (2.10), gives $ht_1 = t_1h$ for all $t_1 \in \mathfrak{R}$ and $h \in \mathfrak{J}_H$. Replacing t_1 by t_2 gives

$$ht_2 = t_2h, \quad \text{for all } t_2 \in \mathfrak{R} \text{ and } \quad h \in \mathfrak{J}_H. \tag{2.12}$$

Replacing h by $t_1 + \sigma(t_1)$ in (2.12), we get

$$(t_1 + \sigma(t_1))t_2 = t_2(t_1 + \sigma(t_1)) \ \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.13}$$

Now we take $s \in \mathfrak{J}_S \cap \mathfrak{J}_Z$, then $s(t_1 - \sigma(t_1)) \in \mathfrak{J}_H$ and using (2.12), we have $s\{(t_1 - \sigma(t_1))t_2 - t_2(t_1 - \sigma(t_1))\} = 0$ for all $t_1, t_2 \in \mathfrak{R}$. By Lemma 2.4, we have either $s = 0$ or $(t_1 - \sigma(t_1))t_2 = t_2(t_1 - \sigma(t_1))$ but first case is not possible, since σ is of the second kind and latter case implies

$$(t_1 - \sigma(t_1))t_2 = t_2(t_1 - \sigma(t_1)) \ \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.14}$$

The last relation together with (2.13), we get $t_1t_2 = t_2t_1$ for all $t_1, t_2 \in \mathfrak{R}$. □

Fact 2.3 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime rings with involution σ which is of the second kind, if σ is centralizing then $\mathfrak{R}$ is commutative.*

Proof: By the given condition

$$[t_1, \sigma(t_1)] \in \mathfrak{J}_Z \ \text{ for all } \ t_1 \in \mathfrak{R}. \tag{2.15}$$

Linearizing (2.15)

$$[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)] \in \mathfrak{J}_Z \ \text{ for all } \ t_1, t_2 \in \mathfrak{R}. \tag{2.16}$$

Replacing t_2 by $\sigma(t_2)$, we get

$$[[t_1, t_2], t_1] + [[\sigma(t_2), \sigma(t_1)], t_1] = 0 \ \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.17}$$

Displacing t_2 by t_2t_1 in (2.17), we get

$$\begin{aligned} &[[t_1, t_2], t_1]t_1 + \sigma(t_1)[[\sigma(t_2), \sigma(t_1)], t_1] + \\ &[\sigma(t_1), t_1][\sigma(t_2), \sigma(t_1)] = 0 \ \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.18}$$

Using (2.17) in (2.18), we receive

$$\begin{aligned} &[[t_1, t_2], t_1]t_1 - \sigma(t_1)[[t_2, t_1], t_1] + \\ &[\sigma(t_1), t_1][\sigma(t_2), \sigma(t_1)] = 0 \ \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.19}$$

Taking t_2t_1 for t_2 in the above equation, we attain

$$\begin{aligned} &[[t_1, t_2], t_1]t_1^2 - \sigma(t_1)[[t_2, t_1], t_1]t_1 \\ &+ [\sigma(t_1), t_1]\sigma(t_1)[\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.20}$$

Using (2.19) in (2.20), and replacing t_1 by $\sigma(t_1)$ and t_2 by $\sigma(t_2)$, we have

$$[t_1, \sigma(t_1)]\{t_1[t_2, t_1] - [t_2, t_1]\sigma(t_1)\} = 0 \ \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.21}$$

Exchanging t_2 by t_2t_1 in (2.21), we capture

$$[t_1,\sigma(t_1)]\{t_1[t_2,t_1]t_1-[t_2,t_1]t_1\sigma(t_1)\}=0 \text{ for all } t_1,t_2\in\mathfrak{R}. \tag{2.22}$$

Invoking (2.21) in (2.22), we obtain

$$[t_1,\sigma(t_1)][t_2,t_1]\{-t_1\sigma(t_1)+\sigma(t_1)t_1\}=0 \text{ for all } t_1,t_2\in\mathfrak{R}. \tag{2.23}$$

The last relation further implies that

$$[t_1,\sigma(t_1)]^2\mathfrak{R}[t_2,t_1]=(0) \text{ for all } t_1,t_2\in\mathfrak{R}. \tag{2.24}$$

Replacing t_1 by $\sigma(t_1)$ and t_2 by $\sigma(t_2)$ in (2.24), we find

$$[t_1,\sigma(t_1)]^2\mathfrak{R}[t_2,t_1]=(0)= \quad [t_1,\sigma(t_1)]^2\mathfrak{R}\ \sigma\{[t_2,t_1]\}, \text{ for all } t_1,t_2\in\mathfrak{R}. \tag{2.25}$$

By the definition of σ-prime ring, we get

$$[t_1,\sigma(t_1)]^2=0 \quad \text{or} \quad [t_1,t_2]=0 \text{ for all } t_1,t_2\in\mathfrak{R}. \tag{2.26}$$

The Latter case suggests that $\mathfrak{R}$ is commutative, the first case implies that

$$[t_1,\sigma(t_1)]^2=0 \text{ for all } t_1\in\mathfrak{R}. \tag{2.27}$$

Since $[t_1,\sigma(t_1)]\in\mathfrak{J}_Z\cap\mathfrak{J}_H$ and by Lemma 2.4, we get

$$[t_1,\sigma(t_1)]=0 \text{ for all } t_1\in\mathfrak{R}. \tag{2.28}$$

Using Fact 2.2, $\mathfrak{R}$ is commutative. □

Fact 2.4 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind, if $t_1\circ\sigma(t_1)\in\mathfrak{J}_Z$ for all $t_1\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof: By the given condition

$$t_1\circ\sigma(t_1)\in\mathfrak{J}_Z \text{ for all } t_1\in\mathfrak{R}. \tag{2.29}$$

Linearizing the above relation

$$t_1\circ\sigma(t_2)+t_2\circ\sigma(t_1)\in\mathfrak{J}_Z \text{ for all } t_1,t_2\in\mathfrak{R}. \tag{2.30}$$

Last relation further implies

$$[t_1\circ\sigma(t_2),r]+[t_2\circ\sigma(t_1),r]=0 \quad \text{for all } t_1,t_2,r\in\mathfrak{R}. \tag{2.31}$$

Replacing t_2 by $\sigma(t_2)$ in (2.31), we found

$$[t_1\circ t_2,r]+[\sigma(t_2)\circ\sigma(t_1),r]=0 \text{ for all } t_1,t_2,r\in\mathfrak{R}. \tag{2.32}$$

Taking t_1 in place of t_2 in (2.32), we grasp

$$[t_1^2,r]+[\sigma(t_1)^2,r]=0 \text{ for all } t_1,r\in\mathfrak{R}. \tag{2.33}$$

Assuming $t_2\in\mathfrak{J}_Z\setminus\{0\}$ and $t_1=t_1^2$ in (2.31), we have

$$[t_1^2,r]t_2+[\sigma(t_1)^2,r]\sigma(t_2)=0 \text{ for all } t_1,r\in\mathfrak{R}. \tag{2.34}$$

Making use of (2.33) in (2.34), we obtain

$$[t_1^2,r]\{t_2-\sigma(t_2)\}=0 \quad \text{for all } t_1,t_2,r\in\mathfrak{R}. \tag{2.35}$$

$\{t_2-\sigma(t_2)\}\in\mathfrak{J}_S\cap\mathfrak{J}_Z$, by using Lemma 2.4, we have either $[t_1^2,r]=0$ or $\{t_2-\sigma(t_2)\}=0$, latter case is not possible since σ is of the second kind, first case implies

$$[t_1^2,r]=0 \quad \text{for all } t_1,r\in\mathfrak{R}. \tag{2.36}$$

So, $t_1^2\in\mathfrak{Z}(\mathfrak{R})$ for all $t_1\in\mathfrak{R}$. Using Lemma 2.6, $\mathfrak{R}$ is commutative. □

Fact 2.5 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D be generalized derivation associated with derivation ψ on $\mathfrak{R}$, if $D(t_1) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$ then either $\mathfrak{R}$ is commutative or $D = 0$.*

Proof: By the given condition

$$D(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.37}$$

If $\mathfrak{J}_Z = 0$ then $D = 0$, so we have $\mathfrak{J}_Z \neq 0$, taking t_2t_1 in place of t_1

$$D(t_2t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.38}$$

Commutes the above relation with t_1, we obtain

$$[t_2\psi(t_1), t_1] = 0 \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.39}$$

Replacing t_2 by $(0 \neq z) \in \mathfrak{J}_Z$, we obtain

$$z[\psi(t_1), t_1] = 0 \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.40}$$

The relation further implies

$$z\mathfrak{R}[\psi(t_1), t_1] = (0) = \sigma(z)\mathfrak{R}[\psi(t_1), t_1] \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.41}$$

By the defination of σ-prime rings, we have either $z = 0$ or $[\psi(t_1), t_1] = 0$, first case is not possible by our assumption, later case implies

$$[\psi(t_1), t_1] = 0 \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.42}$$

By *[14, theorem 1]*, $\mathfrak{R}$ is commutative or $\psi = 0$, replacing t_1 by t_1u where $u \in \mathfrak{R}$ in (2.37) and using $\psi = 0$, we obtain

$$D(t_1)u \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, u \in \mathfrak{R}. \tag{2.43}$$

Last relation further implies

$$D(t_1)\sigma(u) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, u \in \mathfrak{R}. \tag{2.44}$$

Last relation together with (2.43) and using Lemma 2.2, we obtain

$$\text{either} \quad D(t_1) = 0 \text{ for all} \quad t_1 \in \mathfrak{R}, \quad \text{or} \quad u \in \mathfrak{J}_Z \quad \text{for all} \quad u \in \mathfrak{R}. \tag{2.45}$$

Later case implies commutativity of $\mathfrak{R}$ and first case implies $D = 0$. □

Theorem 2.6 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D be a generalized derivation associated with derivation ψ on $\mathfrak{R}$, if $D(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = 0$.*

Proof: Given that

$$D(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z \quad \text{for all} \ \ t_1 \in \mathfrak{R}. \tag{2.46}$$

Repacing t_1 by $t_1 + t_2$ in the above equation, we get

$$D(\nabla[t_1, \sigma(t_2)]) + D(\nabla[t_2, \sigma(t_1)]) \in \mathfrak{J}_Z \quad \text{for all} \ \ t_1, t_2 \in \mathfrak{R}. \tag{2.47}$$

Replacing t_1 by t_1h in (2.47) and using it, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we achieve

$$(\nabla[t_1, \sigma(t_2)] + \nabla[t_2, \sigma(t_1)])\psi(h) \in \mathfrak{J}_Z \quad \text{for all} \ \ t_1, t_2 \in \mathfrak{R}. \tag{2.48}$$

Replacing t_2 by t_2s, where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, we have

$$s\{(-\nabla\ [t_1, \sigma(t_2)] + t_2\sigma(t_1) + \sigma(t_1)\sigma(t_2))\psi(h)\} \in \mathfrak{J}_Z \quad \text{for all} \ \ t_1, t_2 \in \mathfrak{R}. \tag{2.49}$$

By Lemma 2.1, we get

$$(-\nabla\ [t_1,\sigma(t_2)]+t_2\sigma(t_1)+\sigma(t_1)\sigma(t_2))\psi(h)\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \tag{2.50}$$

By combining (2.48) and (2.50), we obtain

$$\nabla[t_2,\sigma(t_1)]\psi(h)\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \tag{2.51}$$

Replacing t_1 by $\sigma(t_1)$ in the last relation, we obtain

$$\sigma(\nabla[t_2,\sigma(t_1)])\psi(h)\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \tag{2.52}$$

Last relation together with (2.51) and using Lemma 2.1, we obtain
$\nabla[t_2,\sigma(t_1)]\in\mathfrak{J}_Z$ or $\psi(h)=0$, first case implies

$$\nabla[t_2,\sigma(t_1)]\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \tag{2.53}$$

Taking $t_2=t_1$, in above relation we have

$$\nabla[t_1,\sigma(t_1)]\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \tag{2.54}$$

By Fact 2.1, $\mathfrak{R}$ is commutative. Later case implies

$$\psi(z)=0 \quad \text{for all} \quad z\in\mathfrak{J}_Z. \tag{2.55}$$

Equation (2.47) further implies

$$\begin{aligned} &D(t_1\sigma(t_2)-\sigma(t_2)\sigma(t_1))+D(t_2\sigma(t_1)-\sigma(t_1)\sigma(t_2)) \\ &\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \end{aligned} \tag{2.56}$$

Replacing t_1 by t_1k where $0\neq k\in\mathfrak{J}_Z\cap\mathfrak{J}_S$ in above relation we have

$$\begin{aligned} &\{D(t_1\sigma(t_2)+\sigma(t_2)\sigma(t_1))-D(t_2\sigma(t_1)-\sigma(t_1)\sigma(t_2))\}k \\ &\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \end{aligned} \tag{2.57}$$

By Lemma 2.1, we obtain

$$\begin{aligned} &D(t_1\sigma(t_2)+\sigma(t_2)\sigma(t_1))-D(t_2\sigma(t_1)-\sigma(t_1)\sigma(t_2)) \\ &\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \end{aligned} \tag{2.58}$$

Last relation together with (2.56), we obtain

$$D(t_1\sigma(t_2))\in\mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2\in\mathfrak{R}. \tag{2.59}$$

Replacing t_2 by h, where $0\neq h\in\mathfrak{J}_Z\cap\mathfrak{J}_H$ in above equation we obtain

$$D(t_1)h\in\mathfrak{J}_Z \quad \text{for all} \quad t_1\in\mathfrak{R}. \tag{2.60}$$

By Lemma 2.1, we obtain

$$D(t_1)\in\mathfrak{J}_Z \quad \text{for all} \quad t_1\in\mathfrak{R}. \tag{2.61}$$

By Fact 2.5, results hold. □

Corollary 2.1 [3, Theorem 2], *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution σ which is of the second kind and D be a generalized derivation associated with derivation ψ on $\mathfrak{R}$, if $D(\nabla[t_1,\sigma(t_1)])\in\mathfrak{J}_Z$ for all $t_1\in\mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D=0$.*

Corollary 2.2 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D and F be a generalized derivation associated with the derivation ψ and ϕ on $\mathfrak{R}$, if $D(\nabla[t_1, \sigma(t_1)]) \pm F(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = \pm F$.*

Proof: By the given hypothesis

$D(\nabla[t_1, \sigma(t_1)]) \pm F(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Last relation further implies

$(D \pm F)(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Taking $G = D \pm F$, where G is generalized derivative associated with the derivation $\psi \pm \phi$, then by Theorem 2.6, $\mathfrak{R}$ is commutative or $D = \pm F$.

□

Corollary 2.3 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D be a generalized derivation associated with the derivation on $\mathfrak{R}$, if $D(\nabla[t_1, \sigma(t_1)]) \pm \nabla[t_1, \sigma(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = \pm I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping on $\mathfrak{R}$.*

Proof: By the given hypothesis

$D(\nabla[t_1, \sigma(t_1)]) \pm \nabla[t_1, \sigma(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Last relation further implies

$(D \pm I_{\mathfrak{R}})(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Taking $G = D \pm I_{\mathfrak{R}}$, where G is generalized derivative associated with the derivation ψ, then by Theorem 2.6, $\mathfrak{R}$ is commutative or $D = \pm I_{\mathfrak{R}}$. □

Theorem 2.7 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D be a generalized derivation associated with the derivation ψ on $\mathfrak{R}$, if $\nabla[t_1, D(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = 0$.*

Proof: Given that

$$\nabla[t_1, D(t_1)] \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.62}$$

Repacing t_1 by $t_1 + t_2$ in above equation, we get

$$\nabla[t_1, D(t_2)] + \nabla[t_2, D(t_1)] \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.63}$$

Last relation further implies

$$\begin{aligned} &t_1 D(t_2) - D(t_2)\sigma(t_1) + t_2 D(t_1) - D(t_1)\sigma(t_2) \\ &\in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.64}$$

Replacing t_1 by $t_1 h$ in (2.64) and using it, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we achieve

$$(t_2 t_1 - t_1 \sigma(t_2))\psi(h) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.65}$$

Replacing t_1 by $\sigma(t_1)$, in the last relation we obtain

$$\sigma(t_2 t_1 - t_1 \sigma(t_2))\psi(h) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.66}$$

By using (2.65), (2.66) and Lemma 2.2, we obtain $t_2t_1 - t_1\sigma(t_2) \in \mathfrak{J}_Z$ for all $t_1, t_2 \in \mathfrak{R}$ or $\psi(h) = 0$, first case implies $\nabla[t_2, t_1] \in \mathfrak{J}_Z$ for all $t_1, t_2 \in \mathfrak{R}$, on taking $t_2 = \sigma(t_1)$ $\mathfrak{R}$ is commutative by Fact 2.1. Now we take $\psi(h) = 0$ for all $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$ then $\psi(z) = 0$ for all $z \in \mathfrak{J}_Z$.

Replacing t_1 by t_1k in (2.64), where $0 \neq k \in \mathfrak{J}_Z \cap \mathfrak{J}_S$ and using $\psi(z) = 0$ for all $z \in \mathfrak{J}_Z$, we have

$$\{t_1D(t_2) + D(t_2)\sigma(t_1) + t_2D(t_1) - D(t_1)\sigma(t_2)\}k \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.67}$$

By using Lemma 2.1, we obtain

$$t_1D(t_2) + D(t_2)\sigma(t_1) + t_2D(t_1) - D(t_1)\sigma(t_2) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.68}$$

Combining (2.64) and (2.68), we obtain

$$2D(t_2)\sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.69}$$

Since, $\mathfrak{R}$ is 2-torsion free, we get

$$D(t_2)\sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.70}$$

Putting h in place of t_1 in above relation $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$ and by using Lemma 2.1, we obtain

$$D(t_2) \in \mathfrak{J}_Z \quad \text{for all} \quad t_2 \in \mathfrak{R}. \tag{2.71}$$

By Fact 2.5, result hold. □

Corollary 2.4 [3, Theorem 2], *Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution σ which is of the second kind and D be a generalized derivation associated with the derivation ψ on $\mathfrak{R}$, if $\nabla[t_1, D(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = 0$.*

Corollary 2.5 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D and F be a generalized derivation associated with the derivation ψ and ϕ on $\mathfrak{R}$, if $\nabla[t_1, D(t_1)]) \pm \nabla[t_1, F(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = \pm F$.*

Proof: By the given hypothesis

$\nabla[t_1, D(t_1)]) \pm \nabla[t_1, F(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Last relation further implies

$\nabla[t_1, (D \pm F)(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Taking $G = D \pm F$, where G is generalized derivative associated with the derivation $\psi \pm \phi$, then by Theorem 2.7, $\mathfrak{R}$ is commutative or $D = \pm F$. □

Corollary 2.6 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D be a generalized derivation associated with the derivation on $\mathfrak{R}$, if $\nabla[t_1, D(t_1)]) \pm \nabla[t_1, t_1] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = \pm I_{\mathfrak{R}}$, where $I_{\mathfrak{R}}$ is the identity mapping on $\mathfrak{R}$.*

Proof: By the given hypothesis

$\nabla[t_1, D(t_1)]) \pm \nabla[t_1, t_1] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Last relation further implies

$\nabla[t_1, (D \pm I_{\mathfrak{R}})(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$.

Taking $G = D \pm I_{\mathfrak{R}}$, where G is generalized derivative associated with the derivation ψ, then by Theorem 2.7, $\mathfrak{R}$ is commutative or $D = \pm I_{\mathfrak{R}}$. □

Theorem 2.8 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D and F be a generalized derivation associated with derivation ψ and ϕ on $\mathfrak{R}$, if $D(\nabla[t_1, \sigma(t_1)]) + \nabla[t_1, F(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = F = 0$.*

Proof: Given that

$$D(\nabla[t_1, \sigma(t_1)]) + \nabla[t_1, F(t_1)] \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.72}$$

Last relation further implies, we get

$$\begin{aligned} &D(t_1)\sigma(t_1) + t_1\psi(\sigma(t_1)) - D(\sigma(t_1))\sigma(t_1) - \sigma(t_1)\psi(\sigma(t_1)) \\ &+ t_1F(t_1) - F(t_1)\sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \end{aligned} \tag{2.73}$$

Replacing t_1 by $t_1 h$ in (2.73) and using it where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we obtain

$$\begin{aligned} &\{2t_1\sigma(t_1) - (\sigma(t_1))^2 - (t_1)^2\}\psi(h) + \{(t_1)^2 - t_1\sigma(t_1\}\phi(h) \\ &\in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \end{aligned} \tag{2.74}$$

Replacing t_1 by $t_1 s$ in (2.74), where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, we obtain

$$\begin{aligned} &(\{-2t_1\sigma(t_1) - (\sigma(t_1))^2 - (t_1)^2\}\psi(h) + \{(t_1)^2 + t_1\sigma(t_1\}\phi(h))s^2 \\ &\in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \end{aligned} \tag{2.75}$$

By Lemma 2.1, we obtain

$$\begin{aligned} &(\{-2t_1\sigma(t_1) - (\sigma(t_1))^2 - (t_1)^2\}\psi(h) + \{(t_1)^2 + t_1\sigma(t_1\}\phi(h)) \\ &\in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \end{aligned} \tag{2.76}$$

By combining (2.74) and (2.76), we get

$$t_1\sigma(t_1)\{2\psi(h) - \phi(h)\} \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.77}$$

Last relation further implies

$$\sigma(t_1\sigma(t_1))\{2\psi(h) - \phi(h)\} \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.78}$$

By (2.77), (2.78) and using Lemma 2.2, we have either $t_1\sigma(t_1) \in \mathfrak{J}_Z$ or $2\psi(h) - \phi(h) = 0$. On replacing t_1 by $\sigma(t_1)$ in first implies $[t_1, \sigma(t_1)] \in \mathfrak{J}_Z$, by Fact 2.3, $\mathfrak{R}$ is commutative. Later case together with Lemma 2.5, implies

$$2\psi(z) = \phi(z) \quad \text{for all} \quad z \in \mathfrak{J}_Z. \tag{2.79}$$

Last relation together with (2.74) implies

$$(t_1^2 - \sigma(t_1)^2)\psi(h) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.80}$$

Last relation further implies

$$\sigma(t_1^2 - \sigma(t_1)^2)\psi(h) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.81}$$

By (2.80), (2.81) and using Lemma 2.2, we have either $t_1^2-\sigma(t_1)^2 \in \mathfrak{J}_Z$ or $\psi(h)=0$ for all $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$. On replacing t_1 by t_1+t_2 in first case implies

$$t_1t_2+t_2t_1-\sigma(t_1)\sigma(t_2)-\sigma(t_2)\sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}. \tag{2.82}$$

Replacing t_1 by t_1s in (2.82), where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, we obtain

$$(t_1t_2+t_2t_1+\sigma(t_1)\sigma(t_2)+\sigma(t_2)\sigma(t_1))s \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}. \tag{2.83}$$

By using Lemma 2.4, we obtain

$$t_1t_2+t_2t_1+\sigma(t_1)\sigma(t_2)+\sigma(t_2)\sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}. \tag{2.84}$$

Last relation together with (2.83), implies

$$t_1t_2+t_2t_1 \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}. \tag{2.85}$$

Taking $t_2=\sigma(t_1)$, in above equation we obtain

$$t_1 \circ \sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}. \tag{2.86}$$

By Fact 2.4, $\mathfrak{R}$ is commutative.

Now we take later condition $\psi(h)=0$, implies $\phi(h)=0$ and by Lemma 2.5, $\psi(z)=0$ and $\phi(z)=0$, for all $z \in \mathfrak{J}_Z$.

Replacing t_1 by t_1s in (2.73), where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, we obtain

$$\begin{aligned}&(-D(t_1)\sigma(t_1)-t_1\psi(\sigma(t_1))-D(\sigma(t_1))\sigma(t_1)-\sigma(t_1)\psi(\sigma(t_1))\\ &+t_1F(t_1)+F(t_1)\sigma(t_1))s^2 \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}.\end{aligned} \tag{2.87}$$

By using Lemma 2.1, we obtain

$$\begin{aligned}&-D(t_1)\sigma(t_1)-t_1\psi(\sigma(t_1))-D(\sigma(t_1))\sigma(t_1)-\sigma(t_1)\psi(\sigma(t_1))\\ &+t_1F(t_1)+F(t_1)\sigma(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}.\end{aligned} \tag{2.88}$$

Last relation together with (2.73), implies

$$D(\sigma(t_1))\sigma(t_1)+\sigma(t_1)\psi(\sigma(t_1))-t_1F(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.89}$$

Linearizing above equation and using it, we obtain

$$\begin{aligned}&D(\sigma(t_1))\sigma(t_2)+D(\sigma(t_2))\sigma(t_1)+\sigma(t_1)\psi(\sigma(t_2))+\sigma(t_2)\psi(\sigma(t_1)\\ &-t_1F(t_2)-t_2F(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}.\end{aligned} \tag{2.90}$$

Replacing t_1 by t_1s in (2.90), where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, we obtain

$$\begin{aligned}&\{-D(\sigma(t_1))\sigma(t_2)-D(\sigma(t_2))\sigma(t_1)-\sigma(t_1)\psi(\sigma(t_2))-\sigma(t_2)\psi(\sigma(t_1))\\ &-t_1F(t_2)-t_2F(t_1)\}s \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}.\end{aligned} \tag{2.91}$$

By Lemma 2.4, we obtain

$$\begin{aligned}&-D(\sigma(t_1))\sigma(t_2)-D(\sigma(t_2))\sigma(t_1)-\sigma(t_1)\psi(\sigma(t_2))-\sigma(t_2)\psi(\sigma(t_1))\\ &-t_1F(t_2)-t_2F(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}.\end{aligned} \tag{2.92}$$

Combining (2.92) and (2.90), we obtain

$$t_1F(t_2)+t_2F(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1,t_2 \in \mathfrak{R}. \tag{2.93}$$

Putting $t_1 = t_2 = h$ in above equation where, $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we obtain

$$2hF(h) \in \mathfrak{J}_Z \quad \text{for all} \;\; 0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H. \tag{2.94}$$

By Lemma 2.4, we obtain

$$F(h) \in \mathfrak{J}_Z \quad \text{for all} \;\; 0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H. \tag{2.95}$$

Replacing t_2 by h in (2.93), we obtain

$$t_1F(h) + hF(t_1) \in \mathfrak{J}_Z \quad \text{for all} \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.96}$$

Commuting above relation with t_1 and by Lemma 2.4, we get

$$[F(t_1), t_1] = 0 \quad \text{for all} \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.97}$$

Replacing t_1 by $t_1 + t_2$, in the above relation we obtain

$$[F(t_1), t_2] + [F(t_2), t_1] = 0 \quad \text{for all} \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.98}$$

Replacing t_1 by ht_2, in the above relation and using Lemma 2.4, we obtain

$$[\phi(t_2), t_2] = 0 \quad \text{for all} \;\; t_2 \in \mathfrak{R}. \tag{2.99}$$

By *[14, Theorem 1]*, $\mathfrak{R}$ is commutative or $\phi = 0$, means $\mathfrak{R}$ is noncommutative. Replacing t_1 by t_1u in (2.98) and using $\phi = 0$, we obtain

$$F(t_1)[u, t_2] + t_1[F(t_2), u] = 0 \quad \text{for all} \;\; t_1, t_2, u \in \mathfrak{R}. \tag{2.100}$$

Replacing t_1 by vt_1, we obtain

$$F(v)t_1[u, t_2] + vt_1[F(t_2), u] = 0 \quad \text{for all} \;\; t_1, t_2, u, v \in \mathfrak{R}. \tag{2.101}$$

Multiplying (2.100), by v from left we obtain

$$vF(t_1)[u, t_2] + vt_1[F(t_2), u] = 0 \quad \text{for all} \;\; t_1, t_2, u, v \in \mathfrak{R}. \tag{2.102}$$

Last relation together with (2.101), implies

$$t_1F(v) - vF(t_1)[u, t_2] = 0 \quad \text{for all} \;\; t_1, t_2, u, v \in \mathfrak{R}. \tag{2.103}$$

Replacing u by ru, in the above relation we get

$$t_1F(v) - vF(t_1)r[u, t_2] = 0 \quad \text{for all} \;\; t_1, t_2, u, v, r \in \mathfrak{R}. \tag{2.104}$$

Last relation further implies

$$t_1F(v) - vF(t_1)\mathfrak{R}[u, t_2] = (0) = t_1F(v) - vF(t_1)\mathfrak{R}\sigma([u, t_2]) \tag{2.105}$$

For all $t_1, t_2, u, v \in \mathfrak{R}$, so by definition of σ-prime ring, we obtain

$$t_1F(v) = vF(t_1) \in \mathfrak{J}_Z \;\; \text{for all} \;\; t_1, v \in \mathfrak{R}. \tag{2.106}$$

Last relation together with (2.93), implies

$$2t_2F(t_1) \in \mathfrak{J}_Z \;\; \text{for all} \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.107}$$

Taking $t_2 = h$ and using Lemma 2.4, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we get

$$F(t_1) \in \mathfrak{J}_Z \;\; \text{for all} \;\; t_1 \in \mathfrak{R}. \tag{2.108}$$

By Fact 2.5, $F = 0$. Equation (2.72), implies $D(\nabla[t_1, \sigma(t_1)]) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, so by Theorem 2.6, $D = 0$. □

Corollary 2.7 *Let $\mathfrak{R}$ be a 2-torsion free σ-prime ring with involution σ which is of the second kind and D be a generalized derivation associated with derivation ψ on $\mathfrak{R}$, if $D(\nabla[t_1, \sigma(t_1)]) + \nabla[t_1, D(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = 0$.*

Corollary 2.8 [3, Theorem 2]*, Let $\mathfrak{R}$ be a 2-torsion free prime ring with involution σ which is of the second kind and D be a generalized derivation associated with derivation ψ on $\mathfrak{R}$, if $D(\nabla[t_1, \sigma(t_1)]) + \nabla[t_1, F(t_1)] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative or $D = F = 0$.*

As it is well-known that the zero-divisor is impossible in the center of a prime ring, but in σ-prime rings centre is not free from zero divisor. The follwing example explain that the above fact.

Example 2.9 *Consider $\mathfrak{R} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Big| a, b \in \mathbb{Z} \right\}$, define σ in such away, $\sigma\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is σ-prime ring with involution σ. For any non zero a, $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{J}_Z$, and for any nonzero b, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathfrak{R}$ and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows the fact.*

The following examples shows that the second kind is necessary in Theorem 2.7.

Example 2.10 *Consider $\mathfrak{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| a, b, c, d \in \mathbb{Z} \right\}$, define σ in such away, $\sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is σ-prime ring with involution σ of the first kind. Moreover, we define generalized derivation D and derivation ψ as $D\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$ and $\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$, here D is generalized derivation associted with the derivation ψ satisfy the condition of Theorem 2.7, however $\mathfrak{R}$ is noncommutative.*

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