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Sobolev spaces on canonical Banach spaces and Fourier transformations

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ABSTRACT: In this article, Sobolev spaces on canonical Banach spaces has been discussed. The Hilbert structure of the Sobolev spaces are discussed in this settings. Finally, in application, we discuss the Fourier transform and its relevance for Sobolev spaces on canonical Banach spaces.

Key Words: Test functions, Weak derivative, Schwartz spaces, Fourier transform, Sobolev spaces.

Contents

1. Introduction

An L^p-norm of a function and its derivatives up to a certain order are combined to form the norm of a vector space of functions, which is known as a Sobolev space. The space, which is a Banach space, is complete when the derivatives are understood in an appropriate weak meaning. As one may assume, a Sobolev space is a set of functions with a norm that quantifies the size and regularity of a function, and with enough derivatives for a certain application domain, like partial differential equations. Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense. Sobolev spaces play an important role in modern analysis. Since their discovery by Sergei Sobolev in the 1930's, they have become the basis for the study of many subjects, such as partial differential equations and calculus of variatons. The general idea of study of Sobolev space is to make use of metrical analysis while taking into account the presence of a linear structure. The theory of Sobolev spaces on metric measure spaces is quite developed now (see [1,7,10,11,12] and references therein). For a detailed treatment and for references to the literature on the subject, one may refer to the cite [4,5,6,11] and references therein.

T. L. Gill et al. in [2] had constructed the corresponding version of Lebesgue measure for every Banach space with an S-basis. A general theory of distributions on canonical Banach spaces, the Schwartz space, and the Fourier transform on canonical Banach spaces are discussed.

The new version of Lebesgue measure of [2], motivated by to developed Sobolev spaces over uniformly convex spaces.

The Structure of the article is as follows: In Section 2, we recall several known definitions. In Section 3, we discuss test functions and weak derivatives. The relationship between the test function spaces and $L^p(B)$ will be established where B is uniformly convex Banach space. In Section 4, we introduce Sobolev spaces on uniformly convex Banach spaces. The completeness of the Sobolev spaces is discussed in these

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settings. The Hilbert spaces and the structure of the Sobolev spaces are also discussed. Finally, in Section 5, we extend the Fourier transform to $S^{k,2}(B)$, where B is a canonical Banach space.

2. Preliminaries

Throughout the article, we will refer to $\mathbb{M}$ as the class of measurable functions on B. If B is a Banach space and B' is its dual space, then S-basis is defined as follows:

Definition 2.1 *[2, Definition 2.36] A sequence $(e_n) \in B$ is called a Schauder basis (S-basis) for B if $||e_n||_B = 1$ and for each $x \in B$, there is a unique sequence (x_n) of scalars such that*

$$x = \lim_{k\to\infty} \sum_{n=1}^{k} x_n e_n = \sum_{n=1}^{\infty} x_n e_n.$$

We can find from the definition of a Schauder basis that, for any sequence (x_n) of scalars associated with a $x \in B$, $\lim\limits_{n\to\infty} x_n = 0$.

Before delving into our concept, consider the virtual approach of the Lebesgue measure on $\mathbb{R}^\infty$ as follows:
In certain, if $\mathcal{I}_0 = [\frac{-1}{2}, \frac{1}{2}]^{\aleph_0}$, the Lebesgue measure μ_∞ must satisfy $\mu_\infty(\mathcal{I}_0) = 1$. When $\mathfrak{B}(\mathbb{R}^n)$ is Borel σ-algebra for $\mathbb{R}^n$ and $\mathcal{I} = [\frac{-1}{2}, \frac{1}{2}]$ and $A_n = A \times \mathcal{I}_n,\ B_n = B \times \mathcal{I}_n$ the n^{th} order box sets in $\mathbb{R}^n$, then

1. $A_n \cup B_n = (A \cup B) \times \mathcal{I}_n$
2. $A_n \cup B_n = (A \cup B) \times \mathcal{I}_n$
3. $\overline{B_n} = \overline{B} \times \mathcal{I}_n$.

Under the condition $\mathbb{R}^n_{\mathcal{I}} = \mathbb{R}^n \times \mathcal{I}_n$ with $\mathfrak{B}(\mathbb{R}^n_{\mathcal{I}})$, the Borel σ-algebra for $\mathbb{R}^n_{\mathcal{I}}$ the topology for $\mathbb{R}^n_{\mathcal{I}}$ can be defined as $\mathbb{T}_n = \Bigg\{\mathbb{U} \times \mathcal{I}_n :\ \mathbb{U} \text{ is open in } \mathbb{R}^n\Bigg\}$. This gives, $\mu_\infty(.)$ is measure on $\mathfrak{B}(\mathbb{R}^n_{\mathcal{I}})$, equivalent to n-dimensional Lebesgue measure on $\mathbb{R}^n_{\mathcal{I}}$. For detailed about $\mu_\infty(.)$ on $\mathbb{R}^\infty_{\mathcal{I}}$ one can follow [2,3,8].

Let us assume $J_k = [-\frac{1}{2In(k+1)}, \frac{1}{2In(k+1)}]$ and $J^n = \Pi_{k=n+1}^{\infty} J_k,\ J = \Pi_{k=1}^{\infty} J_k$. If $\{e_k\}$ be an S-basis for B and let $x = \sum_{n=1}^{\infty} x_n e_n$. Recalling that $P_n(x) = \sum_{k=1}^{n} x_n e_k$ and define $Q_n x = (x_1, x_2, .., x_n)$, we define B^n_J by

$$B^n_J = \{Q_n(x) :\ x \in B\} \times J^n$$

with norm

$$||(x_k)||_{B^n_J} = \max_{1\le k\le n} ||\sum_{i=1}^{k} x_i e_i|| = \max_{1\le k\le n} ||P_n(x)||_B.$$

Since $B^n_J \subset B^{n+1}_J$ we set $B^\infty_J = \bigcup_{n=1}^{\infty} B^n_J$. We define B_J by

$$B_J = \{(x_1, x_2, ..) :\ \sum_{k=1}^{\infty} x_k e_k \in B\} \subset B^\infty_J$$

and define a norm on B_J by

$$||x||_{B_J} = \sup_n ||P_n(x)||_B = |||x|||_B. \tag{2.1}$$

Let $\mathfrak{B}(B^\infty_J)$ be the smallest $\sigma-$algebra containing B^∞_J and define
$\mathfrak{B}(B_J) = \mathfrak{B}(B^\infty_J) \cap B_J$. Using the [2, Theorem 1.61] we can find,

$$|||x|||_B = \sup_n ||\sum_{k=1}^{n} x_k e_k||_B \tag{2.2}$$

is an equivalent norm on B. When B carries the equivalent norm (2.2), the operator $T : (B, |||.|||_B) \to (B_J, ||.||_{B_J})$ defined by $T(x) = (x_k)$ is an isometric isomorphism from B onto B_J. B_J is called canonical representation of B (see [2, page 67]). This means that every Banach space B with an S-basis has a natural embedding in $\mathbb{R}^\infty_{\mathcal{I}}$. In under the isometric isomorphism from B to B_J, in our work, we use the canonical Banach spaces B_J. For simplicity of the notation, we write B_J by B.
The σ-algebra generated by B and associated with $\mathfrak{B}(B_J)$ is

$$\begin{aligned}\mathfrak{B}_J(B) &= \{T^{-1}(A) : \ A \in \mathfrak{B}(B_J)\}\\ &= T^{-1}\{\mathfrak{B}(B_J)\}\end{aligned}$$

Definition 2.2 [2, Definition 2.42]
Define $\overline{v}_k$, $\overline{\gamma}_k$ on $A \in \mathfrak{B}(\mathbb{R})$ by $\overline{v}_k(A) = \frac{\mu(A)}{\mu(J_k)}$, $\overline{\gamma}_k(A) = \frac{\mu(A\cap J_k)}{\mu(J_k)}$ for elementary sets $A = \Pi_{k=1}^\infty B_k$, $A \in \mathfrak{B}(B_J^n)$, define $\overline{v}_J^n$ by:

$$\overline{v}_J^n(A) = \Pi_{k=1}^n \overline{v}_k(A_k) \times \Pi_{k=n+1}^\infty \overline{\gamma}_k(B_k).$$

If B is a Banach space with an S-basis and $A \in \mathfrak{B}_J(B)$. We define $\mu_B(A) = v_J(T(A))$ for $A \in \mathfrak{B}_J(B)$ and $v_J(B) = \lim_{n\to\infty} v_J^n(B)$ for all $B \in \mathfrak{B}(B_J)$.

2.1. The integrable functions over B

Here, we will discuss the nature of the integrable functions over B. Since $B_J \subset \mathbb{R}^\infty_{\mathcal{I}}$, it suffices to discuss functions on $\mathbb{R}^\infty_{\mathcal{I}}$. Consider $x = (x_1, x_2, ..) \in \mathbb{R}^\infty_{\mathcal{I}}$, $\mathcal{I}_n = \Pi_{k=n+1}^\infty[\frac{-1}{2}, \frac{1}{2}]$, $h_n(x) = \otimes_{k=n+1}^\infty \chi_{\mathcal{I}}(x_k)$, where $\chi_{\mathcal{I}}$ is the characteristic function for the interval $\mathcal{I} = [\frac{-1}{2}, \frac{1}{2}]$.

Let M^n represent the class of measurable functions on $\mathbb{R}^n$. If $x \in \mathbb{R}^\infty_{\mathcal{I}}$ and $f^n \in M^n$, let $\overline{x} = (x_i)_{i=1}^n$, $\widehat{x} = (x_i)_{i=n+1}^\infty$, then $f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x})$ and

$$M^n_{\mathcal{I}} = \left\{ f(x) : \ f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}, \ x \in \mathbb{R}^\infty_{\mathcal{I}} \right\}.$$

Definition 2.3 [2, Definition 2.47]A function $f : \mathbb{R}^\infty_{\mathcal{I}} \to \mathbb{R}$ is said to be measurable if there is a sequence $\{f_n \in M^n_{\mathcal{I}}\}$ such that $\lim_{n\to\infty} f_n(x) \to f(x)$ μ_∞-a.e.

Since μ_∞ restricted to $\mathfrak{B}(\mathbb{R}^n_{\mathcal{I}})$ is equivalent to μ_n. Recalling μ_∞ in $\mathbb{R}^\infty_{\mathcal{I}}$ is not unique. Also, the family $\{J_n\}$ ensures that every Banach space with an S-basis can be embedded as a closed subspace of B_J in $\mathbb{R}^\infty_{\mathcal{I}}$.

Definition 2.4 [2, Definition 2.55]Let $f : B \to [0, \infty]$ be a measurable function and let μ_B be constructed using the family $\{J_k\}$. If $\{s_n\} \subset \mathbb{M}$ is a increasing family of non negative simple functions with $s_n \in \mathbb{M}^n_J$, for each n and $\lim_{n\to\infty} s_n(x) = f(x)$, μ_B - a.e., the integral of f over B by

$$\int_B f(x) d\mu_B = \lim_{n\to\infty} \int_B \left[s_n(x) \Pi_{i=1}^n \mu(J_i) \right] d\mu_B(x).$$

Hence, μ_B restricted to $\mathfrak{B}(B_J^n)$ is equivalent to μ_n. We denote μ_B as the canonical version of Lebesgue measure associated with B.

2.2. L^p spaces

We recall let B be a Banach space with an S-basis and let $L^1(\widehat{B}) = \bigcup_{n=1}^\infty L^1(B^k)$ and $C_0(\widehat{B}) = \bigcup_{n=1}^\infty C_0(B^n)$.

We say that a measurable function $f \in L^1(B)$ if there exists a Cauchy-sequence $\{f_m\} \subset L^1(\widehat{B})$, such that

$$\lim_{m\to\infty} \int_B \left| f_m(x) - f(x) \right| d\mu_B(x) = 0.$$

Definition 2.5 [2, Def 2.65]Let B be a Banach space with an S-basis, let $L^p(\widehat{B}) = \bigcup_{n=1}^\infty L^p(B^k)$ and $C_0(\widehat{B}) = \bigcup_{n=1}^\infty C_0(B^n)$.

1. We say that a measurable function $f \in L^p(B)$ if there exists a Cauchy-sequence $\{f_m\} \subset L^p(\widehat{B})$, such that
$$\lim_{m\to\infty} \int_B \Big| f_m(x) - f(x) \Big|^p d\mu_B(x) = 0.$$

2. We say that a measurable function $f \in C_0(B)$, the space of continuous functions that vanish at infinity, if there exists a Cauchy sequence $\{f_m\} \subset C_0(\widehat{B})$, such that $\lim_{m\to\infty} \sup_{x\in B} \big| f_m(x) - f(x) \big| = 0$.

Theorem 2.1 *$C_c(B)$ is dense in $L^p(B)$.*

3. Test Functions and weak derivatives

In this section, we will look at the test functions and weak derivatives of B. The relationship between the test function spaces and $L^p(B)$ will be established.

Recall for two sets A and B, $A \subset\subset B$ means that the closure of A is a relatively compact subset of B. For example:
$$(0,\infty) \subset \mathbb{R} \text{ but } (0,\infty) \not\subset\not\subset \mathbb{R}, \text{ where as } (0,1) \subset \mathbb{R} \text{ and } (0,1) \subset\subset \mathbb{R}.$$

Let $C_c(B^n_J)$ be the class of continuous functions on B^n_J which vanish outside the compact sets. We say that a measurable function $f \in C_c(B^\infty_J)$, if there exists a Cauchy sequence $\{f_n\} \subset \bigcup_{n=1}^{\infty} C_c(B^n_J) = C_c(\widehat{B}^\infty_J)$ such that $\lim_{n\to\infty} ||f_n - f||_\infty = 0$. We define $C_0(B^\infty_J)$, the continuous functions that vanish at ∞, and $C^\infty_0(B^\infty_J)$ the compactly supported smooth functions.

Let $\mathbb{N}^\alpha_0$ be the set of all multi-index infinite tuples $\alpha = (\alpha_1, \alpha_2, ..)$ with $\alpha_i \in \mathbb{N}$ and all but a finite number of entries are zero.
We define the operator D^α and D_α by
$$D^\alpha = \Pi^\infty_{k=1} \frac{\partial^{\alpha_k}}{\partial {x_i}^{\alpha_k}}$$
and
$$D_\alpha = \Pi^\infty_{k=1} \left(\frac{1}{2\pi i} \frac{\partial}{\partial x_k} \right)^{\alpha_k}.$$

Next we define set of test functions in our settings as follows.

Definition 3.1 *We define the set of test functions (or C^∞_c-functions with compact support on B as*
$$D_t(B) = \big\{ \phi \in C^\infty_c(B) : \ supp(\phi) = \overline{\{x : \ \phi(x) \neq 0\}} \subseteq B \ \ is\ compact \big\}.$$

We defined a given function $f \in D_t(B)$ to be a measurable if and only if there exists a sequence of functions $\{f_m\} \in D_t(\widehat{B}) = \bigcup_{n=1}^{\infty} B_t(B^n)$ and a compact set $\mathbb{K} \subset B$, which contains the support of $f - f_m$, for all m, and $D^\alpha f_m \to D^\alpha f$ uniformly on $\mathbb{K}$, for every multi-index $\alpha \in \mathbb{N}^\alpha_0$.

We called $supp(\phi)$ be the support of ϕ. The topology of $D_t(B)$ will be the compact sequential limit topology. We denote $D'_t(B)$ as the dual space of $D_t(B)$ in our work. The space of distributions on B is the set of all continuous linear functionals $T \in D'_t(B)$, the dual space of $D_t(B)$. A family of distributions $T_i \subset D'_t(B)$ is said to converge to $T \in D'_t(B)$ if $T_i(\phi)$ converge to $T(\phi)$ for every $\phi \in D_t(B)$.

Definition 3.2 [2, Definition 2.84]If α is a multi-index and $u, v \in L^1_{loc}(B)$, v is the α^{th} weak partial derivative of u provided that
$$\int_B u(D^\alpha \phi) d\mu_B = (-1)^{|\alpha|} \int_B \phi v d\mu_B$$
for all functions $\phi \in C^\infty_c(B)$.

Let $B \subset \mathbb{R}^\infty_\mathcal{I}$ be an open and $\epsilon > 0$. Let ∂B is the boundary of B, then

$$B_\epsilon = \Big\{ x \in B : \ dist(x, \partial B) > \epsilon \Big\}.$$

Lemma 3.1 *The space of test functions $D_t(B)$ is dense in $L^p(B)$ for $1 \le p < \infty$.*

Proof: Let $f \in L^p(B)$. Let us define a mollifier $f_\epsilon = \int_B \theta_\epsilon(x-y) f(y) d\mu_B(y)$ where $\theta_\epsilon(x) = \frac{1}{\epsilon}\theta(\frac{x}{\epsilon})$ and

$$\theta(x) = \begin{cases} c.\exp\left(\left(|x|^2 - 1\right)^{-1}\right) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \ge 1. \end{cases}$$

Now, the property of mollifiers gives $f_\epsilon \in C^\infty(B_\epsilon)$ and $f_\epsilon \to f$ a.e. as $\epsilon \to 0$. Let us assume an open set $V \subset B$ and another open set W so that $V \subset\subset W \subset\subset B$. Then,

$$\begin{aligned} |f_\epsilon(x)| &= \left| \int_{B_\epsilon} \theta_\epsilon(x-y) f(y) d\mu_B(y) \right| \\ &\le \int_{B_\epsilon} \theta_\epsilon^{1-\frac{1}{p}}(x-y)\theta_\epsilon^{\frac{1}{p}}(x-y)|f(y)| d\mu_B(y) \\ &\le \left(\int_{B_\epsilon} \theta_\epsilon(x-y) d\mu_B(y) \right)^{1-\frac{1}{p}} \left(\int_{B_\epsilon} \theta_\epsilon(x-y)|f(y)|^p d\mu_B(y) \right)^{\frac{1}{p}}. \end{aligned}$$

Since $\int_{B_\epsilon} \theta_\epsilon(x-y) d\mu_B(y) = 1$ so,

$$\begin{aligned} \int_V |f_\epsilon(x)|^p d\mu_B(x) &\le \int_V \left(\int_{B_\epsilon} \theta_\epsilon(x-y)|f(y)|^p d\mu_B(y) \right) d\mu_B(x) \\ &\le \int_W |f(y)|^p \left(\int_{B_{y,\epsilon}} \theta_\epsilon(x-y) d\mu_B(x) \right) d\mu_B(y) \\ &= \int_W |f(y)|^p d\mu_B(y) \end{aligned}$$

provided $\epsilon \to 0$. So, $||f_\epsilon||_{L^p(V)} \le ||f||_{L^p(W)}$. So, for $1 \le p < \infty$, $f_\epsilon \to f$ in $L^p(B)$. Hence, $D_t(B)$ is dense in $L^p(B)$ for $1 \le p < \infty$. □

Lemma 3.2 *$C_0^\infty(B')$ is dense in $L^p(B')$.*

Proof: Taking $\phi \in C_0^\infty(B')$, $\phi \ge 0$ and $\int_{B'} \phi d\mu_B = 1$. Define $\phi_\epsilon(x) = \epsilon^{-1}\phi(\frac{x}{\epsilon})$. If $f \in L^p(B')$ with compact support then $\phi_\epsilon * f$ has compact support is of the class $C^\infty(B')$ and $\phi_\epsilon * f$ converges to f in $L^p(B')$. □

Theorem 3.1 (Fundamental lemma of the Calculus of variations) If $f \in L^1_{loc}(B)$ satisfies $\int_B f\phi d\mu_B = 0$ for every $\phi \in C_0^\infty(B)$, then $f = 0$ a.e. in B.

Proof: Let $v_1 \in L^1_{loc}(B)$ and $v_2 \in L^1_{loc}(B)$ be weak αth partial derivatives of u, then

$$\begin{aligned} \int_B u D^\alpha \phi d\mu_B &= (-1)^{|\alpha|} \int_B v_1 \phi d\mu_B \\ &= (-1)^{|\alpha|} \int_B v_2 \phi d\mu_B \end{aligned}$$

for every $\phi \in C_0^\infty(B)$. We have,

$$\int_B (v_1 - v_2)\phi d\mu_B = 0 \text{ for every } \phi \in C_0^\infty(B).$$

Let B' is open and $\overline{B}'$ is a compact subset of B. Since $C_0^\infty(B')$ is dense in $L^p(B')$, then there exists a sequence of functions (ϕ_i) in $C_0^\infty(B')$ such that $|\phi_i| \leq \alpha$ in B' and $\phi_i \to sgn(v_1 - v_2)$ a.e. in B' as $i \to \infty$. Now from dominated convergence theorem, with the majorant $|(v_1 - v_2)\phi_i| \leq 2(|v_1| + |v_2|) \in L^1(B')$, gives

$$\begin{aligned} 0 &= \lim_{i\to\infty} \int_{B'} (v_1 - v_2)\phi_i d\mu_B \\ &= \int_{B'} \lim_{i\to\infty} (v_1 - v_2)\phi_i d\mu_B \\ &= \int_{B'} (v_1 - v_2) sgn(v_1 - v_2) d\mu_B \\ &= \int_{B'} |v_1 - v_2| d\mu_B. \end{aligned}$$

This implies that $v_1 = v_2$ a.e. in B' for every $B' \subset B$. Thus $v_1 = v_2$ a.e. in B.
Consequently, if $f \in L^1_{loc}(B)$ satisfies $\int_B f\phi d\mu_B = 0$ for every $\phi \in C_0^\infty(B)$ then $f = 0$ a.e. in B. □

Definition 3.3 [2, Definition 2.87]A function $f \in C^\infty(B)$ is called a Schwartz function, or $f \in \mathbb{S}(B)$, iff, for all multi-indices α and β in $\mathbb{N}_0^\alpha$, the seminorm $\rho_{\alpha,\beta}(f)$ is finite, where

$$\rho_{\alpha,\beta}(f) = \sup_{x\in B} |x^\alpha D^\beta f(x)|.$$

$\mathbb{S}(B)$ (respectively $\mathbb{S}(B')$) is a Fréchet space, which is dense in $C_0(B)$. The test function space $D_t(B)$ is subspace of $\mathbb{S}(B)$. By Lemma 3.1, $\mathbb{S}(B)$ is dense in $L^p(B)$.

4. Sobolev space on canonical Banach spaces

In this section, we discuss Sobolev space $S^{k,p}(B)$ on canonical Banach space B.

Since, $S^{k,p}(B_J^n) \subset S^{k,p}(B_J^{n+1})$, we assume $S^{k,p}(\widehat{B}) = \bigcup_{n=1}^\infty S^{k,p}(B^k)$.

We say a given function $f \in S^{k,p}(B)$ to be a measurable function if there exists a Cauchy sequence $\{f_m\} \subset S^{k,p}(\widehat{B})$ such that

$$\sum_{|\alpha|\leq k} \lim_{m\to\infty} \int_B \left| D^\alpha f_m(x) - D^\alpha f(x) \right|^p d\mu_B(x) = 0.$$

Thus the Sobolev space $S^{k,p}(B)$ consists of those functions of $L^p(B)$ that have weak partial derivatives upto order k and they belong to $L^p(B)$. Equivalently, we can state the following definition of Sobolev space.

Definition 4.1 *The Sobolev space $S^{k,p}(B)$ consists of function $f \in L^p(B)$ suct that for every multi-iindex α with $|\alpha| \leq k$, the weak derivative $D^\alpha f$ exists and $D^\alpha f \in L^p(B)$. Thus*

$$S^{k,p}(B) = \left\{ f \in L^p(B) : \ D^\alpha f \in L^p(B) : \ |\alpha| \leq k \right\}.$$

In our setting, we will find the norm of the Sobolev space as follows.

Proposition 4.1 *The expression $\sum_{|\alpha|\leq k} ||D^\alpha f||_{L^p(B)}$, $1 \leq p \leq \infty$ is a norm on $S^{k,p}(B)$.*

Proof: To prove the expression is a norm, we need the following:

1. The expression

$$\begin{aligned}
&\sum_{|\alpha|\leq k} ||D^\alpha f||_{L^p(B)} = 0\\
&\Rightarrow ||D^\alpha f||_{L^p(B)} = 0\\
&\Rightarrow ||f||_{L^p(B)} = 0 \ a.e.\\
&\Rightarrow f = 0 \ a.e. \ in \ B.
\end{aligned}$$

Now, if $f = 0 \in L^1_{loc}(B)$ a.e. in B. Now from the [2, Definition 2.84] we have

$$\int_B f(D^\alpha \phi) d\mu_B = (-1)^{|\alpha|} \int_B \phi g d\mu_B = 0$$

for all $\phi \in C_c^\infty(B),\ g$ is in the dual space $D_t'(B)$ of $D_t(B)$ with $g \in L^1_{loc}(B)$. Now from the Theorem 3.1, $D^\alpha f = 0$ a.e. in B for all $\alpha,\ |\alpha| \leq k$.

2. Clearly, $||\alpha f||_{S^{k,p}(B)} = |\alpha| ||f||_{S^{k,p}(B)},\ \alpha \in \mathbb{R}$.

3. For the triangle inequality for $1 \leq p < \infty$, using the elementary inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha,\ a, b \geq 0,\ 0 < \alpha \leq 1$ and Minkowski inequality we have

$$\begin{aligned}
||f+g||_{S^{k,p}(B)} &= \left(\sum_{|\alpha|\leq k} ||D^\alpha f + D^\alpha g||^p_{L^p(B)} \right)^{\frac{1}{p}}\\
&\leq \left(\sum_{|\alpha|\leq k} \left(||D^\alpha f||_{L^p(B)} + ||D^\alpha g||_{L^p(B)} \right)^p \right)^{\frac{1}{p}}\\
&\leq \left(\sum_{|\alpha|\leq k} ||D^\alpha f||^p_{L^p(B)} \right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\leq k} ||D^\alpha g||^p_{L^p(B)} \right)^{\frac{1}{p}}.
\end{aligned}$$

□

We denote this norm as $||f||_{S^{k,p}(B)} = \sum_{|\alpha|\leq k} ||D^\alpha f||_{L^p(B)},\ 1 \leq p \leq \infty$.

Theorem 4.1 *The Sobolev space* $\left(S^{k,p}(B), ||.||_{S^{k,p}(B)} \right)$ *is a Banach space for* $1 \leq p \leq \infty$.

Proof: Let (f_i) be a Cauchy sequence in $S^{k,p}(B)$. Since,

$$||D^\alpha f_i - D^\alpha f_j||_{L^p(B)} \leq ||f_i - f_j||_{S^{k,p}(B)},\ |\alpha| \leq k.$$

So, $D^\alpha f_i \to f_\alpha \in L^p(B)$. Again,

$$\begin{aligned}
\int_B f D^\alpha \phi d\mu_B &= \lim_{i\to\infty} \int_B f_i D^\alpha \phi d\mu_B\\
&= \lim_{i\to\infty} (-1)^{|\alpha|} \int_B D^\alpha f_i \phi d\mu_B\\
&= (-1)^{|\alpha|} \int_B f_\alpha \phi d\mu_B
\end{aligned}$$

for every $\phi \in C_0^\infty(B)$.

Case 1 For $1 < p < \infty$, let $\phi \in C_0^\infty(B)$. Using Holder's inequality we have

$$\begin{aligned}
\left| \int_B f_i D^\alpha \phi d\mu_B - \int_B f D^\alpha \phi d\mu_B \right| &= \left| \int_B (f_i - f) D^\alpha \phi d\mu_B \right|\\
&\leq ||f_i - f||_{L^p(B)} ||D^\alpha \phi||_{L^{p'}(B)} \to 0.
\end{aligned}$$

So, $D^\alpha f = f_\alpha,\ |\alpha| \leq k$.

Hence, $||D^\alpha f_i - D^\alpha f||_{L^p(B)} \to 0$ gives $||f_i - f||_{S^{k,p}(B)} \to 0$. Therefore, $f_i \to f$ in $S^{k,p}(B)$.

Case 2 For $p = 1,\ p = \infty$, the proof is very straight, so we have omitted. □

Lemma 4.1 *The space of test functions $D_t(B)$ is dense in $S^{k,p}(B)$ for $1 \leq p < \infty$.*

Proof: The proof is similar to the proof of the Lemma (3.1), so we omit the proof. □

Proposition 4.2 *$\mathbb{S}(B)$ is dense in $S^{k,p}(B)$.*

Proof: It is well known that $\mathbb{S}(B)$ is dense in $C_0(B)$. So, there exists a Cauchy sequence $\{f_m\} \subset C_0(\widehat{B})$ such that

$$\lim_{m\to\infty} \sup_{x\in B} |f_m(x) - f(x)| = 0.$$

Now,

$$\lim_{m\to\infty} |D^\alpha f_m(x) - D^\alpha f(x)| = 0$$

if and only if $\{f_m\} \subset S^{k,p}(\widehat{B})$. So,

$$\sum_{|\alpha|\leq k} \lim_{m\to\infty} \int_B |D^\alpha f_m(x) - D^\alpha f(x)|^p d\mu_B(x) = 0.$$

Hence, $f \in S^{k,p}(B)$. Consequently, $\mathbb{S}(B)$ is dense in $S^{k,p}(B)$. □

Corollary 4.1 *$\mathbb{S}(B')$ is dense in $S^{k,p}(B')$.*

Definition 4.2
1. *The Sobolev space $S^{k,2}(B)$ consists of functions $u \in L^2(B)$ such that for every multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^2(B)$. Thus*

$$S^{k,2}(B) = \big\{u \in L^2(B):\ D^\alpha u \in L^2(B),\ |\alpha| \leq k\big\}.$$

2. *We assume the inner product on $S^{k,p}(B)$ as:*

$$\big\langle f \mid g\big\rangle_{S^{k,2}} = \sum_{|\alpha|\leq m} \big\langle D^{(\alpha)} f \mid D^{(\alpha)} g\big\rangle_{L^2} \tag{4.1}$$

$$H^{k,2}(\overline{B}) = \overline{C^k(\overline{B}) \cap S^{k,2}(B)},$$

where the closure is with respect to the norm induced by $< . \mid . >_{S^{k,2}}$.

3. *$H_0^{k,2}(B) = \overline{D_t(B)}$, with respect to the induced norm on $S^{k,2}$.*

Theorem 4.2 *$S^{k,2}(B)$ is a Hilbert space with the inner product (4.1).*

Proposition 4.3
1. *$\mathbb{S}(B')$ is dense in $S^{k,2}(B)$.*
2. *$\mathbb{S}(B')$ is dense in $S^{k,2}(B')$.*

5. The Transform on Sobolev space of canonical Banach spaces

In this section we study the Fourier transform when the Sobolev space is on B, a canonical space with an S-basis. In the case of finite dimensional Euclidean space, it is very natural framework. We consider for the case of infinite dimensional. We define the Fourier transform as follows:

Definition 5.1 *For each $f \in L^1(B)$, we define*

$$F_r(f)(y) = \widehat{f}(y) = \int_B e^{-2\pi ixy} f(x) d\mu_B(x) \tag{5.1}$$

where $x \in B$ and $y \in B'$ and the notation xy for the scalar product of x with y. The operator F_r is called the Fourier transform.

Theorem 5.1 *Let $f \in L^1(B)$. Then $F_r(f) \in C_0(B')$.*

Proof: Let $f \in L^1(B)$. To prove $F_r \in C^0(B')$, let (τ_n) be a sequence in B' with $\tau_n \to \tau$. Using the continuity of $\exp(x)$ or $\exp(y)$ and the scalar product we obtain

$$|e^{-ix\tau_n} - e^{-ix\tau}| \to 0, \ \forall \ x \in B, \ y \in B'$$

or

$$|e^{-iy\tau_n} - e^{-iy\tau}| \to 0 \ \forall \ x \in B, \ y \in B'.$$

Now from the Definition (5.1) of the Fourier transform and using dominated convergence, we have

$$|F_r(\tau_n) - F_r(\tau)| \leq \int_B |f(x)||e^{-ixy\tau_n} - e^{-ixy\tau}| d\mu_B(x) \to 0.$$

Since the L^1-function dominates the integrand, $F_r(f)$ is continuous and vanishes at infinity. Hence, $F_r(f) \in C_0(B')$. □

Proposition 5.1 *If $f \in \mathbb{S}(B)$, then $F_r(f) \in \mathbb{S}(B)$.*

Proof: Let $f \in \mathbb{S}(B)$. Now,

$$\begin{aligned} D^\alpha(F_r f)(x) &= \frac{\partial^\alpha}{\partial x^\alpha} \int_B f(x) e^{-2\pi ixy} d\mu_B(x) \\ &= (-i)^{|\alpha|} \int_B f(x) x^\alpha e^{-2\pi ixy} d\mu_B(x) \\ &= (-i)^{|\alpha|} F_r(x^\alpha f)(x). \end{aligned}$$

When we allow differentiation into the integral sign in second step, the dominated convergence theorem gives $x^\alpha f \in \mathbb{S}(B)$. So, $F_r(f) \in C^\infty(B)$. Let $P(x)$ be a polynomial, using Leibnitz formula and Closed graph theorem, $f(x) \to P(x)f(x), \ f(x) \to x^\alpha D^\beta f(x)$ are continuous linear mapping of $\mathbb{S}(B)$ into $\mathbb{S}(B)$. Let $\widehat{\mathbb{S}}(B')$ be the set of all $\widehat{f}(y) = F_r(f)(y)$ for $f \in \mathbb{S}(B)$, then $F_r(Pf)(y) \in \widehat{\mathbb{S}}(B')$. It is easy to see F_r is surjective, $\widehat{\mathbb{S}}(B') = \mathbb{S}(B')$ and F_r^{-1} is continuous. Since, $\mathbb{S}(B)$ is dense in $L^1(B)$, it is clear that every $f \in L^1(B)$ is the limit of a sequence $\{f_n\}$ in $\mathbb{S}(B)$. Hence, for every $f \in \mathbb{S}(B)$, $\widehat{f} \in \mathbb{S}(B') \subset C_0(B')$. Consequently, $\widehat{f} \in \mathbb{S}(B)$. □

Theorem 5.2 *The mapping $F_r : \mathbb{S}(B) \to \mathbb{S}(B')$ extends to a continous linear isometry of $\mathcal{U} : S^{k,2}(B) \to S^{k,2}(B')$ satisfying the following*

$$\int_B |D^\alpha f(x)|^2 d\mu_B(x) = \int_B |D^\alpha \widehat{f}(y)|^2 d\mu_B(y). \tag{5.2}$$

Proof: To prove the Equation 5.2, we have

$$\int_B f(x)\overline{g(x)}d\mu_B(x) = \int_B \overline{g(x)}\Big\{\int_{B'} \widehat{f}(y)e^{2\pi ixy}d\mu_{B'}(y)\Big\}d\mu_B(x)$$
$$= \int_{B'} \widehat{f}(y)\Big\{\int_B \overline{g(x)}e^{2\pi ixy}d\mu_B(x)\Big\}d\mu_{B'}(y)$$

So, $\int_B f(x)\overline{g(x)}d\mu_B(x) = \int_{B'} \widehat{f}(y)\overline{\widehat{g}(y)}d\mu_{B'}(y)$. Again, $f(x) \in \mathbb{S}(B)$ then using Leibnitz formula and closed graph theorem, the transform $f(x) \to D^\alpha f(x)$ are continous linear mapping of $\mathbb{S}(B)$ into $\mathbb{S}(B)$. Taking $f = g$,

$$\int_B |D^\alpha f(x)|^2 d\mu_B(x) = \int_B |D^\alpha \widehat{f}(y)|^2 d\mu_B(y).$$

It is known from the Proposition 4.3, that $\mathbb{S}(B)$ is dense in $S^{k,2}(B)$ and $\mathbb{S}(B)$ is dense in $S^{k,2}(B')$. We see the Definition (5.1) of the Fourier transform, relative to the $S^{k,2}$ metric, the mapping $F_r : f \to \widehat{f}$ is a linear isometry of $\mathbb{S}(B) \subset S^{k,2}(B)$ onto $\mathbb{S}(B') \subset S^{k,2}(B')$. It is now follows that F_r has a unique extension $\mathcal{U} = \overline{F_r}$; $\mathcal{U} : S^{k,2}(B) \to S^{k,2}(B')$. □

Theorem 5.3 *$F_r(f)$ is bijective and isometric with respect to $||.||_2$ on subspace $\mathbb{S}(B)$ of $L^2(B)$ and (inversion) $\{F_r(f)\}^{-1} : \mathbb{S}(B') \to \mathbb{S}(B)$ is also continuous.*

Lemma 5.1 *Let $f \in \mathbb{S}(B)$ then*

$$\big(F_r F_r f\big)(x) = f(-x) \;\forall\; x \in B.$$

Proof: Let $f \in \mathbb{S}(B)$ then $F_r(f) \in \mathbb{S}(B)$. Using Fubini's theorem, we have

$$\int_B F_r(f)(x)g(x)d\mu_B(x) = \int_B\int_{B'} f(y)g(x)e^{-2\pi ixy}d\mu_B(x)d\mu_{B'}(y)$$
$$= \int_B f(x)F_r(g)(x)d\mu_B(x).$$

As $xy \to f(y)g(x)e^{-2\pi ixy}$ is integrable. Let $g(x) = e^{-2\pi ixy_0}\gamma(ax)$ with $y_0 \in B'$ and $a > 0$. Then,

$$(F_r g)(y) = \int_B e^{-2\pi ixy_0}\gamma(ax)d\mu_B(x)$$
$$= (F_r\gamma_a)(y + y_0).$$

Now,

$$\int_B F_r(f)(x)e^{-2\pi ixy_0}\gamma(ax)d\mu_B(x) = \int_B f(x)(F_r(g))(x)d\mu_B(x)$$
$$= \int_B f(x)\frac{1}{a^n}F_r(\gamma)\Big(\frac{x+y_0}{a}\Big)d\mu_B(x)$$
$$= \int_B f(au - y_0)\gamma(u)d\mu_B(u),\; where\; u = \frac{x+y_0}{a}.$$

When $a \to 0$, using dominated convergence theorem $\big(F_r F_r(f)\big)(x) \to f(-x) \;\forall\; x \in B.$ □

Since, $\mathbb{S}(B)$ is dense in $L^2(B)$ so we can extend F_r to an isometric operator on $L^2(B)$. The Theorem 5.3 implies the following Theorem.

Theorem 5.4 *$F_r : L^2(B) \to L^2(B')$ is isometric and*

$$\big\langle F_r(f) \mid F_r(g)\big\rangle_{L^2(B)} = \big\langle f \mid g\big\rangle_{L^2(B')},\;\forall\; f \in L^2(B),\; g \in L^2(B').$$

Proof: The Lemma 5.1, gives

$$\begin{aligned}\big(F_r^{-1}f\big)(x) &= \big(F_r^2(F_r f)\big)(x)\\ &= (F_r f)(-x).\end{aligned}$$

So, we obtain

$$\int_B F_r(f)\overline{F_r(g)(x)}d\mu_B(x) = \int_B f(x)\big(F_r\overline{(F_r(g)}\big)(x)d\mu_B(x).$$

Hence,

$$\big\langle F_r(f) \mid F_r(g)\big\rangle_{L^2(B)} = \big\langle f \mid g\big\rangle_{L^2(B')} \;\forall\; f \in L^2(B),\; g \in L^2(B').$$

□

Finally, $\mathbb{S}(B)$ is dense in $S^{k,2}(B)$, encourage us to extend F_r to an isometric operator on $S^{k,2}(B)$ as follows:

Theorem 5.5 *$F_r : S^{k,p}(B) \to S^{k,p}(B')$ is isometric and*

$$\big\langle F_r(f) \mid F_r(g)\big\rangle_{S^{k,p}(B)} = \big\langle f \mid g\big\rangle_{S^{k,p}(B')} \;\forall\; f \in S^{k,p}(B),\; g \in S^{k,p}(B').$$

Proof: The proof is similar to the proof of the theorem 5.4, so we have omitted it. □

Next we shall illustrate our results with an example.

Example 4.9. Consider a Schwartz function $f(x) = e^{-x^2}$.

The $F_rf(\xi) = e^{\frac{-\xi^2}{2}} = f(\xi)$ and $F_r(e^{-\frac{x^2}{2}}) = \frac{1}{a^n}f(\frac{\xi}{a})$ for $a > 0$ with $f_a(x) = f(ax)$. Using the Lemma 4.6, F_r^2 is the reflection. The Theorem 4.7 gives $||F_rf||_2 = ||f||_2$ for all $f \in \mathbb{S}(B)$. Since, $\mathbb{S}(B)$ is dense in $S^{k,2}(B)$, so $f(x) = e^{-\frac{x^2}{2}} \in S^{k,2}(B)$. Finally, using the Theorem 4.8, $||F_rf||_{S^{k,p}(B)} = ||f||_{S^{k,p}(B)}$.

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