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Family of Surfaces with a Common Special Polynomial Curves

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ABSTRACT: Polynomial curves play an important role in statistics, mathematics, engineering, computer science, and many other fields. It is widely used to model and analyze data. By using it to best simulate data points, it is possible to gain better insights and predictions on data sets. In engineering applications, it is used to model material properties, vibration analysis, temperature changes and similar factors. Computer graphics and animations are frequently used to represent the surfaces of objects, especially in 3D modeling and animation software for rendering surfaces and objects. In this paper, we establish the necessary and sufficient conditions to parameterize a surface family on which the polynomial curve of any given curve lies as an isogeodesic, isoasymptotic and curvature line in $\mathbb{E}^3$. We also extend this idea on ruled surfaces. Finally, we present some examples and picture the corresponding surfaces.

Key Words: Polynomial curves, isoparametricity, asymptoticity, geodecity, curvature line, ruled surface.

Contents

1. Introduction

In differential geometry, curves have many important consequences and properties, [14,20]. Researches follow labours about the curves. In the light of the existing studies, authors always introduce new curves. One of the most significant curve on a surface is geodesic curve. Geodesics are important in the relativistic description of gravity. Einstein's principle of equivalence tells us that geodesics represent the paths of freely falling particles in a given space. In architecture, some special curves have nice properties in terms of structural functionality and manufacturing cost. One example is planar curves in vertical planes, which can be used as support elements. The asymptotic curve on a surface has been a long-term research topic in Differential Geometry. Asymptotic curves are encountered in architecture, astronomy, astrophysics and CAD, [13]. Another special characteristic for a curve is to be the curvature line on a given surface. The curvature lines have been of interest to researchers for many years, and they have especially been used for the process of extracting information from geometrical objects which is known in short as shape interrogation. With the help of the procedure on generating such new surface families, researchers began to examine the methods to construct new surfaces with some specific curves lying on the surface as geodesic, asymptotic or curvature line, [4]-[16]. By this study, it is of interest for us to form the family of surfaces with a common polynomial curve as isogeodesic, isoasymptotic and line of curvature. To do so, we first remind some basic concepts in Section 2. In Section 3.1, we express the surface family with a linear combination of Flc vectors and provide the necessary conditions for a surface such that polynomial curve lies as isogeodesic, isoasymptotic and line of curvature on it. In Section 3.2, the idea is attributed to the ruled surfaces. Some examples are also presented at the end of this section.

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2. Preliminaries

In this section, we remind some basic concepts that will be used throughout the paper. Let $\lambda = \lambda(s)$ be a regular space curve satisfying non-degenerate condition $\lambda^{'}(s) \wedge \lambda^{''}(s) \neq 0$. Then, the orthonormal vector system called Frenet frame is defined by

$$T(s) = \frac{\lambda^{'}(s)}{\|\lambda^{'}(s)\|}, \ B(s) = \frac{\lambda^{'}(s) \wedge \lambda^{''}(s)}{\|\lambda^{'}(s) \wedge \lambda^{''}(s)\|}, \ N(s) = B(s) \wedge T(s) \tag{2.1}$$

where T is tangent, N is principal normal, and B is binormal vector field. The Frenet formulas are given by

$$T' = \kappa\eta N, \ N' = -\kappa\eta T + \tau\eta B, \ B' = -\tau\eta N \ , \ \|\lambda'\| = \eta \tag{2.2}$$

where the curvature κ and torsion τ of the curve are, [14]

$$\kappa = \frac{\left\|\lambda^{'}(s) \wedge \lambda^{''}(s)\right\|}{\|\lambda^{'}(s)\|^{3}}, \qquad \tau = \frac{\left\langle \lambda^{'}(s) \wedge \lambda^{''}(s), \lambda^{'''}(s) \right\rangle}{\|\lambda^{'}(s) \wedge \lambda^{''}(s)\|^{2}}. \tag{2.3}$$

The n^{th} degree polynomial with parameter s is defined as

$$P(s) = \lambda_n s^n + \lambda_{n-1} s^{n-1} + ... + \lambda_1 s^1 + \lambda_0, \qquad \lambda_n \neq 0$$

where $n \in \mathbb{N}_0$, $\lambda_i \in \mathbb{R}$, $(0 \leq i \leq n)$, [17]. Now let us define a curve such that, $\lambda : [a,b] \to E^n$, $\lambda(s) = (\lambda_1(s), \lambda_2(s), ..., \lambda_n(s))$. If each $\lambda_i(s)$ are polynomials for $1 \leq i \leq n$, then $\lambda_s \in \mathbb{R}[s]$ is defined to be an $n-$dimensional polynomial curve [15]. The degree of such a polynomial curve as $\lambda(s)$ is given by

$$\deg \lambda(s) = max \left\{\deg\left(\lambda_1(s)\right), \deg\left(\lambda_2(s)\right), ..., \deg\left(\lambda_n(s)\right)\right\}, \ [15].$$

The definition of the Flc frame of a polynomial space curve $\lambda = \lambda(s)$ given by Dede in [12] is as follows

$$T(s) = \frac{\lambda^{'}(s)}{\|\lambda^{'}(s)\|}, \ D_1(s) = \frac{\lambda^{'}(s) \wedge \lambda^{(n)}(s)}{\left\|\lambda^{'}(s) \wedge \lambda^{(n)}(s)\right\|}, \ D_2(s) = D_1(s) \wedge T(s), \tag{2.4}$$

where the prime $'$ indicates the differentiation with respect to s and $^{(n)}$ stands for the n^{th} derivative. The new vectors D_1 and D_2 are called binormal-like vector and normal-like vector, respectively. The curvatures of the Flc-frame d_1, d_2, and d_3 are given by

$$d_1 = \frac{\langle T', D_2 \rangle}{\eta}, \ d_2 = \frac{\langle T', D_1 \rangle}{\eta}, \ d_3 = \frac{\left\langle {D_2}', D_1 \right\rangle}{\eta}, \tag{2.5}$$

where $\|\lambda'\| = \eta$. The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} T' \\ {D_2}' \\ {D_1}' \end{bmatrix} = \eta \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}. \tag{2.6}$$

The relationship between the Frenet and Frenet like frame (Flc) is given by

$$\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.7}$$

and the relations between the curvatures of two frames are

$$d_1 = \kappa cos\theta, \qquad d_2 = -\kappa sin\theta, \qquad \theta = arctan\left(-\frac{d_2}{d_1}\right), \qquad d_3 = \frac{d\theta}{\eta} + \tau \tag{2.8}$$

where $\theta = \sphericalangle(N, D_2)$.

3. Main Result

3.1. Surfaces with a common polynomial curve as isogeodesic, isoasymptotic and line of curvature

Let us denote $\{T(s), D_2(s), D_1(s)\}$ as the Flc frame of the curve λ lying on the surface $\varphi(s,v)$. The parametric equation of the surface is given by

$$\begin{aligned} \varphi(s,v) &= \lambda(s) + [x(s,v)T(s) + y(s,v)D_2(s) + z(s,v)D_1(s)], \\ & L_1 \le s \le L_2, \quad K_1 \le v \le K_2. \end{aligned} \tag{3.1}$$

Here $x(s,v)$, $y(s,v)$ and $z(s,v)$ are called differentiable marching-scale functions. By recalling the relation (2.7), we reform the parameterization given above as

$$\begin{aligned} \varphi(s,v) &= \lambda(s) + x(s,v)T(s) \\ &\quad + \big(y(s,v)cos\theta - z(s,v)sin\theta\big)N(s) \\ &\quad + \big(y(s,v)sin\theta + z(s,v)cos\theta\big)B(s). \end{aligned} \tag{3.2}$$

Theorem 3.1 *λ is isogeodesic on the surface $\varphi(s,v)$ if and only if*

$$\begin{cases} x(s,v_0) = & y(s,v_0) = z(s,v_0) \equiv 0, \\ \frac{\partial y}{\partial v}(s,v_0) = & \beta_1(s)sin\theta, \qquad \beta_1(s) \neq 0, \quad \sphericalangle(N, D_2) = \theta. \\ \frac{\partial z}{\partial v}(s,v_0) = & \beta_1(s)cos\theta, \end{cases} \tag{3.3}$$

Proof: Since the polynomial curves are parametric, we first write

$$x(s,v_0) = y(s,v_0) = z(s,v_0) \equiv 0, \tag{3.4}$$

for $v = v_0$. The normal vector, n of the surface φ is calculated by

$$\begin{aligned} \overrightarrow{n}(s,v) &= \frac{\partial \varphi(s,v)}{\partial s} \times \frac{\partial \varphi(s,v)}{\partial v} \\ &= -\eta(s)\left[\frac{\partial y(s,v)}{\partial v}sin\theta + \frac{\partial z(s,v)}{\partial v}cos\theta\right]N(s) \\ &\quad + \eta(s)\left[\frac{\partial y(s,v)}{\partial v}cos\theta - \frac{\partial z(s,v)}{\partial v}sin\theta\right]B(s). \end{aligned} \tag{3.5}$$

By taking $v = v_0$, we reform n as

$$n(s,v_0) = \phi_1(s,v_0)T(s) + \phi_2(s,v_0)N(s) + \phi_3(s,v_0)B(s), \tag{3.6}$$

where

$$\begin{cases} \phi_1(s,v_0) = & 0, \\ \phi_2(s,v_0) = & -\eta(s)\left[\frac{\partial y(s,v)}{\partial v}sin\theta + \frac{\partial z(s,v)}{\partial v}cos\theta\right], \quad (\eta(s) \neq 0) \\ \phi_3(s,v_0) = & \eta(s)\left[\frac{\partial y(s,v)}{\partial v}cos\theta - \frac{\partial z(s,v)}{\partial v}sin\theta\right]. \end{cases} \tag{3.7}$$

Now, recall the geodesicity condition that is $n \parallel N$, we write

$$\begin{aligned} -\frac{\partial y(s,v)}{\partial v}\Big|_{v_0} sin\theta - \frac{\partial z(s,v)}{\partial v}\Big|_{v_0} cos\theta \neq 0, \\ \frac{\partial y(s,v)}{\partial v}\Big|_{v_0} cos\theta - \frac{\partial z(s,v)}{\partial v}\Big|_{v_0} sin\theta = 0. \end{aligned} \tag{3.8}$$

This last expression can be restated with a given function, $\beta_1(s) \neq 0$

$$\begin{cases} \frac{\partial y}{\partial v}(s,v_0) = & \beta_1(s)sin\theta, \\ \frac{\partial z}{\partial v}(s,v_0) = & \beta_1(s)cos\theta, \end{cases} \tag{3.9}$$

which completes the proof. □

Theorem 3.2 *λ is isoasymptotic on the surface $\varphi(s,v)$ if and only if*

$$\begin{cases} x(s,v_0) = & y(s,v_0) = \ z(s,v_0) \equiv 0, \\ \frac{\partial y}{\partial v}(s,v_0) = & \beta_2(s)cos\theta, \qquad\qquad \beta_2(s) \neq 0, \quad \sphericalangle(N, D_2) = \theta. \\ \frac{\partial z}{\partial v}(s,v_0) = & -\beta_2(s)sin\theta \end{cases} \tag{3.10}$$

Proof: As polynomial curves are parametric, the relations (3.4),(3.5), (3.6) and (3.7) hold. Now, this time recalling the asymptoticity condition allows us to write

$$\begin{aligned} \langle \frac{\partial n}{\partial s}(s,v), T(s)\rangle = 0 \iff & \frac{\partial(\phi_1(s,v))}{\partial s} - \kappa(s)\phi_2(s,v) = 0, \qquad \kappa(s) \neq 0 \\ \iff & \phi_2(s,v) = 0. \end{aligned} \tag{3.11}$$

From (3.7), $\lambda(s)$ is isoasymptotic on the surface $\varphi(s,v)$, if and only if

$$\kappa(s)\eta(s)\left[\frac{\partial y(s,v_0)}{\partial v}sin\theta + \frac{\partial z(s,v_0)}{\partial v}cos\theta\right] = 0, \qquad \kappa(s), \eta(s) \neq 0. \tag{3.12}$$

The latter can be rewritten with $\beta_2(s) \neq 0$

$$\begin{cases} \frac{\partial y}{\partial v}(s,v_0) = & \beta_2(s)cos\theta, \\ \frac{\partial z}{\partial v}(s,v_0) = & -\beta_2(s)sin\theta \end{cases} \tag{3.13}$$

which completes the proof. □

Theorem 3.3 *The polynomial curve, α is a line of curvature on the surface $\varphi(s,v)$, if and only if*

$$\begin{cases} x(s,v_0) = y(s,v_0) = \ z(s,v_0) \equiv 0, \\ \\ (\gamma+\theta)(s) = -\eta \int \tau(s)ds, \\ \\ \frac{\partial y}{\partial v}(s,v_0) = \left(\frac{\beta_3}{\eta}\right)(s)sin\gamma(s), \\ \\ \frac{\partial z}{\partial v}(s,v_0) = -\left(\frac{\beta_3}{\eta}\right)(s)cos\gamma(s) \end{cases} \tag{3.14}$$

where $\left(\frac{\beta_3}{\eta}\right)(s) \neq 0, \quad \sphericalangle(D_2, \mu) = \gamma$ and $\mu(s)$ denotes the orthogonal vector field of the surface.

Proof: As $\mu(s)$ is orthogonal vector field of the surface on which the polynomial curve lies, we write

$$\mu(s) = cos\gamma(s)D_2(s) + sin\gamma(s)D_1(s).$$

By recalling the relation (2.7), we reform the parameterization given above as

$$\begin{aligned} \mu(s) = & \left[cos\gamma(s)cos\theta(s) - sin\gamma(s)sin\theta(s)\right]N(s) \\ & + \left[cos\gamma(s)sin\theta(s) + sin\gamma(s)cos\theta(s)\right]B(s). \end{aligned}$$

The condition for λ to be a line of curvature on the surface is twofold. First is that $\mu(s) \parallel n(s, v_0)$ which results the following:

$$\frac{-\eta(s)\left[\frac{\partial y(s,v)}{\partial v}sin\theta + \frac{\partial z(s,v)}{\partial v}cos\theta\right]}{cos(\gamma+\theta)} = \frac{\eta(s)\left[\frac{\partial y(s,v)}{\partial v}cos\theta - \frac{\partial z(s,v)}{\partial v}sin\theta\right]}{sin(\gamma+\theta)} = \beta_3(s).$$

When the necessary arrangements are made here

$$\frac{\partial y}{\partial v}(s, v_0) = (\frac{\beta_3}{\eta})(s)sin\gamma(s),$$

$$\frac{\partial z}{\partial v}(s, v_0) = -(\frac{\beta_3}{\eta})(s)cos\gamma(s)$$

is found. Second, the surface accepting λ as the base curve and $\mu(s)$ as the director, $\Phi(s, v) = \lambda(s)+v\mu(s)$ should be developable, that is $det(\lambda', \mu, \mu') = 0$. From this condition, we get

$$\eta(s)\left[(\gamma + \theta)'(s) + \tau(s)\right] = 0 \quad \Longrightarrow \quad \gamma + \theta = -\eta\int\tau(s)ds,$$

which completes the proof. □

3.2. Ruled surfaces with a common polynomial isogeodesic and isoasymptotic curve

The parametric equation of a ruled surface having λ as the base vector is given as

$$\varphi(s, v) = \varphi(s, v_0) + (v - v_0)R(s). \tag{3.15}$$

Now by considering the parametricity condition given in (3.4) and using it in the latter expression, we get

$$\varphi(s, v_0) = \lambda(s). \tag{3.16}$$

When substituted (3.2) and (3.16) in (3.15), we form the following relation

$$x(s, v)T(s) + y(s, v)D_2(s) + z(s, v)D_1(s) = (v - v_0)R(s).$$

The inner product of this last expression with T, D_2, D_1 results

$$\begin{aligned} x(s, v) &= (v - v_0) < R, T >,\\ y(s, v) &= (v - v_0) < R, D_2 >,\\ z(s, v) &= (v - v_0) < R, D_1 > . \end{aligned} \tag{3.17}$$

By taking into account (3.3), we write

$$\begin{aligned} x(s, v) &= (v - v_0)f(s),\\ y(s, v) &= (v - v_0)\beta_1(s)sin\theta,\\ z(s, v) &= (v - v_0)\beta_1(s)cos\theta \end{aligned} \tag{3.18}$$

where $\beta_1(s) \neq 0$ and $f(s)$ are any real valued functions. Substituting this in (3.15) we define the parametric representation of the ruled surfaces with a common polynomial isogeodesic curve as

$$\varphi(s, v) = \lambda(s) + (v - v_0)\Big(f(s)T(s) + \beta_1(s)sin\theta D_2(s) + \beta_1(s)cos\theta D_1(s)\Big). \tag{3.19}$$

When considered both (3.10) and (3.17), we write

$$\begin{aligned} x(s,v) &= (v-v_0)h(s), \\ y(s,v) &= (v-v_0)\beta_2(s)cos\theta, \\ z(s,v) &= -(v-v_0)\beta_2(s)sin\theta \end{aligned} \tag{3.20}$$

where $\beta_2(s) \neq 0$ and $h(s)$ are any real valued functions. Substituting this in (3.15) the parametrization of the ruled surfaces with a common polynomial isoasymptotic curve as

$$\varphi(s,v) = \lambda(s) + (v-v_0)\Big(h(s)T(s) + \beta_2(s)cos\theta D_2(s) - \beta_2(s)sin\theta D_1(s)\Big). \tag{3.21}$$

3.3. Examples of Generating Simple Surfaces and Ruled Surface with Common Polynomial Curve

Example 3.1 *Let us consider,* $\lambda(s) = \left(s, \frac{s^2}{2}, \frac{s^3}{6}\right)$ *and the surface* $\varphi(s,v) = \left(s, \frac{s^2}{2}, \frac{s^3}{6} + 6v\right)$. *The Flc frame and the curvatures of* λ *are given as*

$$\begin{cases} T(s) = & \left(\frac{2}{s^2+2}, \frac{2s}{s^2+2}, \frac{s^2}{s^2+2}\right), \\ D_2(s) = & \left(-\frac{s^2}{\sqrt{s^2+1}(s^2+2)}, -\frac{s^3}{\sqrt{s^2+1}(s^2+2),}, \frac{2\sqrt{s^2+1}}{s^2+2}\right), \\ D_1(s) = & \left(\frac{s}{\sqrt{s^2+1}}, -\frac{1}{\sqrt{s^2+1}}, 0\right) \end{cases}$$

and its curvatures are $d_1 = \dfrac{s}{\sqrt{s^2+1}}, d_2 = -\dfrac{1}{\sqrt{s^2+1}}$ *and* $d_3 = \dfrac{s^2}{2s^2+2}$. *The Frenet vectors of the curve* $\lambda(s)$ *are*

$$\begin{cases} T(s) = & \left(\frac{2}{s^2+2}, \frac{2s}{s^2+2}, \frac{s^2}{s^2+2}\right), \\ N(s) = & \left(-\frac{2s}{s^2+2}, -\frac{s^2-2}{s^2+2}, \frac{2s}{s^2+2}\right), \\ B(s) = & \left(\frac{s^2}{s^2+2}, -\frac{2s}{s^2+2}, \frac{2}{s^2+2}\right) \end{cases}$$

and its curvatures are $\kappa = \frac{4}{(s^2+2)^2}$ *and* $\tau = \frac{4}{(s^2+2)^2}$. *From this line of the paper, in order to form surfaces, we consider manipulating marching scale functions such that they satisfy the geodesicity, asymptoticity and line of curvature conditions defined in equations 3.3, 3.10 and 3.14, respectively.*

Case 1.
a) *By choosing* $x(s,v) = v$, $y(s,v) = svsin(arctan(\frac{1}{s}))$, $z(s,v) = svcos(arctan(\frac{1}{s}))$ *and* $\beta_1(s) = s$, $v_0 = 0$, *one of the surfaces with a common polynomial isogeodesic curve is given as fallows:*

$$\varphi^1_{ig}(s,v) = \left(\frac{s^3v + s^3 + 2s + 2v}{s^2+2}, \frac{\left(s^3 - 4sv + 2s + 4v\right)s}{2(s^2+2)}, \frac{s\left(s^4 + 2s^2 + 6sv + 12v\right)}{6(s^2+2)}\right)$$

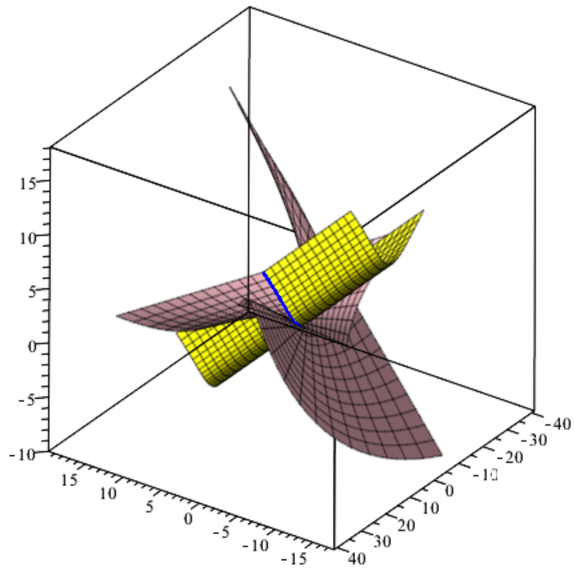

Figure 1: The surface φ_{ig}^{1} with isogeodesic curve

b) *On the other hand, by taking* $x(s,v)=v$, $y(s,v)=svcos(arctan(\frac{1}{s}))$, $z(s,v)=-svsin(arctan(\frac{1}{s}))$ *and* $\beta_2(s)=s$, $v_0=0$, *we get the following parametrization for the surface with a common polynomial isoasymptotic curve as:*

$$\varphi_{ia}^{1}(s,v)=\Big(\frac{s^3-2vs^2+2s+2v}{s^2+2},\frac{\left(s^3-2vs^2+2s+8v\right)s}{2(s^2+2)},\frac{s^2\left(s^3+2s+18v\right)}{6(s^2+2)}\Big)$$

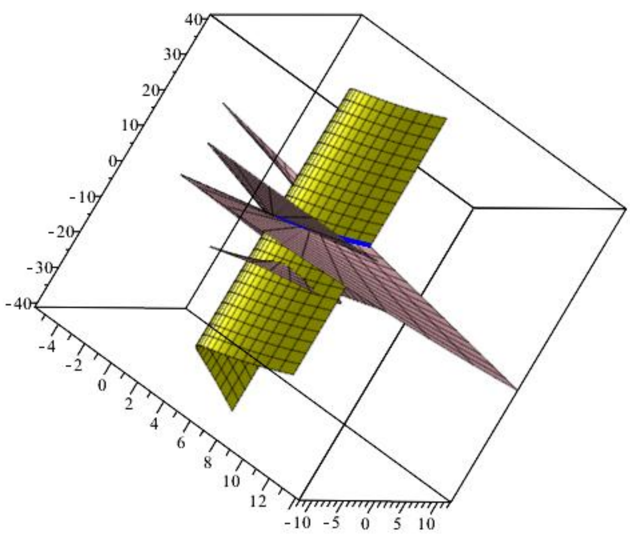

Figure 2: The surface φ_{ia}^{1} with isoasymptotic curve

c) *Now if marching scales are chosen to be that* $x(s,v)=v$, $y(s,v)=\frac{2sv}{s^2+2}sin(-\frac{4s}{(s^2+2)^2}-\arctan\left(\frac{1}{s}\right))$, $z(s,v)=-\frac{2sv}{s^2+2}cos(-\frac{4s}{(s^2+2)^2}-\arctan\left(\frac{1}{s}\right))$ *with* $\theta(s)=\arctan\left(\frac{1}{s}\right)$, $\eta(s)=\frac{s^2+2}{2}$, $\beta_3(s)=s$, $v_0=0$ *and* $\gamma(s)=-\frac{2s}{s^2+2}-\arctan\left(\frac{1}{s}\right)$, *we obtain the following parametrization for the surface on which* λ *lies as a line of curvature:*

$$\varphi^1_{lc}(s,v)=\Big(s+\frac{2v}{s^2+2}+\frac{2s^3v\sin\left(\frac{2s}{s^2+2}+\arctan\left(\frac{1}{s}\right)\right)}{\left(s^2+2\right)^2\sqrt{s^2+1}}-\frac{2s^2v\cos\left(\frac{2s}{s^2+2}+\arctan\left(\frac{1}{s}\right)\right)}{\left(s^2+2\right)^2\sqrt{s^2+1}},$$

$$\frac{s^2}{2}+\frac{2sv}{s^2+2}+\frac{2s^4v\sin\left(\frac{2s}{s^2+2}+\arctan\left(\frac{1}{s}\right)\right)}{\left(s^2+2\right)^2\sqrt{s^2+1}}+\frac{2sv\cos\left(\frac{2s}{s^2+2}+\arctan\left(\frac{1}{s}\right)\right)}{\left(s^2+2\right)^2\sqrt{s^2+1}},$$

$$\frac{s^3}{6}+\frac{s^2v}{s^2+2}\frac{-4sv\sin\left(\frac{2s}{s^2+2}+\arctan\left(\frac{1}{s}\right)\right)\sqrt{s^2+1}}{\left(s^2+2\right)^2}\Big)$$

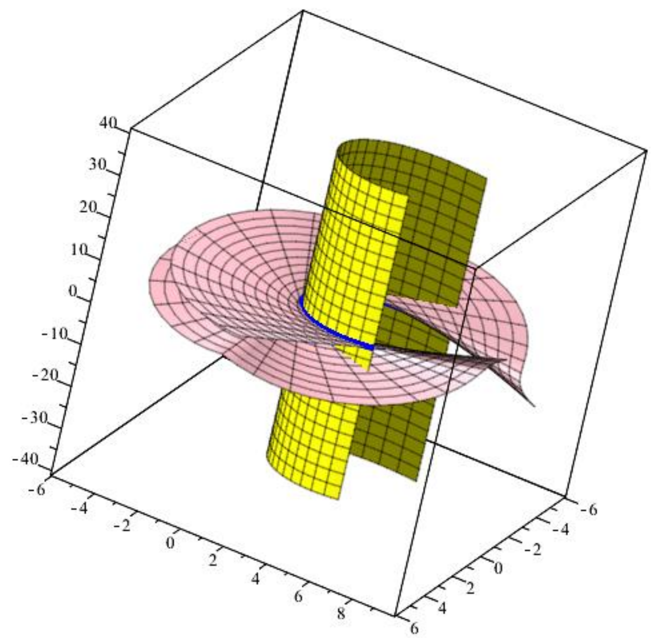

Figure 3: The surface φ^1_{lc} with line of curvature

Case 2.

a) *Another series of functions* $x(s,v)=0, y(s,v)=vsin(s)sin(arctan(\frac{1}{s}))$, $z(s,v)=vsin(s)cos(arctan(\frac{1}{s}))$ *and* $\beta_1(s)=sin(s)$, $v_0=0$ *satisfying the geodesicity condition given by (3.3), produce the following parametrization for the surface as*

$$\varphi^2_{ig}(s,v)=\Big(\frac{s^2vsin(s)+s^3+2s}{s^2+2},\frac{\left(s^3-4vsin(s)+2s\right)s}{2(s^2+2)},\frac{s^5+2s^3+12vsin(s)}{6(s^2+2)}\Big)$$

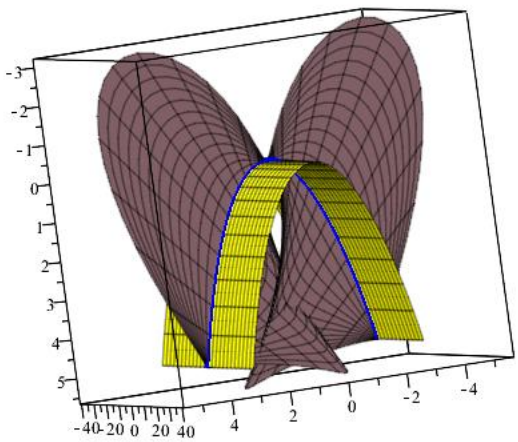

Figure 4: The surface φ_{ig}^2 with isogeodesic curve

b) *Similarly, the asymptoticity condition given in (3.10) is valid for the functions* $x(s,v) = 0$, $y(s,v) = vsin(s)cos(arctan(\frac{1}{s}))$, $z(s,v) = -vsin(s)sin(arctan(\frac{1}{s}))$ *and* $\beta_2(s) = sin(s)$, $v_0 = 0$. *This results the following parametric equation:*

$$\varphi_{ia}{}^2(s,v) = \Big(\frac{s^3 - 2svsin(s) + 2s}{s^2+2}, \frac{s^4 + 2s^2 + 4vsin(s) - 2s^2vsin(s)}{2(s^2+2)}, \frac{s^5 + 2s^3 + 12svsin(s)}{6(s^2+2)}\Big)$$

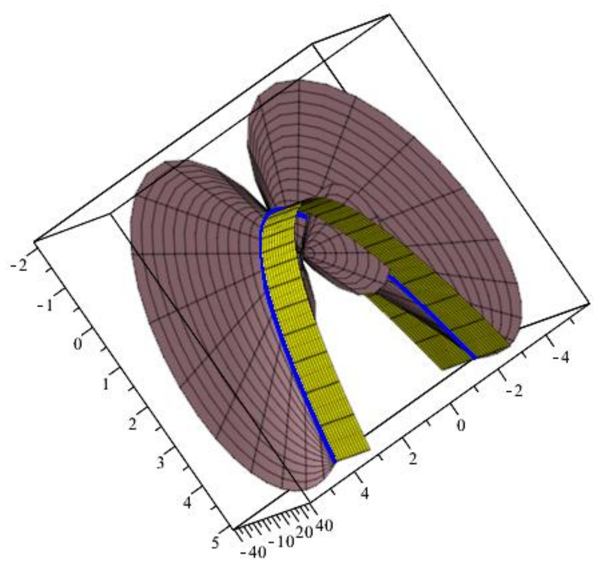

Figure 5: The surface φ_{ia}^2 with isoasymptotic curve

c) *Finally, we form another surface with* λ *as a line of curvature, by choosing* $x(s,v) = 0$, $y(s,v) = \frac{2v}{s^2+2} sin(s) sin(-\frac{4s}{(s^2+2)^2} - \arctan\left(\frac{1}{s}\right))$, $z(s,v) = -\frac{2v}{s^2+2} sin(s) cos(-\frac{4s}{(s^2+2)^2} - \arctan\left(\frac{1}{s}\right))$ *with* $\theta(s) = \arctan\left(\frac{1}{s}\right)$, $\eta(s) = \frac{s^2+2}{2}, \beta_3(s) = sin(s)$, $v_0 = 0$ *and* $\gamma(s) = -\frac{2s}{s^2+2} - \arctan\left(\frac{1}{s}\right)$,

$$\varphi^2_{lc}(s,v) = \Big(s + \frac{2s^2 v sin(s) \sin\left(\frac{2s}{s^2+2} + \arctan\left(\frac{1}{s}\right)\right)}{(s^2+2)^2\sqrt{s^2+1}} - \frac{2sv sin(s)\cos\left(\frac{2s}{s^2+2} + \arctan\left(\frac{1}{s}\right)\right)}{(s^2+2)^2\sqrt{s^2+1}},$$

$$\frac{s^2}{2} + \frac{2s^3 v sin(s)\sin\left(\frac{2s}{s^2+2} + \arctan\left(\frac{1}{s}\right)\right)}{(s^2+2)^2\sqrt{s^2+1}} + \frac{2v sin(s)\cos\left(\frac{2s}{s^2+2} + \arctan\left(\frac{1}{s}\right)\right)}{(s^2+2)^2\sqrt{s^2+1}},$$

$$\frac{s^3}{6} - \frac{4v sin(s)\sin\left(\frac{2s}{s^2+2} + \arctan\left(\frac{1}{s}\right)\right)\sqrt{s^2+1}}{(s^2+2)^2}\Big)$$

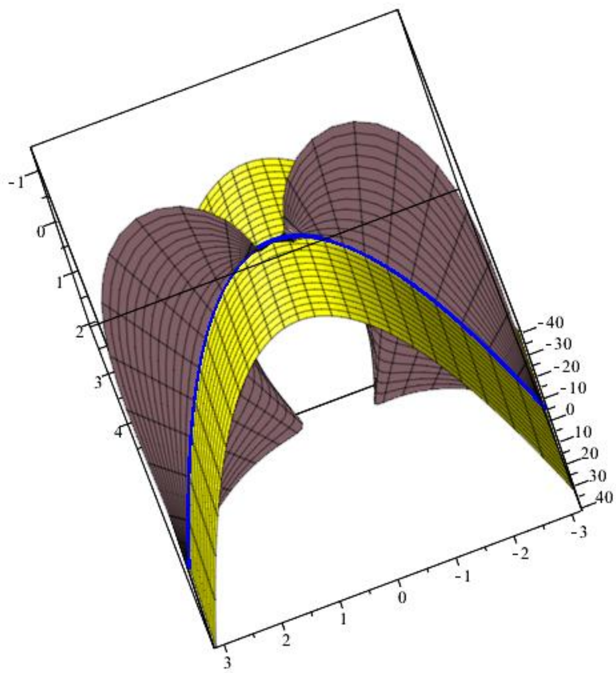

Figure 6: The surface φ^2_{lc} with line of curvature

Case 3.
a) *For a ruled surface with a common polynomial isogeodesic curve, we choose the functions as* $f(s) = s$ *and* $\beta_1(s) = s$. *The corresponding parametrization for this surface is*

$$\varphi^1_{igr}(s,v) = \big(sv + s, \frac{s^2}{2}, \frac{s^3 + 6sv}{6}\big)$$

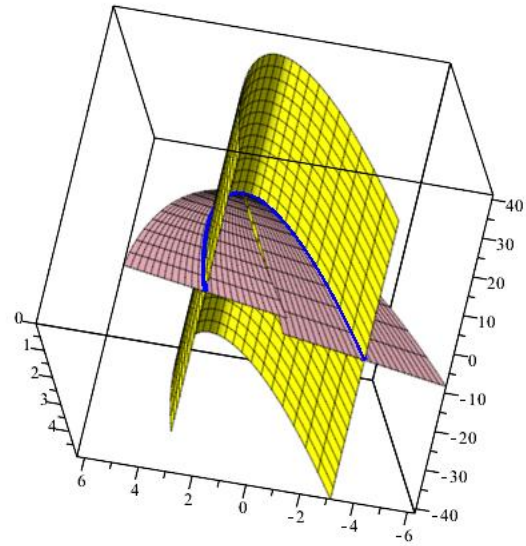

Figure 7: The ruled surface φ^1_{igr} with isogeodesic curve

b) *By taking $h(s) = s$ and $\beta_2(s) = s$, we form the following equation for the ruled surface with a polynomial isoasymptotic curve as:*

$$\varphi^1_{iar}(s,v) = \left(\frac{s^3 - 2s^2v + 2sv + 2s}{s^2+2}, \frac{s^4 - 2s^3v + 4s^2v + 2s^2 + 4sv}{2s^2+4}, \frac{s^5 + 6s^3v + 2s^3 + 12s^2v}{6s^2+12}\right)$$

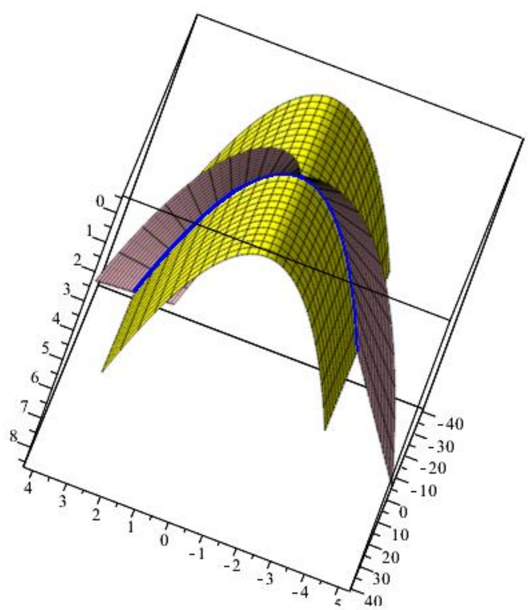

Figure 8: The ruled surface φ^1_{iar} with isoasymptotic curve

Case 4.

a) *To illustrate more, a final example for an isogeodesic polynomial curve included ruled surface could be given as*

$$\varphi^2_{igr}(s,v) = \left(\frac{s^4v + s^3 + 2s}{s^2+2}, \frac{s^4 - 4s^3v + 2s^2}{2s^2+4}, \frac{s^5 + 2s^3 + 12s^2v}{6s^2+12}\right)$$

by taking $f(s) = 0$ and $\beta_1(s) = s^2$.

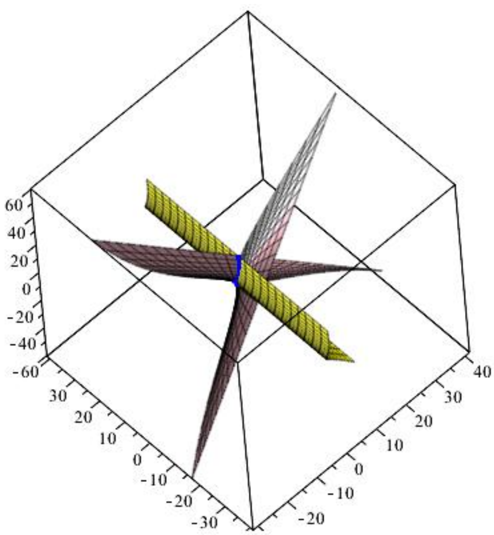

Figure 9: The ruled surface φ^2_{igr} with isogeodesic curve

b) *For an isoasymptotic polynomial curve included ruled surface, we have the parametrization:*

$$\varphi^2_{iar}(s,v) = \left(\frac{s^3 + 2s - 2s^3v}{s^2+2}, \frac{s^4 + 4s^2v + 2s^2 - 2s^4v}{2s^2+4}, \frac{s^5 + 12s^3v + 2s^3}{6s^2+12}\right)$$

with the functions, $h(s) = 0$ and $\beta_2(s) = s^2$.

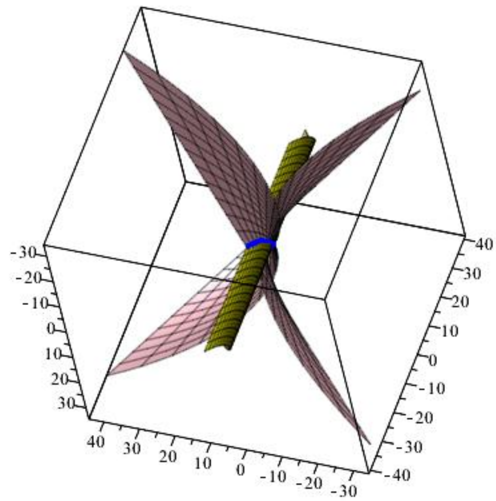

Figure 10: The ruled surface φ^2_{iar} with isoasymptotic curve

4. Conclusion

Polynomial curves are a versatile and powerful tool in data analysis and modeling processes. By using polynomials of different degrees, you can conveniently approximate data in a variety of ways. This is useful for making predictions, recognizing patterns, and making decisions. In this study, polynomial curves were defined according to the Flc framework and surface families that accept this curve as a geodesic, asymptotic and curvature line were obtained. Afterwards, this study was adapted to ruled surfaces, which are a special type of surface, and ruled surfaces that accept this polynomial curve as geodesic and asymptotic were designed. At the end of the subject, various surface examples were obtained by selecting the deviation functions of the surface, which accepts the polynomial curve as a geodesic, asymptotic and curvature line, to meet the conditions. When these surface examples are examined, it is important to create surfaces that have the same polynomial curve by adhering only to the selection of deviation functions. In future studies, different results can be obtained by using different surfaces (such as timelike or lightlike surfaces). It is also possible to expand this study by generalizing it to dual spaces and high-dimensional spaces.

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