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# Some Characterizations on Gradient Almost $\eta$-Ricci-Bourguignon Solitons

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ABSTRACT: We characterize a Riemannian manifold with gradient almost $\eta$-Ricci-Bourguignon solitons structures. We show that a gradient almost $\eta$-Ricci-Bourguignon soliton is gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton with the potential function $k$. Moreover, we investigate that a gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton is isometric to a standard unit sphere $\mathbb{S}^n$, hyperbolic space $\mathbb{H}^n$ and Euclidean space $\mathbb{R}^n$ with constant scalar curvature or its associated vector field is conformal. Finally, we deduce some properties of integral formulas for the gradient compact case.

Key Words: Ricci-Bourguignon soliton, conformal vector field, traceless Ricci soliton, space form.

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## 1. Introduction

In differential geometry, the idea of Ricci-Bourguignon flow introduced by Jean Pierre Bourguignon [9] is defined as an extension of Ricci flow [18]. Richard S. Hamilton [17] defined the Ricci solitons as a self-similar solutions of Ricci flow. For generalizing and particularizing gradient Ricci-Bourguignon solitons many examples were given. Then Catino-Mazzieri [11] investigated an important results for Einstein manifold, the idea is to give the well-known results for rotationally symmetric and asymptotically cylindrical in either compact or non compact condition. Therefore, Thomas Ivey [20] investigated a complete classification for Ricci solitons on compact three-manifolds, while in a gradient shrinking solitons Ni and Wallach [22] studied in complete manifold which is an alternate proof to a result of Perelman [23]. Further, it is proved by Perelman, a compact Ricci soliton is always gradient. In fact, in the non compact case [21] there does not exist gradient Ricci solitons. Moreover for a compact gradient almost Ricci-Bourguignon soliton, some integral formulas were derived in [14]. As a consequence of integral formula, the author investigated that there exists an isometry between Euclidean sphere and compact gradient soliton when the scalar curvature is constant or the related potential vector field of the manifold is conformal.

Besides, several authors studied the almost $\eta$-Ricci and $\eta$-Ricci-Bourguigon solitons ([3], [12], [25], [26]). Moreover, generalization results of Ricci solitons were given in [14]. In this study, inspiring the work of Ricci almost solitons, he initiated the concept of almost Ricci-Bourguignon solitons. He gave some important results which were qualified as the generalizing results for Ricci almost solitons. The almost $\eta$-Ricci-Bourguignon solitons provided some special potential vector fields on a doubly and sequential warped product were studied in ([4], [29]). The almost $\eta$-Ricci-Bourguignon solitons on compact and non compact case with some special vector field were investigated by Traore et al. ([28]). In addition, Barros and Riberio ([6], [7]) presented an integral formula for the compact almost Ricci solitons and generalized $m$-quasi Einstein metrics. Further, Gomes et al. [15] studied the $h$-almost Ricci soliton, where they showed that there exists an isomety between a standard sphere and compact nontrivial case.

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Throughout this manuscript we study some properties for gradient almost $\eta$-Ricci-Bourguignon solitons in Riemannian manifolds. Therefore, some characterizations and definitions of an almost and gradient $\eta$-Ricci-Bourguignon solitons are given in section 2. Moreover in section 3, we introduce the standard unit sphere, hyperbolic space and Euclidean space which allow us to deduce that a gradient almost $\eta$-Ricci Bourguignon soliton is gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton. Further, in the last section we also investigate that the gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton in case its related potential vector field is nontrivial conformal will be isometric to $\mathbb{S}^n(1)$.

## 2. Preliminaries

In this part, we recall the fundamental definitions and concepts for the further study. Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold, then we defined on $M^n$ the *Ricci-Bourguignon solitons* as a self-similar solutions to *Ricci-Bourguignon flow* [10] defined by:

$$\frac{\partial}{\partial t} g(t) = -2(\mathrm{Ric} - \rho r g), \tag{2.1}$$

where $r$ is the scalar curvature of the Riemannian metric $g$, Ric is the Ricci curvature tensor of the metric, and $\rho$ is a real constant. When $\rho = 0$ in (2.1), then we get a Ricci flow. Remark that for some values of $\rho$ in equation (2.1), $\mathrm{Ric} - \rho r g$ evolve into the following situation:

(1) If $\rho = \frac{1}{2}$, then it is an Einstein tensor $\mathrm{Ric} - \dfrac{r}{2} g$,

(2) If $\rho = \frac{1}{n}$, then it is a traceless Ricci tensor $\mathrm{Ric} - \dfrac{r}{n} g$.

**Definition 2.1.** *Let $(M^n, g)$ be a Riemannian manifold of dimension $n \geq 3$. Then it is called Ricci-Bourguignon soliton [2] if*

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi g = (\lambda + \rho r) g, \tag{2.2}$$

*where $\pounds_\xi$ denotes the Lie derivative operator along the vector field $\xi$ which is called soliton or potential, $\rho$ and $\lambda$ are real constants and the soliton structure is denoted by $(M, g, \xi, \lambda)$.*

If $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, then the Ricci-Bourguignon soliton is called expanding, steady and shrinking respectively.

Considering the 1-form $\eta$, $(M^n, g)$ is called *$\eta$-Ricci-Bourguignon soliton* [25] if the following equation holds

$$\mathrm{Ric} + \frac{1}{2}\pounds_\xi g = (\lambda + \rho r) g + \omega \eta \otimes \eta, \tag{2.3}$$

for a vector field $\xi$, where $\lambda, \omega$ are real constants. Particularly, taking $\rho = 0$ in equation (2.3), we get the *$\eta$-Ricci soliton* [3].

In (2.3), if $\xi$ is the gradient of a function $k$ on $M$, then we have a *gradient $\eta$-Ricci-Bourguignon soliton*. Then, equation (2.3) can be rewritten as

$$\mathrm{Ric} + \nabla^2 k = (\lambda + \rho r) g + \omega dk \otimes dk, \tag{2.4}$$

where $\nabla^2 k$ is the Hessian of $k$. If $\lambda$ is smooth function in equation (2.4), then we get the *gradient almost $\eta$-Ricci-Bourguignon soliton* and it is denoted by $(M^n, g, \nabla k, \lambda, \omega)$.

Moreover, contracting the equation (2.4), we get

$$(1 - n\rho) r + \Delta k = n\lambda + \omega |\nabla k|^2. \tag{2.5}$$

Hence, we derive that

$$(1 - n\rho) g(\nabla k, \nabla r) + g(\nabla k, \nabla \Delta k) = n g(\nabla k, \nabla \lambda) + \omega g(\nabla |\nabla k|^2, \nabla k). \tag{2.6}$$

If $k$ is constant, then the gradient almost $\eta$-Ricci-Bourguignon soliton is called trivial. On the other hand, for a nontrivial gradient almost $\eta$-Ricci-Bourguignon soliton $k$ is not trivial. Hence a gradient almost $\eta$-Ricci-Bourguignon soliton is an Einstein manifold when $n \geq 3$ and $k$ is constant.

**Definition 2.2.** *[16] Let $(M^n, g)$ be a Riemannian manifold. Then it is called $h$-almost Ricci-Bourguignon soliton if there exist a vector field $\xi$, the smooth functions $\lambda$ and $h$ such that*

$$\mathrm{Ric} + \frac{h}{2}\pounds_\xi g = (\lambda + \rho r)g. \tag{2.7}$$

*Hence it is denoted by $(M^n, g, \xi, h, \lambda)$.*

Therefore, if $\xi = \nabla k$, we get a gradient $h$-almost Ricci-Bourguignon soliton. Hence, equation (2.7) is rewritten as

$$\mathrm{Ric} + h\nabla^2 k = (\lambda + \rho r)g, \tag{2.8}$$

here $\nabla^2 k$ is the Hessian of $k$. Thus for $\rho = \frac{1}{2}$, we have a gradient $h$-almost Einstein soliton and for $\rho = \frac{1}{n}$, we get a gradient $h$-almost traceless Ricci soliton.

## 3. Main Results

Firstly we give a characterization for a gradient almost $\eta$-Ricci-Bourguignon soliton when $\omega$ is positive in equation (2.4) in order to obtain a gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton.

Considering the non constant function $u = e^{-\omega k}$ on $M^n$, where $\omega$ is a positive function, we obtain that

$$\nabla u = -\omega e^{-\omega k}\nabla k,$$

and

$$-\frac{\nabla^2 u}{\omega u} = \nabla^2 k - \omega dk \otimes dk. \tag{3.1}$$

Therefore the equation (2.4) is rewritten as

$$\mathrm{Ric} - \frac{\nabla^2 u}{\omega u} = (\lambda + \rho r)g. \tag{3.2}$$

Then from the previous equation we can state the next proposition.

**Proposition 3.1.** *If $u = e^{-\omega k}$ is a function defined on a gradient almost $\eta$-Ricci-Bourguignon soliton, then the soliton becomes gradient almost $(-\frac{1}{\omega u})$-Ricci-Bourguignon soliton.*

If $\rho = \frac{1}{n}$ in (3.2), then we obtain a gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton defined by

$$\mathrm{Ric} - \frac{\nabla^2 u}{\omega u} = (\lambda + \frac{r}{n})g. \tag{3.3}$$

If we take the trace of equation (3.2), we obtain

$$\frac{\Delta u}{\omega u} = \Big((1 - n\rho)r - n\lambda\Big). \tag{3.4}$$

Let $(M^n, g)$ be a Riemannian manifold. If $\nabla u$ is written as $\frac{1}{2}\pounds_{\nabla u} g = \varphi g$, where $\varphi$ is a smooth function on $\mathbb{R}$, then it is called *conformal vector field*. If $\varphi \neq 0$, then the conformal vector field is nontrivial. If we suppose that $\nabla u$ is nontrivial, then we shall write $\frac{1}{2}\pounds_{\nabla u} g = \nabla^2 u = \frac{\Delta u}{n} g$. Putting equation (3.4) in equation (3.2), we obtain

$$\mathrm{Ric} = \frac{r}{n} g, \tag{3.5}$$

where $r$ is a constant, then $M$ is Einstein manifold. Thus, we deduce:

**Remark 3.1.** The manifold $M^n$ is Einstein if and only if its associated vector field $\nabla u$ is conformal.

Before presenting our main results, we defined the standard unit sphere $(\mathbb{S}^n(1), g_0)$, $n \geq 2$ as follows:

$$\mathbb{S}^n(1) = \{v \in \mathbb{R}_0^{n+1}; g_0(v,v) = 1\},$$

and the standard hyperbolic space $(\mathbb{H}^n(-1), g_0)$ as

$$\mathbb{H}^n(-1) = \{v \in \mathbb{R}_1^{n+1}; g_0(v,v) = -1; v_1 > 0\},$$

which are the hypersurfaces in $\mathbb{R}^{n+1}$.

The height function $h_v$ on $\mathbb{R}^n$ is defined by $h_v(x) = g_0(x, v)$, where $v$ is fixed unit vector $v \in \mathbb{R}^{n+1}$.

We point out that the following theorems are motivated by Proposition 3.1 and the corresponding Example 1, 2 and 3 [6] of Barros-Ribeiro.

**Theorem 3.2.** *Let $(\mathbb{S}^n(1), g_0)$, $n \geq 2$ be a standard Euclidean sphere admitting a gradient almost $\eta$-Ricci-Bourguignon soliton of the form $(\mathbb{S}^n(1), g_0, \nabla u, \lambda, \omega)$ with a potential function $k = -\frac{1}{\omega}\ln(\phi - \frac{h_v}{n})$, $(\omega > 0)$, where $u = e^{-\omega k} = \phi - \frac{h_v}{n}$ and $\phi$ is a real parameter lying in $(\frac{1}{n}, +\infty)$. Then $(\mathbb{S}^n(1), g_0, \nabla u, \lambda, \omega)$ is a shrinking gradient almost $(-\frac{1}{\omega u})$-traceless Ricci soliton.*

*Proof.* Taking the differential of $k$ and using $\nabla^2 h_v = -h_v g_0$, we obtain $dk = \frac{dh_v}{\omega(n\phi - h_v)}$ and

$$\begin{aligned}
\nabla^2 k = \nabla dk =& \frac{\nabla dh_v}{\omega(n\phi - h_v)} + d\left(\frac{1}{\omega(n\phi - h_v)^2}\right) \otimes dh_v \\
=& \frac{\nabla^2 h_v}{\omega(n\phi - h_v)} + \frac{dh_v \otimes dh_v}{\omega(n\phi - h_v)^2} \\
=& -\frac{h_v}{\omega(n\phi - h_v)} g_0 + \frac{dh_v \otimes dh_v}{\omega(n\phi - h_v)^2}.
\end{aligned}$$

Thus we can write

$$-\frac{\nabla^2 u}{\omega u} = \nabla^2 k - \omega dk \otimes dk = -\frac{h_v}{\omega(n\phi - h_v)} g_0,$$

from equation (3.1). Since $\mathrm{Ric} = (n-1)g_0$, then equation (3.2) becomes

$$\mathrm{Ric} - \frac{\nabla^2 u}{\omega u} = \left((n-1) - \frac{h_v}{\omega(n\phi - h_v)}\right) g_0 = (\lambda + \rho r) g_0, \tag{3.6}$$

where $\lambda = (n-1) - \frac{1}{\omega}(\frac{\phi - u}{u}) - \rho r$. Therefore if $\rho = \frac{1}{n}$ and $r = n(n-1)$, then $\lambda = -\frac{1}{\omega}(\frac{\phi - u}{u}) > 0$, which completes the proof. □

**Theorem 3.3.** *Let $(\mathbb{H}^n(-1), g_0)$, $n \geq 2$ be a standard Hyperbolic space admitting a gradient almost $\eta$-Ricci-Bourguignon soliton of the form $(\mathbb{H}^n(-1), g_0, \nabla u, \lambda, \omega)$ with a potential function $k = -\frac{1}{\omega}\ln(\phi + \frac{h_v}{n})$, $(\omega > 0)$, where $u = e^{-\omega k} = \phi + \frac{h_v}{n}$ and $\phi$ is a real parameter lying in $(\frac{1}{n}, +\infty)$. Then $(\mathbb{H}^n(-1), g_0, \nabla u, \lambda, \omega)$ is an expanding gradient almost $(-\frac{1}{\omega u})$-traceless Ricci soliton.*

*Proof.* We prove this theorem like Theorem 3.2. It is sufficient to take $\nabla^2 h_v = h_v g_0$ and using the fact that $\mathrm{Ric} = -(n-1)g_0$, then equation (3.2) becomes

$$\mathrm{Ric} - \frac{\nabla^2 u}{\omega u} = \left(-(n-1) - \frac{h_v}{\omega(n\phi - h_v)}\right) g_0 = (\lambda + \rho r) g_0, \tag{3.7}$$

where $\lambda = -(n-1) - \frac{1}{\omega}(\frac{u - \phi}{u}) - \rho r$. Therefore if $\rho = \frac{1}{n}$ and $r = -n(n-1)$, then $\lambda = -\frac{1}{\omega}(\frac{u - \phi}{u}) < 0$, which completes the proof. □

For the similar one of the previous theorems, we consider the Euclidean space $(\mathbb{R}^n, g_0)$, where $g_0$ is a canonical metric on $\mathbb{R}^n$.

**Theorem 3.4.** *Let $(\mathbb{R}^n, g_0)$, $n \geq 2$ be an Euclidean space admitting a gradient almost $\eta$-Ricci-Bourguignon soliton of the form $(\mathbb{R}^n, g_0, \nabla u, \lambda, \omega)$ with a potential function $k = -\dfrac{1}{\omega}\ln(\phi + |x|^2)$, $(\omega > 0)$, where $u = e^{-\omega k} = \phi + |x|^2$ and $\phi$ is a positive real parameter and $|x|$ is the Euclidean norm of $x$. Then $(\mathbb{R}^n, g_0, \nabla u, \lambda, \omega)$ is an expanding gradient almost $(-\frac{1}{\omega u})$-traceless Ricci soliton.*

*Proof.* It is enough to recall that $\nabla^2|x|^2 = 2g_0$ and by a straightforward computation, we obtain

$$\mathrm{Ric} - \frac{\nabla^2 u}{\omega u} = -\frac{2}{\omega u} g_0 = (\lambda + \rho r) g_0,$$

where $\lambda = -\dfrac{2}{\omega u} - \rho r$. As the Ricci tensor of a Euclidean space is flat, its scalar curvature $r = 0$ and $\lambda = -\dfrac{2}{\omega u} < 0$. Thus, we get the desired result. $\square$

In the rest of this manuscript, the gradient almost $\eta$-Ricci-Bourguignon soliton with the potential function $k$ given in the previous theorems will be called gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton.

We give the following lemma which was proved in [24].

**Lemma 3.5.** *Let $(M^n, g)$ be a Riemannian manifold. Then the followings identities hold:*

$$2\,\mathrm{div}(\nabla^2 k)(\nabla k) = \frac{1}{2}\Delta|\nabla k|^2 - |\nabla^2 k|^2 + \mathrm{Ric}(\nabla k, \nabla k) + g(\nabla k, \nabla \Delta k), \tag{3.8}$$

*or in $(1,1)$-tensor notation*

$$\mathrm{div}\,\nabla^2 k = \mathrm{Ric}(\nabla k) + \nabla \Delta k. \tag{3.9}$$

Considering $u = e^{-\omega k}$, we can give the following lemma:

**Lemma 3.6.** *Let $(M^n, g, \nabla u, \lambda, \omega)$ be a nontrivial gradient almost $\eta$-Ricci-Bourguignon soliton with $n \geq 3$. If $(M^n, g)$ is a trivial manifold, then the function $u$ satisfies*

$$\nabla^2 u = (-ku + b)g, \tag{3.10}$$

*where $k = -\dfrac{r}{n(n-1)}$ and $b = c\omega$ are constant coefficients.*

*Proof.* We know that the manifold $(M^n, g)$ is trivial and $n \geq 3$. Then for a constant $r$, we find $\mathrm{Ric} = \dfrac{r}{n} g$. From equation (3.2), we have

$$\nabla^2 u = \omega\left(\frac{r}{n}u - \rho r u - \lambda u\right) g. \tag{3.11}$$

Hence from equation (3.9), we deduce that

$$\mathrm{Ric}(\nabla u) + \nabla \Delta u = \omega\left(\frac{r}{n}\nabla u - \rho r \nabla u - \nabla(\lambda u)\right). \tag{3.12}$$

Therefore we obtain

$$\frac{r}{n}\nabla u + \nabla \Delta u = \omega\left(\frac{r}{n}\nabla u - \rho r \nabla u - \nabla(\lambda u)\right). \tag{3.13}$$

Thus with the help of (3.2), we find

$$\Delta u = \omega\Big(r\nabla u - n\rho r \nabla u - n\nabla(\lambda u)\Big). \tag{3.14}$$

Hence putting (3.14) into (3.13), we obtain

$$\nabla(\lambda u) = r\frac{(1+\omega n-\omega)}{\omega n(n-1)}\nabla u - \rho r\nabla u. \tag{3.15}$$

Thus $\lambda u = r\dfrac{(1+\omega n-\omega)}{\omega n(n-1)}u - \rho r u - c$, where $c$ is constant. Putting the value of $\lambda u$ in equation (3.11), we obtain the desired result. □

It is known when the Ricci soliton is shrinking, compact and dimension $n \geq 2$, it is isometric to a standard sphere [13]. Besides, the authors [19] proved that a complete Riemannian manifold is conformally equivalent to the Euclidean sphere $\mathbb{S}^n$ with a nontrivial function $k$ such that $\nabla^2 k = \dfrac{\Delta k}{n} g$. From these results and Theorem 1 in [6], we give the next theorem:

**Theorem 3.7.** *Let $(M^n, g, \nabla u, (-\frac{1}{\omega u}), \lambda, \omega)$ be a gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton with $n \geq 3$. If $\nabla u$ is nontrivial conformal, then we have the following cases:*

1. *The manifold $M^n$ is isometric to $\mathbb{S}^n$ if it is shrinking with a positive constant scalar curvature. Moreover, $k$ is, up to constant, given in Theorem (3.2).*
2. *The manifold $M^n$ is isometric to $\mathbb{H}^n$, provided $u$ has only one critical point if it is expanding with a negative constant scalar curvature. Moreover, $k$ is, up to constant given in Theorem (3.3).*
3. *The manifold $M^n$ is isometric to $\mathbb{R}^n$ if it is expanding with a zero scalar curvature. Moreover, $k$ is, up to change of coordinates, given in Theorem (3.4).*

*Proof.* Since $\nabla u$ is conformal, then there exists a potential function $\varphi$ , such that $\dfrac{1}{2}\pounds_{\nabla u} g = \nabla^2 u = \dfrac{\Delta u}{n} g$. Thus $M^n$ is Einstein manifold. Therefore we infer

$$\mathrm{Ric} = \left((\lambda+\rho r) + \frac{\Delta u}{nu\omega}\right)g,$$

from equation (3.2). For $n \geq 3$, using Schur's Lemma, we have $r = n(\lambda + \rho r) + \dfrac{\Delta u}{u\omega}$ which is constant. Therefore we can use Lemma 3.6, Theorem 2 of Tashiro [27] and jointly with the result due to Theorem 1 of [6] to prove that $M^n$ is isometric to $\mathbb{S}^n$, to $\mathbb{H}^n$ and $\mathbb{R}^n$, when the scalar curvature $r > 0$, $r < 0$ and $r = 0$ respectively. □

We give the following lemma which is a generalization for Ricci soliton obtained in [24] .

**Lemma 3.8.** *If $(M^n, g, \nabla k, \lambda, \omega)$ is a gradient almost $\eta$-Ricci-Bourguignon soliton. Then the following equations holds:*

$$\frac{1}{2}\Delta|\nabla k|^2 = |\nabla^2 k|^2 - \mathrm{Ric}(\nabla k, \nabla k) + 2\omega|\nabla k|^2\Delta k - (n-2)\Big(g(\nabla\lambda, \nabla k) + \rho g(\nabla r, \nabla k)\Big). \tag{3.16}$$

*Proof.* From (2.5), we have $\mathrm{Ric} + \nabla^2 k = (\lambda + \rho r)g + \omega\eta\otimes\eta$, where $\eta$ is the $g$-dual 1-form of $\xi = \nabla k$. Therefore using the fact that

$$\mathrm{div}(\eta\otimes\eta) = \Delta k\nabla k + \nabla_{\nabla k}\nabla k$$

and the second contracted Bianchi identity

$$dr = 2\,\mathrm{div}\,\mathrm{Ric}$$

and applying the divergence to the equation (2.4), we obtain

$$dr + 2\,\mathrm{div}(\nabla^2 k) = 2d\lambda + 2\rho dr + 2\omega\Delta k\nabla k + 2\omega\nabla_{\nabla k}\nabla k. \tag{3.17}$$

Then, we get that

$$2\operatorname{div}(\nabla^2 k)(\nabla k) = -g(\nabla r, \nabla k) + 2\omega\Delta k|\nabla k|^2 + 2\omega g(\nabla_{\nabla k}\nabla k, \nabla k) + 2g(\nabla\lambda, \nabla k) + 2\rho g(\nabla r, \nabla k). \tag{3.18}$$

Next using the equations (2.6), (3.8) and (3.18), we obtain that

$$\frac{1}{2}\Delta|\nabla k|^2 = |\nabla^2 k|^2 - \operatorname{Ric}(\nabla k, \nabla k) + 2\omega|\nabla k|^2\Delta k - (n-2)g(\nabla\lambda, \nabla k) - \rho(n-2)g(\nabla r, \nabla k), \tag{3.19}$$

which allow us to get the desired result. □

**Lemma 3.9.** *Let $\nabla k$ be a gradient conformal vector field on a Riemannian manifold $(M^n, g)$. If $M^n$ is compact then*

$$\int_M |\nabla k|^2\Delta k \, d\mu = 0, \tag{3.20}$$

*where $d\mu$ is the canonical measure.*

*Proof.* Since $\nabla k$ is gradient on $M$, then there exists a smooth function $\varphi$ on $M$, such that

$$\pounds_{\nabla k} g = 2\varphi g, \tag{3.21}$$

where $\varphi = \frac{1}{n}\operatorname{div}\nabla k$. From which we obtain

$$|\nabla k|^2 \operatorname{div}\nabla k = ng(\nabla_{\nabla k}\nabla k, \nabla k). \tag{3.22}$$

Hence

$$\begin{aligned}\operatorname{div}(\nabla k|\nabla k|^2) &= |\nabla k|^2 \operatorname{div}\nabla k + 2g(\nabla_{\nabla k}\nabla k, \nabla k)\\ &= \frac{n+2}{n}|\nabla k|^2\Delta k.\end{aligned}$$

Therefore using Stokes' formula, we obtain

$$\int_M |\nabla k|^2\Delta k \, d\mu = 0. \tag{3.23}$$

Then the proof of the lemma is completed. □

The next theorem is an analogous of Theorem 2 in [6] and Theorem 1 in [8].

**Theorem 3.10.** *Let $(M^n, g, \nabla k, \lambda, \omega)$ be a compact, connected, oriented without boundary gradient almost $\eta$-Ricci-Bourguignon soliton with $n \geq 3$. If one of the following conditions hold then $M^n$ is trivial:*

1. *$\nabla k$ is conformal and* $$\int_M \operatorname{Ric}(\nabla k, \nabla k)\, d\mu \leq -(n-2)\int_M \Big(g(\nabla\lambda, \nabla k) + \rho g(\nabla r, \nabla k)\Big)\, d\mu.$$
2. *$|\nabla k|$ is constant and* $$\int_M \operatorname{Ric}(\nabla k, \nabla k)\, d\mu \leq -(n-2)\int_M \Big(g(\nabla\lambda, \nabla k) + \rho g(\nabla r, \nabla k)\Big)\, d\mu.$$
3. $r \geq \dfrac{\lambda n}{1 - n\rho}$ *or* $r \leq \dfrac{\lambda n}{1 - n\rho}$.

*Proof.* Firstly integrating (3.16) of Lemma 3.8, we obtain

$$\begin{aligned}\int_M \Delta|\nabla k|^2\,d\mu &= \int_M |\nabla^2 k|^2\,d\mu - \int_M \mathrm{Ric}(\nabla k,\nabla k)\,d\mu + \int_M 2\omega|\nabla k|^2\Delta k\,d\mu\\ &\quad -(n-2)\int_M \Big(g(\nabla\lambda,\nabla k)+\rho g(\nabla r,\nabla k)\Big)\,d\mu.\end{aligned} \tag{3.24}$$

Since $M$ is compact and $\nabla k$ is a conformal, thus from Lemma 3.9, we obtain

$$\int_M |\nabla^2 k|^2\,d\mu = \int_M \mathrm{Ric}(\nabla k,\nabla k)\,d\mu + (n-2)\int_M \Big(g(\nabla\lambda,\nabla k)+\rho g(\nabla R,\nabla k)\Big)\,d\mu. \tag{3.25}$$

As we are assuming that the right-hand side of equation (3.25) is $\leq 0$, then we have $\nabla^2 k = 0$. Consequently, $\Delta k = 0$ and by using the Hopf's theorem we conclude that $k$ is constant which completes the item 1.

For the second one, if $|\nabla k|$ is constant, using Stokes' formula in (3.24) then we obtain that

$$\int_M |\nabla^2 k|^2\,d\mu = \int_M \mathrm{Ric}(\nabla k,\nabla k)\,d\mu + (n-2)\int_M \Big(g(\nabla\lambda,\nabla k)+\rho g(\nabla r,\nabla k)\Big)\,d\mu. \tag{3.26}$$

Thus the second item is completed.

Further, taking the trace of equation (3.2), we obtain

$$\mathrm{div}(\nabla u) = \omega u\big((1-n\rho)r - n\lambda\big).$$

Thus $\omega u$ is positive and $(1-n\rho)r - n\lambda \leq 0 (\geq 0)$. Since $M^n$ is compact, connected, oriented without boundary we obtain that $u$ is constant from Hopf's theorem, which gives that $k$ is constant. Then the proof is completed. □

## 4. Integral formulas for compact gradient$(-\frac{1}{\omega u})$-almost traceless Ricci soliton

Now we give integral formulas for a compact gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton.

Using the theory of the classical tensorial calculus, we can write the equation (3.2) as follows:

$$R_{ij} - \frac{\nabla_i\nabla_j u}{\omega u} = (\lambda+\rho r)g_{ij}. \tag{4.1}$$

First we deduce for gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton, an important formula deduce for Ricci soliton. The next lemma can be found in [7] and [16] for almost Ricci soliton and for $h$-almost Ricci soliton respectively.

**Lemma 4.1.** *Let $(M^n, g, \nabla u, (-\frac{1}{\omega u}), \lambda, \omega)$ be a gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton. Then we have:*

$$\frac{\Delta u}{\omega u} = -n\lambda, \tag{4.2}$$

$$\left(\frac{2-n}{n}\right)\nabla_i r = -\frac{2}{\omega u}R_{ij}\nabla^j u + 2(n-1)\nabla_i\lambda + 2\nabla_i\Big(\frac{1}{\omega u}\Big)\Delta u - 2g^{jl}\nabla_l\Big(\frac{1}{\omega u}\Big)\nabla_i\nabla_j u, \tag{4.3}$$

$$\left(\frac{2-n}{n}\right)\nabla r + \frac{1}{(\omega u)^2}\nabla|\nabla u|^2 = -\frac{2}{\omega u}\Big(\lambda+\frac{r}{n}\Big)\nabla u + 2(n-1)\nabla\lambda + 2\nabla\Big(\frac{1}{\omega u}\Big)\Delta u - 2g^{jl}\nabla_l\Big(\frac{1}{\omega u}\Big)\nabla_i\nabla_j u. \tag{4.4}$$

*Proof.* For proving equation (4.2), it is sufficient to take the trace of the equation (4.1) with $\rho = \frac{1}{n}$.

For the equation (4.3), we recall that $2\,\mathrm{div}\,\mathrm{Ric} = dr$ and the equation (4.1), we obtain

$$\begin{aligned}
\frac{1}{2}\nabla_i r &= \mathrm{div}\,\mathrm{Ric}_{ij} \\
&= g^{jl}\nabla_l R_{ij} \\
&= g^{jl}\nabla_l(\frac{1}{\omega u}\nabla_i\nabla_j u + \lambda g_{ij} + \rho r g_{ij}) \\
&= g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u + \frac{1}{\omega u}g^{jl}\nabla_l\nabla_i\nabla_j u + g^{jl}(\nabla_l\lambda)g_{ij} + \rho g^{jl}(\nabla_l r)g_{ij} \\
&= g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u + \frac{1}{\omega u}g^{jl}\nabla_l\nabla_i\nabla_j u + \frac{1}{\omega u}g^{jl}R_{lijs}\nabla^s u + \nabla_i\lambda + \rho\nabla_i r \\
&= g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u + \frac{1}{\omega u}\nabla_i\Delta u + \frac{1}{\omega u}R_{is}\nabla^s k + \nabla_i\lambda + \rho\nabla_i r \\
&= g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u + (1+\rho(1-n))\nabla_i r + (1-n)\nabla_i\lambda - \nabla_i(\frac{1}{\omega u})\Delta u \\
&+ \frac{1}{\omega u}R_{is}\nabla^s u.
\end{aligned}$$

Thus taking $\rho = \frac{1}{n}$, the equation (4.3) follows

$$\left(\frac{2-n}{n}\right)\nabla r = -\frac{2}{\omega u}R_{ij}\nabla^j u + 2(n-1)\nabla_i\lambda + 2\nabla_i(\frac{1}{\omega u})\Delta u - 2g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u.$$

Using the equation (4.3), recalling that $\nabla|\nabla u|^2 = 2\nabla_{\nabla u}\nabla u$ and $\frac{1}{\omega u}\nabla_{\nabla k}\nabla k = -(\lambda+\rho r)\nabla u + \mathrm{Ric}(\nabla u)$, we obtain

$$\begin{aligned}
\left(\frac{2-n}{2n}\right)\nabla r + \frac{1}{2(\omega u)^2}\nabla|\nabla u|^2 &= -\frac{1}{\omega u}\,\mathrm{Ric}(\nabla u) + \frac{1}{(\omega u)^2}\nabla_{\nabla u}\nabla u + (n-1)\nabla\lambda \\
&+ \nabla(\frac{1}{\omega u})\Delta u - g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u \\
&= -\frac{1}{\omega u}(\lambda+\rho r)\nabla u + (n-1)\nabla\lambda \\
&+ \nabla(\frac{1}{\omega u})\Delta u - g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j u,
\end{aligned}$$

which completes the proof of the lemma. □

Before deriving our integral theorem, we present the next lemma.

**Lemma 4.2.** *[5] Let $(M, g)$ be a Riemannian manifold and $S$ is a $(0,2)$-tensor on $M$. Then*

$$\mathrm{div}(S(\phi\xi)) = \phi(\mathrm{div}\,S)(\xi) + \phi\langle\nabla\xi, S\rangle + S(\nabla\phi, \xi), \tag{4.5}$$

*for all vector field $\xi$ on $M$.*

Lastly, for compact gradient $(-\frac{1}{\omega u})$-almost traceless Ricci solitons, we derive an integral formula as a generalization of the result obtained in [1] and [16].

**Theorem 4.3.** *Let $(M^n, g, \nabla u, (-\frac{1}{\omega u}), \lambda, \omega)$ be a compact gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton. Then the followings identities hold:*

$$\int_M \frac{1}{\omega u}\left|\nabla^2 u - \frac{\Delta u}{n}g\right|^2 d\mu = \frac{2-n}{2n}\int_M g(\nabla r, \nabla u)\,d\mu, \tag{4.6}$$

*or*

$$\int_M \left|\mathrm{Ric} - \frac{r}{n}g\right|^2 d\mu = \frac{2-n}{2n}\int_M g(\nabla r, \nabla u)\,d\mu. \tag{4.7}$$

*Proof.* It is enough to use Lemma 4.2 to infer

$$\operatorname{div}(\operatorname{Ric}(\nabla k)) = (\operatorname{div}\operatorname{Ric})(\nabla k) + \langle \nabla^2 k, \operatorname{Ric}\rangle. \tag{4.8}$$

From (4.1), the second contracted Bianchi identity and the equation $\left|\nabla^2 k - \frac{\Delta k}{n} g\right|^2 = \left|\nabla^2 k\right|^2 - \frac{(\Delta k)^2}{n} g$, we get

$$\begin{aligned}\operatorname{div}(\operatorname{Ric}(\nabla k)) &= \frac{1}{2} g(\nabla r, \nabla k) + (\lambda + \rho r)\Delta k + \frac{1}{\omega u}|\nabla^2 k|^2 \\ &= \frac{1}{2} g(\nabla r, \nabla k) + (\lambda + \rho r)\Delta k + \frac{1}{\omega u}|\nabla^2 k - \frac{\Delta k}{n} g|^2 + \frac{(\Delta k)^2}{n\omega u} \\ &= \frac{1}{2} g(\nabla r, \nabla k) + \frac{r}{n}\Delta k + \frac{1}{\omega u}|\nabla^2 k - \frac{\Delta k}{n} g|^2. \end{aligned} \tag{4.9}$$

Therefore, using the equation (4.3) of Lemma 4.1, we obtain

$$\frac{1}{\omega u}\operatorname{Ric}(\nabla k) = \left(\frac{n-2}{2n}\right)\nabla r + (n-1)\nabla\lambda + \nabla(\frac{1}{\omega u})\Delta k - g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j k.$$

Thus, we find

$$\frac{1}{\omega u}\operatorname{div}\operatorname{Ric}(\nabla k) = \left(\frac{n-2}{2n}\right)\Delta r + (n-1)\Delta\lambda + \Delta(\frac{1}{\omega u})\Delta k - \operatorname{div}(g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j k). \tag{4.10}$$

Combining the last equation with the equation (4.9), we obtain

$$\begin{aligned}\frac{1}{(\omega u)^2}\left|\nabla^2 k - \frac{\Delta k}{n} g\right|^2 &= \left(\frac{n-2}{2n}\right)\Delta r + (n-1)\Delta\lambda + \Delta(\frac{1}{\omega u})\Delta k \\ &\quad - \frac{1}{2\omega u} g(\nabla r, \nabla k) - \frac{r\Delta k}{n\omega u} - \operatorname{div}(g^{jl}\nabla_l(\frac{1}{\omega u})\nabla_i\nabla_j k). \end{aligned} \tag{4.11}$$

Integrating both sides of equation (4.11), we have

$$\int_M \frac{1}{(\omega u)^2}\left|\nabla^2 k - \frac{\Delta k}{n} g\right|^2 d\mu = -\frac{1}{2\omega u}\int_M g(\nabla r, \nabla k)\, d\mu - \frac{1}{n\omega u}\int_M R\Delta k\, d\mu. \tag{4.12}$$

We know that $M^n$ is compact with $\int_M r\Delta k \ d\mu = -\int_M g(\nabla r, \nabla k)\, d\mu$ and replacing $k$ with $u$ in the equation (4.12), we find

$$\int_M \frac{1}{\omega u}\left|\nabla^2 u - \frac{\Delta u}{n} g\right|^2 d\mu = \frac{2-n}{2n}\int_M g(\nabla u, \nabla k)\, d\mu. \tag{4.13}$$

From the equation (3.2), we get

$$\begin{aligned}\operatorname{Ric} - \frac{r}{n} g &= \frac{1}{\omega u}\nabla^2 u + (\lambda + \rho r - \frac{r}{n} g) \\ &= \frac{1}{\omega u}(\nabla^2 u - \frac{\Delta u}{n} g). \end{aligned} \tag{4.14}$$

Thus the proof is completed. □

Consequently from the previous theorem and a classical theorem due to Tashiro [27], also the results owing to [6], we obtain certain hypothesis for a compact gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton to be isometric to $\mathbb{S}^n$, when $r$ is positive to obtain the next corollary which is an analogous of $h$-almost Ricci soliton in [16].

**Corollary 4.4.** *Let $(M^n, g, \nabla u, (-\frac{1}{\omega u}), \lambda, \omega)$ be a nontrivial compact gradient $(-\frac{1}{\omega u})$-almost traceless Ricci soliton with $n \geq 3$. Then $M^n$ is isometric to $\mathbb{S}^n$, if one of the following conditions holds:*

*(1)* $M^n$ *has constant scalar curvature with* $r > 0$.

*(2)* $\displaystyle\int_M g(\nabla r, \nabla u) d\mu \geq 0.$

*(3)* $M^n$ *is a homogenous manifold.*

*Proof.* Notice that any one of the assertions of corollary 4.4 allow us to write that

$$\int_M |\nabla^2 u - \frac{\Delta u}{n} g|^2 \, d\mu = 0.$$

Hence, we deduce that $\mathrm{Ric} = \dfrac{r}{n} g$. Now from the equation (3.2), we obtain

$$\nabla^2 u = \omega u \bigg( r(\frac{1}{n} - \rho) - \lambda \bigg) g, \tag{4.15}$$

which implies that $\nabla u$ is a nontrivial conformal. Then from Theorem 3.7, we deduce that $M^n$ is isometric to the sphere $\mathbb{S}^n$. □


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## References


1. Aquino, C., Barros, A., Ribeiro, E. Jr., *Some applications of the Hodge-de Rham decomposition to Ricci solitons*, Results Math., 60(1), 235-246, (2011).
2. Aubin, T., *Metriques Riemanniennes et courbure*, J. Differ. Geom., 4(4), 383–424, (1970).
3. Blaga, A.M., Perktaş, S.Y., *Remarks on almost η-Ricci solitons in ϵ-para Sasakian manifolds*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., 68(2), 1621-1628, (2019).
4. Blaga, A.M., Taştan, H.M., *Some results on almost η-Ricci-Bourguignon solitons*, J. Geom. Phys., 168, 104316, (2022).
5. Barros, A., Gomes, J.N., *A compact gradient generalized quasi-Einstein metric with constant scalar curvature*, J. Math. Anal. Appl., 401(2), 702–705, (2013).
6. Barros, A., Ribeiro, E. Jr., *Characterizations and Integral formulae for generalized m-quasi-Einstein metrics*, Bull Braz. Math. Soc, New Series., 45(2), 325-341, (2014).
7. Barros, A., Ribeiro, E.Jr., *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc., 140(3), 1033-1040, (2012).
8. Barros, A., Ribeiro, E. Jr., *Integral formulae on quasi-Einstein manifold and Applications*, Glasgow Math. J., 54(2), 213-223, (2012).
9. Bourguignon, J.P., *Ricci curvature and Einstein metrics*, Global differential geometry and global analysis., 42–63, 1981.
10. Catino, G., Cremaschi, L., Djadli, Z., Mantegazza, C., Mazzieri, L., *The Ricci-Bourguignon flow*, Pac. J. Math., 287(2), 337–370, (2017).
11. Catino, G., L. Mazzieri, L., *Gradient Einstein solitons*, Nonlinear Anal., 132(1), 66–94, (2016).
12. Chaubey S. K., Siddiqi M. D., Prakasha D. G., *Invariant Submanifolds of Hyperbolic Sasakian Manifolds and η-Ricci-Bourguignon Solitons*, Filomat, 36(2), 409–421,(2022).
13. Chen, X., Lu, P., Tian, G., *A note on uniformization of Riemannian surfaces by Ricci flow*, Proc. Amer. Math. Soc., 134, 3391–3393, (2006).
14. Dwivedi, S., *Some results on Ricci-Bourguignon and almost solitons*, Can. Math. Bull., 64(3), 591-604, (2021).
15. Gomes J. N, Wang, Q., Xia, C., *On the h-almost Ricci soliton*, J. Geom. Phys., 144, 216–222, (2017).
16. Ghahremani-Gol, H., *Some results on h-almost Ricci solitons*, J. Geom. Phys., 137, 212–216, (2019).

17. Hamilton, R.S., *Three-manifolds with positive Ricci curvature*, J. Differ. Geom., 17(2), 255-306, (1982).

18. Hamilton, R.S., *The Ricci flow on surfaces, Mathematics and general relativity*, Contemp. Math., 71, 237-262, (1988),.

19. Ishihara, S., Tashiro, Y.: *On Riemannian manifolds admitting a concircular transformation*, Math. J. Okayama Univ., 9, 19–47, (1959).

20. Ivey, T., *Ricci solitons on compact three-manifolds*, Differ. Geom. Appl., 3(4), 301–307, (1993).

21. Lott, J., *On the long time behavior of type-III Ricci flow solutions*, Math. Ann., 339(3), 627-666, (2007).

22. Ni, L., Wallach, N., *On a classification of gradient shrinking solitons*, Math. Res. Lett., 15(5), 941–955, (2008).

23. Perelman, G., *The Entropy formula for the Ricci flow and its Geometric Applications*, arXiv preprint math/0211159 (2002)

24. Petersen, P., Wylie, W., *Rigidity of gradient Ricci solitons*, Pacific J. Math., 241(2), 329-345, (2009).

25. Siddiqi, M.D., Akyol, M.A., *$\eta$-Ricci-Yamabe solitons on Riemannian submersions from Riemannian submanifold*, arXiv:2004.14124 (2020).

26. Siddiqui A. N., Siddiqi M. D., *Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime*, Balk. J. Geom. Appl., 26(2), 126–138, (2021).

27. Tashiro, Y., *Complete Riemannian manifolds and some vector fields*, Trans, Amer. Math.Soc., 117, 673-680, (1965).

28. Traore, M., Taştan, H.M., Gerdan Aydın, S., *On almost $\eta$-Ricci-Bourguignon solitons*, Miskolc. Math. Notes., 25(1), 493–508, (2024).

29. Traore, M., Taştan, H.M., *On sequential warped product $\eta$-Ricci-Bourguignon solitons*, Filomat, 38(19), 6785–6797, (2024).

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