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Existence of Solutions to Hybrid Differential Equations by Nabla Caputo Fractional Operator

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ABSTRACT: In this article, we demonstrate the existence of solutions to hybrid differential equations using fractional Nabla-Caputo derivatives of order $0 < \nu < 1$. Numerous Lipschitz and Caratheodory criteria are used to demonstrate existence. At last, as application, an illustrative example is given to show the applicability of our theoretical results.

Key Words: Hybrid differential equation, Nabla Caputo fractional derivative, fractional differential equations, Arzela-Ascoli's theorem, Caratheodory conditions.

Contents

1. Introduction

Many writers have written about fractional differential equations throughout the last few decades. Fractional derivatives are frequently used to model the mechanical and electrical properties of many real-world materials. In addition, fractional derivatives are often used in a wide range of physical, chemical, and biological processes [1, 11]. Many authors have examined the existence of solutions to fractional boundary value problems using various boundary conditions and methodologies, [3, 6] as examples.

The quadratic perturbations of nonlinear differential equations have received a lot of interest. They are referred to as fractional hybrid differential equations. There have been numerous publications on the theory of hybrid differential equations, and we suggest readers to the papers [5, 10].

Dhage and Lakshmikantham [17] consider the next first order hybrid differential equation:

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T] \\ x(t_0) = x_0 \in \mathbb{R} \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$. They showed the existence, uniqueness outcomes, and certain essential differential inequalities for hybrid differential equations, launching the study of such systems' theory, and demonstrated, using inequalities theory, the existence of extremal solutions and comparison findings.

Zhao et al. [18] have studied the fractional hybrid differential equations using Riemann-Liouville differential operators as follows:

$$\begin{cases} D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T] \\ x(0) = 0 \end{cases}$$

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where $f \in C(J\times\mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in \mathcal{C}(J\times\mathbb{R}, \mathbb{R})$. The authors of [18] discovered the existence theorem for fractional hybrid differential equations and various basic differential inequalities, as well as the existence of extremal solutions.

Benchohra et al. [19] discussed the next boundary value problems for differential equations with fractional order:

$$\begin{cases} {}^{c}D^{\alpha}y(t) = f(t, y(t)) & \text{for each } t \in J = [0, T], 0 < \alpha < 1 \\ ay(0) + by(T) = c \end{cases}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, a, b, c are real constants with $a + b \neq 0$.

Abdeljawad, Atici, and Eloe, as well as Hein et al., have recently investigated and developed nabla fractional calculus in depth [2, 13]. Goodrich and Peterson [4] offer a self-contained treatment to nabla differential derivatives that includes the both Riemann-Liouville as well as Caputo fractional differences (for more see [21–28]).

We investigate the following fractional hybrid differential equations in this study, which is prompted by all of the preceding papers:

$$\begin{cases} {}^{C}\nabla_0^v\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), \quad t \in \mathbb{N}_1, \\ x(0) = 0, \end{cases} \tag{1.1}$$

where $f, g : \mathbb{N}_1 \times \mathbb{R} \to \mathbb{R}$ are continuous with $f(t, 0) \neq 0$ `for all` $t \in \mathbb{N}_1$ and ${}^{c}\nabla_0^v$ signifies the Caputo's nabla fractional of order $v \in (0, 1)$.

The main contribution of this work is to develop the theory of boundary fractional hybrid differential equations involving the nabla Caputo differential operators of order $0 < v < 1$. An existence theorem for boundary fractional hybrid differential equations is proved under Lipschitz and Carathéodory conditions.

This work will be distributed as follows: Section 2 includes some fundamental concepts, notations, lemmas, and theorems needed for this paper. Section 3 will look into the existence of solutions. Section 4 provides an example to demonstrate the theoretical results. Section 5 contains the final conclusion.

2. Preliminaries

In this part, we provide a few basic definitions, nabla notation, results of fractional operations, and various lemmas and theorems that aids in the proof of our research.

Notations

- The set of all real numbers and positive real numbers is denoted by $\mathbb{R}$ and $\mathbb{R}^+$, respectively.
- We define by $\mathbb{N}_a = \{a, a + 1, a + 2, \ldots\}$ for any $a \in \mathbb{R}$.
- The Caratheodory class of functions on $\mathbb{N}_1\times\mathbb{R}$ is denoted by $\mathbf{C}_c(\mathbb{N}_1 \times \mathbb{R}, \mathbb{R})$. i. e. $g \in \mathbf{C}_c(\mathbb{N}_1 \times \mathbb{R}, \mathbb{R})$ iff
  a. For any $x \in \mathbb{R}$, the map $t \to g(t, x)$ is measurable,
  b. For any $t \in \mathbb{N}_1$, the map $x \to g(t, x)$ is continuous.
- The set of bounded real valued functions defined on $\mathbb{N}_1$ is denoted by $\mathbf{C}(\mathbb{N}_1\times \mathbb{R}, \mathbb{R}\backslash\{0\})$, with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{N}_1} |x(t)|. \tag{2.1}$$

Remark 2.1 *$\mathbf{C}(\mathbb{N}_1 \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ is indeed a Banach space whose norm is (2.1).*

Definition 2.1 *[4] The nabla fractional sum for $a \in \mathbb{R}$ and $0 < v < 1$ is given by*

$$\nabla_a^{-v}x(t) = \frac{1}{\Gamma(v)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{v-1}} x(s), t \in \mathbb{N}_{a+1}$$

In this case $\rho(s) = s - 1$. However, ∇_a^{-0} is defined to be the identity operator.

Definition 2.2 *[4] The $v-th$ order nabla fractional difference for $x:\mathbb{N}_{a+n}\to\mathbb{R}$ and $v>0$, and for $n-1<v<n$, is defined as*

$$\nabla_a^v x(t)=\frac{1}{\Gamma(-v)}\sum_{s=a+n}^{t}(t-\rho(s))^{\overline{-v-1}}x(s), t\in\mathbb{N}_{a+n}.$$

The identity operator is ∇_a^0 in this case.

Definition 2.3 *[14] The nabla $v-th$ order Caputo left fractional difference of a function x defined on $\mathbb{N}_a$ and certain points before a is defined as follows:*

$$\begin{aligned}{}^{c}\nabla_a^v x(t) &\triangleq \nabla_a^{-(n-v)}\nabla^n x(t)\\ &=\frac{1}{\Gamma(n-v)}\sum_{s=a+1}^{t-(n-v)}(t-\rho(s))^{\overline{n-v-1}}\nabla^n x(s).\end{aligned}$$

Proposition 2.1 *[4] Allow x to be a real valued function with $\mu, v>0$. So*

$$\nabla_a^{-v}\left[\nabla_a^{-\mu}x(t)\right]=\nabla_a^{-(\mu+v)}x(t)=\nabla_a^{-\mu}\left[\nabla_a^{-v}x(t)\right]$$

Proposition 2.2 *[13] For $v>0$, and x defined in $\mathbb{N}_a$, one has*

$$\begin{aligned}\nabla_a^v\nabla_a^{-v}x(t)&=x(t),\\ \nabla_a^{-v}\nabla_a^v x(t)&=x(t), \quad \text{when } v\notin\mathbb{N}.\\ \nabla_a^{-v}\nabla_a^v x(t)&=x(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k}}{k!}\nabla^k x(a), \quad \text{when } v=n\in\mathbb{N}.\end{aligned} \tag{2.2}$$

Proposition 2.3 *Suppose that $v>0$ and x is defined in appropriate domain $\mathbb{N}_a$. Then,*

$$\nabla_a^{-v^c}\nabla_a^v x(t)=x(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{\bar{k}}}{k!}\nabla^k x(a). \tag{2.3}$$

Specifically, if $0<v\le 1$, so

$$\nabla_a^{-v^c}\nabla_a^v x(t)=x(t)-x(a)$$

Proof: The demonstration of (2.3) is then followed by the definition and use of proposition 2.1 and (2.2) of proposition 2.2. i.e.

$$\begin{aligned}\nabla_a^{-v^C}\nabla_a^v x(t)&=\nabla_a^{-v}\left(\nabla_a^{-(n-v)}\nabla^n x(t)\right),\\ &=\nabla_a^{-v}\nabla_a^{-(n-v)}\left(\nabla^n x(t)\right),\\ &=\nabla_a^{-n}\left(\nabla^n x(t)\right),\\ &=x(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{\bar{k}}}{k!}\nabla^k x(a).\end{aligned}$$

Hence

$$\nabla_a^{-v^c}\nabla_a^v x(t)=x(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{\bar{k}}}{k!}\nabla^k x(a).$$

□

Definition 2.4 [14] The nabla discrete Mittag Leffler functions are defined as follows for $\beta \in \mathbb{R}, |\beta| < 1$ and $v, \mu, t \in \mathbb{C}$ with $\mathsf{R}(v) > 0$,

$$E_{\overline{i,\mu}}(\beta, t) = \sum_{m=0}^{\infty} \beta^m \frac{t^{\overline{mv+\mu-1}}}{\Gamma(mv+\mu)}.$$

The one variable nabla discrete Mittag-Leffler function for $\mu = 1$ may be expressed as

$$E_{\bar{v}}(\beta, t) = \sum_{m=0}^{\infty} \beta^m \frac{t^{\overline{Mv}}}{\Gamma(mv+1)}$$

The following characteristics are stated and shown in [9] and [12].

(P1) For $t \in \mathbb{R}^+, 0 < v \leq 1$, The one and two-variable nabla discrete Mittag-Leffler functions are decreasing functions of t, limited by 1 above. That seems to be, $E_{\bar{v}}(\beta, t) \leq 1$ and $E_{\bar{Q},\bar{v}}(\beta, t) \leq 1$, with $-1 < \lambda < 0$ and $-v < \beta < 0$.

(P2) $\lim_{t\to\infty} E_{\bar{v}}(\beta, t) = \lim_{\mathrm{r}\to\infty} E_{\bar{v},\mathrm{v}}(\beta, t) = 0$.

The following discrete form of Arzelà-Ascoli's theorem is important for the proof of the main theorem.

Theorem 2.1 *[7]. Let $S_0 : f_1, f_2, f_3, \cdots,$ be a series of functions on the closed interval $[a, b]$ that is uniformly bounded and equicontinuous. A subsequence S of S_0 exists, and it converges uniformly on $[a, b]$.*

Lemma 2.1 *[15, 16]. Allow $\mathbf{C}$ to be a nonempty, closed, convex, and bounded subset of the Banach space X and allow $A : C \to X$ and $B : \mathbf{C} \to X$ to be two operators such that:*

i. A is Lipschitz with the Lipschitz constant k,

ii. B is completely continuous,

iii. $x = AxBy \Rightarrow x \in \mathrm{C}$ for all $y \in \mathrm{C}$,

iv. $kM < 1$ where $M = \|B(\mathbf{C})\| = \mathrm{Sup}\{B(x) \mid x \in \mathbf{C}\}$.

Then, there is a solution to the problem $AxBx = x$ in $\mathbf{C}$.

The following assumptions are made for the rest of the work:

(H_1) The function $x \mapsto \frac{x}{f(t,x)}$ is increasing on $\mathbb{R}$ for $t \in \mathbb{N}_1$.

(H_2) There is a constant $k > 0$ that provides

$$|f(t, x) - f(t, y)| \leq k|x - y| \text{ for all } \quad t \in \mathbb{N}_1 \text{ and } x, y \in \mathbb{R}$$

(H_3) There is a function φ such that

$$|g(t, x)| \leq \varphi(t) \quad \text{a. e. } t \in \mathbb{N}_1, \ \texttt{for all } x \in \mathbb{R}.$$

3. Main Result

Before presenting the primary conclusions of our work, we must prove the following essential lemma.

Lemma 3.1 *Assuming the assumption (H_1) is correct, an integral solution of problem (1.1) is a function $x \in C$ which satisfies the following:*

$$x(t) = \frac{f(t,x)}{\Gamma(v)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{v-1}} g(s, x(s)). \tag{3.1}$$

Proof: Given x as the solution to the problem (1.1), apply the ∇-fractional derivative ∇_0^{-v} on both sides of (1.1), we get:

$$\nabla_0^{-v c}\nabla_0^{v}\left(\frac{x(t)}{f(t,x(t))}\right)=\nabla_0^{-v}g(t,x(t)),$$

From the proposition 2.2 and initial conditions,

$$\begin{aligned}\frac{x(t)}{f(t,x(t))}-\frac{x(0)}{f(t,x(0))}&=\nabla_0^{-v}g(t,x(t))\\ \frac{x(t)}{f(t,x(t))}&=\frac{0}{f(t,0)}+\nabla_0^{-v}g(t,x(t))\\ \frac{x(t)}{f(t,x(t))}&=0+\nabla_0^{-v}g(t,x(t))\end{aligned}$$

Thus

$$x(t)=\frac{f(t,x)}{\Gamma(v)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{v-1}}g(s,x(s)).$$

Hence equation (3.1) holds.

Conversely, suppose $x(t)$ satisfies the equation (3.1). Dividing by $f(t,x)$ and use the fractional ∇-Caputo derivative ${}^{C}\nabla_0^{\gamma}$ to the both of the equation's sides (3.1) and we use Proposition 2.3, gives us

$$\begin{aligned}{}^{c}\nabla_0^{v}\left(\frac{x(t)}{f(t,x)}\right)&={}^{c}\nabla_0^{v}\left(\frac{1}{\Gamma(v)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{v-1}}g(s,x(s))\right),\\ {}^{c}\nabla_0^{v}\left(\frac{x(t)}{f(t,x)}\right)&={}^{c}\nabla_0^{v}\nabla_0^{-v}g(t,x(t)),\end{aligned}$$

Thus

$${}^{c}\nabla_0^{v}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)).$$

Finally, we must check if the condition $x(0)=0$ in the expression (1.1) is also true. Put $t=0$ in (3.1) for this,

$$\begin{aligned}x(0)&=\frac{f(0,x)}{\Gamma(v)}\sum_{s=1}^{0}(0-\rho(s))^{\overline{v-1}}g(s,x(s)).\\ x(0)&=0.\end{aligned}$$

Since the map $x\mapsto\frac{x}{f(t,x)}$ is increasing on $\mathbb{R}$ for any $t\in\mathbb{N}_1$, so for the map $x\mapsto\frac{x}{f(0,x)}$ is injective on $\mathbb{R}$ and $x(0)=0$. The demonstration is now complete. □

Theorem 3.1 *Assume that the hypothesis* $(H_1)-(H_3)$ *are true. Further, if*

$$k(\|\varphi\|)<1$$

Then the fractional hybrid differential equation (1.1) *has a solution given in* $\mathbb{N}_1$.

Proof: Allow $E=C(\mathbb{N}_1,\mathbb{R})$ and C_b to be a subset of the space E stated by:

$$C_b=\{x\in E:\|x\|\leq b\}.$$

Where

$$b=\frac{f_0\|\varphi\|}{1-k\|\varphi\|},$$

And

$$f_0 = \sup_{t\in\mathbb{N}_1} f(t,0); \quad \sum_{s=1}^{t} \frac{(t-\rho(s))^{\overline{v-1}}}{\Gamma(v)} = \lim_{t\to+\infty} E_v(1, t-\rho(s)) \leq 1.$$

C_b is clearly a closed, convex, bounded subset of the Banach algebra E.

Using the lemma 3.1, the fractional hybrid differential equation (1.1) is similar to the following non-linear fractional hybrid integral equation

$$x(t) = \frac{f(t,x)}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} g(s, x(s)). \tag{3.2}$$

Allow $\Lambda : E \to E$ and $\mathcal{B} : C_b \to E$ to be two operators defined by

$$\Lambda x(t) = f(t, x(t)),$$

and

$$\mathcal{B}x(t) = \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} g(s, x(s)).$$

The nonlinear fractional hybrid integral equation (3.2) may be transformed into an operator equation as follows:

$$\Lambda x(t)\mathcal{B}x(t) = x(t), \quad t \in \mathbb{N}_1.$$

We prove that the operators Λ and $\mathcal{B}$ meet all of the lemma's criteria 2.1.

We begin by demonstrating that Λ is the Lipschitz operator on E with the Lipschitz constant k.

Let $x, y \in E$, therefore by assumption (H_2) $|\Lambda x(t) - \Lambda y(t)| = |f(t, x(t)) - f(t, y(t))| \leq k|x(t) - y(t)|$ for all $t \in \mathbb{N}_1$, Using supremum over t, we get

$$\|\Lambda x - \Lambda y\| \leq k\|x - y\|, \quad \texttt{for all } x, y \in E.$$

Second, we demonstrate that the operator $\mathcal{B}$ is completely continuous.

To do so, it suffices to demonstrate that the operator $\mathcal{B}$ is continuous and that $\mathcal{B}(C_b)$ is equicontinuous and uniformly bounded.

Now let demonstrate that the operator $\mathcal{B}$ is continuous.

Allow x_n to be a sequence that converges to $x \in C_b$ at C_b, therefore from the Lebesgue dominance convergence theorem, we get

$$\begin{aligned}
\lim_{n\to+\infty} \mathcal{B}x_n(t) &= \lim_{n\to+\infty} \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} g(s, x_n(s)) \\
\lim_{n\to+\infty} \mathcal{B}x_n(t) &= \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} \lim_{n\to+\infty} g(s, x_n(s)) \\
\lim_{n\to+\infty} \mathcal{B}x_n(t) &= \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} g(s, x(s)) \\
\lim_{n\to+\infty} \mathcal{B}x_n(t) &= \mathcal{B}x(t), \ \texttt{for all } t \in \mathbb{N}_1
\end{aligned}$$

As a result, $\mathcal{B}$ is a continuous operator on C_b.

Following that, we demonstrate that $\mathcal{B}(C_b)$ is uniformly bounded. Given $x \in C_b$, then we have

$$\begin{aligned}
|\mathcal{B}x(t)| &= \left| \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} g(s, x(s)) \right|, \\
|\mathcal{B}x(t)| &\leq \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} |g(s, x(s))|,
\end{aligned}$$

Using (H_3) gives

$$|\mathcal{B}x(t)| \leq \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} |\varphi(s)|$$
$$|\mathcal{B}x(t)| \leq |\varphi(t)| \texttt{ for all } t \in \mathbb{N}_1$$

Considering supremum over t, we get

$$\|\mathcal{B}x\| \leq \|\varphi\| \text{ for all } x \in C_b.$$

This demonstrates that $\mathcal{B}$ is uniformly bounded on C_b.

Let us now demonstrate that $\mathcal{B}(C_b)$ is equicontinuous on $\mathbb{N}_1$.

Given $x \in \mathcal{B}(C_b)$ and $t_1, t_2 \in \mathbb{N}_1$ such that $t_1 < t_2$, then we have

$$\begin{aligned}
|\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| &= \left| \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} (t_1-\rho(s))^{\overline{v-1}} g(s,x(s)) - \frac{1}{\Gamma(v)} \sum_{s=1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} g(s,x(s)) \right| \\
&\leq \left| \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} (t_1-\rho(s))^{\overline{v-1}} g(s,x(s)) - \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} (t_2-\rho(s))^{\overline{v-1}} g(s,x(s)) \right| \\
&+ \left| \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} (t_2-\rho(s))^{\overline{v-1}} g(s,x(s)) - \frac{1}{\Gamma(v)} \sum_{s=1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} g(s,x(s)) \right| \\
&\leq \left| \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} \left((t_1-\rho(s))^{\overline{v-1}} - (t_2-\rho(s))^{\overline{v-1}} \right) g(s,x(s)) \right| \\
&+ \left| \frac{1}{\Gamma(v)} \sum_{s=t_1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} g(s,x(s)) \right| \\
&\leq \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} \left((t_1-\rho(s))^{\overline{v-1}} - (t_2-\rho(s))^{\overline{v-1}} \right) |g(s,x(s))| \\
&+ \frac{1}{\Gamma(v)} \sum_{s=t_1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} |g(s,x(s))| \\
&\leq \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} \left((t_1-\rho(s))^{\overline{v-1}} - (t_2-\rho(s))^{\overline{v-1}} \right) |\varphi(s)| \\
&+ \frac{1}{\Gamma(v)} \sum_{s=t_1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} |\varphi(s)| \\
&\leq \frac{1}{\Gamma(v)} \sum_{s=1}^{t_1} \left((t_1-\rho(s))^{\overline{v-1}} - (t_2-\rho(s))^{\overline{v-1}} \right) \|\varphi\| + \frac{1}{\Gamma(v)} \sum_{s=t_1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} \|\varphi\| \\
&\leq \frac{\|\varphi\|}{\Gamma(v)} \left(\sum_{s=1}^{t_1} \left((t_1-\rho(s))^{\overline{v-1}} - (t_2-\rho(s))^{\overline{v-1}} \right) + \sum_{s=t_1}^{t_2} (t_2-\rho(s))^{\overline{v-1}} \right),
\end{aligned}$$

So

$$\lim_{t_1 \to r_2} |\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| = 0.$$

This confirms that $\mathcal{B}(C_b)$ is equi-continuous.

The set $\mathcal{B}(C_b)$ is now uniformly and equicontinuously bounded Thereby also we may derive that $\mathcal{B}(C_b)$ is compact by utilizing the Arzelà-Ascoli Theorem 2.1, This show that the operator $\mathcal{B}$ is completely continuous.

It is now necessary to demonstrate that the third hypothesis in the lemma 2.1 is correct.

Let $x \in E$ and $y \in C_b$ such that $x = \Lambda x \mathcal{B} y$, then by the assumption (H_2), we get

$$\begin{aligned}
|x(t)| &= |\Lambda x(t)||\mathcal{B}y(t)| = |f(t, x(t))| \left| \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} g(s, x(s)) \right|, \\
|x(t)| &\leq (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} |g(s, x(s))|, \\
|x(t)| &\leq (|kx(t)| + |f_0|) \frac{1}{\Gamma(v)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{v-1}} |\varphi(s)|, \\
|x(t)| &\leq (|kx(t)| + |f_0|) \, \|\varphi\|,
\end{aligned}$$

Which implies that

$$|x(t)| \leq \frac{|f_0| \, \|\varphi\|}{1 - k\|\varphi\|}$$

Considering the supremum over t, we have

$$\|x\| \leq \frac{f_0 \|\varphi\|}{1 - k\|\varphi\|} := b.$$

Since

$$M = \|\mathcal{B}(C_b)\| = \sup\{\mathcal{B}(x) : x \in C_b\} \leq \|\varphi\|,$$

Then we get

$$\lambda M \leq k(\|\varphi\|) < 1.$$

Consequently, all the criteria of the lemma 2.1 are fulfilled for the operators Λ and $\mathcal{B}$. As a result, the FHDE (1.1) has a solution stated on $\mathbb{N}_1$. □

4. An Example

We consider the following equation:

$$\begin{cases} \nabla_0^{\frac{1}{2}} \left(\frac{x(t)}{f(t, x(t))} \right) = g(t, u(t)), \quad t \in \mathbb{N}_1, \\ x(0) = 0 \end{cases} \tag{4.1}$$

Where $g(t, x(t)) = \cos(x(t))$ and $f(t, x(t)) = \frac{e^{-t}}{9+e} \left(\frac{|x(t)|}{1+|x(t)|} \right)$. It is obvious that the assumption (H_1) is satisfied.

To prove the hypothesis (H_2), let $t \in \mathbb{N}_1$ and $x, y \in \mathbb{R}$, so we have

$$\begin{aligned}
|f(t, x(t)) - f(t, y(t))| &= \left| \frac{e^{-t}}{9+e^t} \left(\frac{|x(t)|}{1+|x(t)|} \right) - \frac{e^{-t}}{9+e^t} \left(\frac{|y(t)|}{1+|y(t)|} \right) \right|, \\
|f(t, x(t)) - f(t, y(t))| &\leq \left| \frac{e^{-t}}{9+e^t} \right| \left| \left(\frac{|x(t)|}{1+|x(t)|} \right) - \left(\frac{|y(t)|}{1+|y(t)|} \right) \right|, \\
|f(t, x(t)) - f(t, y(t))| &\leq \frac{1}{10} \left| \frac{x(t) - y(t)}{(1+|x(t)|)(1+|y(t)|)} \right|, \\
|f(t, x(t)) - f(t, y(t))| &\leq \frac{1}{10} |x(t) - y(t)|,
\end{aligned}$$

Thus, (H_2) holds with $k = \frac{1}{10}$.

It remains to check the assumption (H_3). Given $t \in \mathbb{N}_1$ and $x \in \mathbb{R}$, then we obtain

$$\begin{aligned}
|g(t, x(t))| &= |\cos(x(t))| \\
|g(t, x(t))| &\leq 1.
\end{aligned}$$

This means that the hypothesis (H_3) is hold with $\varphi(t) = 1$. Additionally, we have

$$k(\|\varphi\|) = \frac{1}{10} \times 1 = \frac{1}{10} \leq 1.$$

Finally, since all the criteria of the theorem 3.1 are satisfied., the fractional hybrid problem (4.1) has a solution on $\mathbb{N}_1$.

5. Conclusion and Future Work

In the current work, the nabla Caputo fractional derivative of order $v \in (0, 1)$ was used to give a solution to the fractional hybrid initial value problem. In addition, some Lipschitz and Carathéodory conditions were used to explain the existence of at least one solution to this problem. Finally, an example is given to illustrate the applicability of the results obtained. In the future work we are going to extend results for this operator in the case of $v \geq 1$.

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