Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.70860

Existence results for an implicit anti-periodic ξ-fractional coupled system involving p-Laplacian operator via topological degree method

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ABSTRACT: In this paper, we investigate the existence and uniqueness of solutions for implicit anti-periodic coupled systems of ξ-Caputo fractional differential equations involving the p-Laplacian operator in a Banach space. Our results are based on the topological degree theory for condensing maps and the Banach contraction principle. An example is provided to illustrate the results.

Key Words: ξ-fractional integral, ξ-Caputo fractional derivative, p-Laplacian operator, condensing maps .

Contents

1. Introduction

The development of fractional calculus and its practical applications are of great importance for the effective modeling of nonlinear complex problems with arbitrary fractional orders. Fractional differential equations have emerged as a crucial area in real-world applications due to their ability to accurately represent a wide range of physical phenomena spanning diverse fields such as chemistry, physics, biology, engineering, viscoelasticity, electrical engineering, signal processing, electrochemistry, and controllability.

In recent years, researchers have been especially driven to investigate fractional differential equations, recognizing their significance in physics and related domains. This has led to the development of fractional calculus, which encompasses various types of fractional derivatives, such as Riemann-Liouville, Caputo [16], Hilfer [14], Erdelyi-Kober [17], Hadamard [1] and Caputo-Katugampola [18].

Recently, Almeida [2] introduced a new type of fractional derivative known as the ξ-Caputo fractional operator. This generalized operator offers substantial flexibility for capturing diverse system dynamics. Moreover, it allows for some flexibility in modeling different phenomena, depending on the choice of the function ξ. For instance, the standard Caputo derivative is recovered when $\xi(t) = t$, while the Caputo-Hadamard derivative corresponds to $\xi(t) = \ln(t)$, and the Caputo-Katugampola derivative arises from $\xi(t) = \frac{t^\sigma}{\sigma}$. This derivative has gained significant popularity in the literature. For more details on these kinds of fractional derivatives, one can refer to recent papers [3,4,5,10]. This operator involves differentiation with respect to another function and has received considerable attention for its practical applications, especially when combined with $p-$Laplacian operators. As a result, researchers have conducted numerous studies to explore the existence of solutions to equations involving this ξ-Caputo fractional derivative. For readers interested in delving deeper into this topic, relevant articles include [3,4,6,7,11,12].

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2010 *Mathematics Subject Classification:* 34A08, 34A12.
Submitted December 28, 2023. Published September 29, 2025

In 2020, Derbazi et al. [9], investigate the existence of solutions for the following fractional coupled systems in a Banach space by employing Mönch's fixed point theorem combined with the technique of measures of noncompactness.

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\alpha_1} x(t) = g_1(t, x(t), y(t)), \quad t \in \mathcal{J}, \\ {}^c\mathcal{D}_{0^+}^{\alpha_2} y(t) = g_2(t, x(t), y(t)), \quad t \in \mathcal{J}, \\ x(a) = x_a, \\ y(a) = y_a, \end{cases}$$

where $\mathcal{J} = [a, b]$, $x_a, y_a \in \mathcal{X}$, ${}^c\mathcal{D}_{0^+}^{\alpha_i}$ is Caputo fractional derivative of order $\alpha_i \in (0, 1]$, $i = 1, 2.$ and $g_i : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $i = 1, 2.$ are a given functions satisfying some assumptions.

In 2023 El Mfadel et al. [12], used the Mönch's fixed point theorem to investigate the existence of results for the following anti-periodic fractional coupled system involving p-Laplacian operator in a Banach space $\mathcal{X}$.

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\beta_1,\xi} \phi_p\big({}^c\mathcal{D}_{0^+}^{\alpha_1,\xi} x(t)\big) = g_1(t, x(t), y(t)), \quad t \in \mathcal{J}, \\ {}^c\mathcal{D}_{0^+}^{\beta_2,\xi} \phi_p\big({}^c\mathcal{D}_{0^+}^{\alpha_2,\xi} y(t)\big) = g_2(t, x(t), y(t)), \quad t \in \mathcal{J}, \\ x(0) = -x(T), \\ y(0) = -y(T), \end{cases}$$

where $\mathcal{J} = [0, T]$ such that $T > 0$, ${}^c\mathcal{D}_{0^+}^{\beta_i,\xi}$ and ${}^c\mathcal{D}_{0^+}^{\alpha_i,\xi}$ are ξ-Caputo fractional derivatives of orders β_i, $\alpha_i \in (0, 1)$ such that $1 < \alpha_i + \beta_i < 2$, $i = 1, 2$ $g_i : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $i = 1, 2.$ are Carathéodory functions and $\phi_p(.)$ is the p-Laplacian operator such that $\phi_p(s) = \|s\|^{p-2}s$ $(p > 1)$.

Inspired by the above works, we investigate the existence and uniqueness of solutions for the following implicit anti-periodic fractional coupled system via topological degree methods in a Banach space $\mathcal{X}$.

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\beta_1,\xi} \phi_p({}^c\mathcal{D}_{0^+}^{\alpha_1,\xi} x(t)) = g_1(t, x(t), y(t), {}^c\mathcal{D}_{0^+}^{\beta_1,\xi} \phi_p({}^c\mathcal{D}_{0^+}^{\alpha_1,\xi} x(t))), \quad t \in \mathcal{J}, \\ {}^c\mathcal{D}_{0^+}^{\beta_2,\xi} \phi_p({}^c\mathcal{D}_{0^+}^{\alpha_2,\xi} y(t)) = g_2(t, x(t), y(t), {}^c\mathcal{D}_{0^+}^{\beta_2,\xi} \phi_p({}^c\mathcal{D}_{0^+}^{\alpha_2 \xi} y(t))), \quad t \in \mathcal{J}, \\ x(0) = -x(T), \\ y(0) = -y(T), \end{cases} \tag{1.1}$$

where $\mathcal{J} = [0, T]$ such that $T > 0$, ${}^c\mathcal{D}_{0^+}^{\beta_i,\xi}$ and ${}^c\mathcal{D}_{0^+}^{\alpha_i,\xi}$ are ξ-Caputo fractional derivatives of orders β_i, $\alpha_i \in (0, 1)$, $i = 1, 2$ and the given $g_i : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, $i = 1, 2.$ are continuous functions that satisfy certain assumptions.
The remainder of this study is organized as follows: In Section 2, we present the basic tools of the ξ-fractional integral and the ξ-Caputo fractional derivative, which will be used throughout the following sections. Section 3 is devoted to proving the existence and uniqueness of the solutions for the coupled system (1.1) using the topological degree method. In Section 4, we provide an illustrative example to validate our theoretical findings.

2. Preliminaries

In this section, we present some of the definitions and results that are essential throughout this work. Let $C(\mathcal{J}, \mathcal{X})$ be a Banach space of all continuous functions $x : \mathcal{J} \to \mathcal{X}$ equipped with the supremum norm

$$\|x\| = \sup_{t \in \mathcal{J}} \|x(t)\|_{\mathcal{X}}.$$

$L^p(\mathcal{J}, \mathcal{X})(1 \leq p < \infty)$ be a Banach space of all Bochner integrable functions $x : \mathcal{J} \to \mathcal{X}$ with the norm

$$\|x\|_{L^p} = \Big(\int_0^t \|x(t)\|_{\mathcal{X}}^p dt \Big)^{\frac{1}{p}}.$$

We set $\mathcal{E} = \big\{(x, y) : (x, y) \in \mathcal{C}(\mathcal{J}, \mathcal{X}) \times \mathcal{C}(\mathcal{J}, \mathcal{X})\big\}$ as a Banach space with the norm

$$\|(x, y)\|_{\mathcal{E}} = \|x\| + \|y\|.$$

Definition 2.1 *[2] Let $u \in L^1(\mathcal{J},\mathbb{R})$ and $\xi \in C^1(\mathcal{J},\mathbb{R})$ with $\xi'(t) > 0$ for every $t \in \mathcal{J}$. The ξ-Riemann-Liouville fractional integral of the function u at order $\alpha > 0$ is given by*

$$\mathcal{I}_{0^+}^{\alpha,\xi}u(t) = \int_0^t \frac{(\xi(t)-\xi(s))^{\alpha-1}}{\Gamma(\alpha)}\xi'(s)u(s)ds.$$

Definition 2.2 *[2] Let $u, \xi \in C^n(\mathcal{J},\mathbb{R})$ with $\xi'(t) > 0$ for every $t \in \mathcal{J}$. The ξ-Caputo fractional derivative of the function u at order $\alpha > 0$ is given by*

$${}^C\mathcal{D}_{0^+}^{\alpha,\xi}u(t) = \int_0^t \frac{(\xi(t)-\xi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}\xi'(s)u_\xi^{[n]}(s)ds,$$

where $u_\xi^{[n]}(s) = \left(\dfrac{1}{\xi'(s)}\dfrac{d}{ds}\right)^n u(s)$ and $n = [\alpha]+1$, such that $[\alpha]$ denotes the integer part of α.

Remark 2.1 *Integrals and derivatives appearing in previous definitions are taken in Bochner sense when u is a function with values in $\mathcal{X}$.*

Proposition 2.1 *[2] For $\alpha > 0$, if $u \in \mathcal{C}^{n-1}(\mathcal{J},\mathbb{R})$, then we have*

1) ${}^c\mathcal{D}_{0^+}^{\alpha,\xi}\mathcal{I}_{0^+}^{\alpha,\xi}u(t) = u(t)$.

2) $\mathcal{I}_{0^+}^{\alpha,\xi}\ {}^c\mathcal{D}_{0^+}^{\alpha,\xi}u(t) = u(t) - \sum_{k=0}^{n-1} c_k(\xi(t)-\xi(0))^k, \quad$ *where* $c_k = \dfrac{u_\xi^{[k]}(0)}{k!}$.

Definition 2.3 *[8] Let $O \subset \mathcal{X}$ be bounded. The map $\lambda\colon O \to \mathbb{R}_+$ given by*

$$\lambda(O) = \inf\{\delta > 0 : O \subseteq \bigcup_{i=1}^n O_i \text{ and } diam(O_i) \leqslant \delta\}.$$

is called Kuratowski measure of non-compactness.

Proposition 2.2 *[8] Let $O, O_1, O_2 \subset \mathcal{X}$ be bounded. Then we have the following properties.*

1. $\lambda(O) = 0 \Leftrightarrow O$ *is relatively compact.*
2. $\lambda(\kappa O) = |\kappa|\lambda(O), \quad \kappa \in \mathbb{R}$.
3. $\lambda(O_1 + O_2) \leq \lambda(O_1) + \lambda(O_2)$.
4. $O_1 \subset O_2 \Rightarrow \lambda(O_1) \leq \lambda(O_2)$.
5. $\lambda(O_1 \cup O_2) = \max\{\lambda(O_1),\ \lambda(O_2)\}$.
6. $\lambda(O) = \lambda(\overline{O}) = \lambda(conv\, O)$ *where $\overline{O}$ and $conv\, O$ represent the closure and the convex hull of O, respectively.*

Definition 2.4 *[8] Let $\mathcal{M} : O \subset \mathcal{X} \to \mathcal{X}$ be a continuous bounded map. $\mathcal{M}$ is called*

- *λ-Lipschitz if there exists $\kappa \geq 0$ such that*

  $$\lambda(\mathcal{M}(O_1)) \leq \kappa\lambda(O_1), \quad \forall O_1 \subset O \text{ bounded.}$$

  Furthermore, $\mathcal{M}$ is called a strict λ-contraction if $\kappa < 1$.

- *λ-condensing if*

  $$\lambda(\mathcal{M}(O_1)) < \lambda(O_1), \quad \forall O_1 \subset O \text{ bounded with } \lambda(O) > 0.$$

Definition 2.5 *[8] Let $\mathcal{M}: O \subset \mathcal{X} \to \mathcal{X}$, Then $\mathcal{M}$ is called Lipschitz if there exists $\kappa > 0$ such that*

$$\|\mathcal{M}x - \mathcal{M}y\| \leq \kappa \|x - y\|, \quad \text{for all } x, y \in O.$$

Furthermore, $\mathcal{M}$ is called a strict contraction if $\kappa < 1$.

Lemma 2.1 *[8] If $\mathcal{M}_1$, $\mathcal{M}_2 : O \subset \mathcal{X} \to \mathcal{X}$ are λ-Lipschitz maps having constants κ_1, κ_2 respectively. Then the map $\mathcal{M}_1 + \mathcal{M}_2 : O \to \mathcal{X}$ is λ-Lipschitz having constant $\kappa_1 + \kappa_2$.*

Lemma 2.2 *[8] Let $\mathcal{M}: O \subset \mathcal{X} \to \mathcal{X}$, then*

- *If $\mathcal{M}$ is Lipschitz with constant κ, then $\mathcal{M}$ is λ-Lipschitz having the constant κ.*
- *If $\mathcal{M}$ is compact, then $\mathcal{M}$ is λ-Lipschitz having constant $\kappa = 0$.*

Theorem 2.1 *[15] Let $\mathcal{M}: \mathcal{X} \to \mathcal{X}$ be λ-condensing, consider the set*

$$\mathcal{S}_\epsilon = \{x \in \mathcal{X} : \text{ there exist } 0 \leq \epsilon \leq 1 \text{ such that } x = \epsilon \mathcal{M}x\}.$$

If $\mathcal{S}_\epsilon$ is a bounded in $\mathcal{X}$, then exists $\mathcal{R} > 0$ such that $\mathcal{S}_\epsilon \subset \mathcal{B}_{\mathcal{R}}(0)$ and

$$deg(I - \epsilon\mathcal{M}, \mathcal{B}_{\mathcal{R}}(0), 0) = 1, \quad \text{for all } \epsilon \in [0, 1].$$

Consequently, $\mathcal{M}$ has at least a fixed point and the fixed points set of $\mathcal{M}$ lies in $B_{\mathcal{R}}(0)$.

Lemma 2.3 *[7] Let ϕ_p be a p-Laplacian operator. Then we have*

- *If $1 < p < 2$, $\quad s_1 s_2 > 0$ and $\|s_1\|, \|s_2\| \geq m > 0$, we have*
$$\|\phi_p(s_1) - \phi_p(s_2)\| \leq (p-1)m^{p-2}\|s_1 - s_2\|.$$
- *If $p > 2$, $s_1 s_2 > 0$ and $\|s_1\|, \|s_2\| \leq M$, we have*
$$\|\phi_p(s_1) - \phi_p(s_2)\| \leq (p-1)M^{p-2}\|s_1 - s_2\|.$$
- *ϕ_p is invertible with $\phi_p^{-1}(s) = \phi_q(s)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

3. Main results

In this section, we study the existence and uniqueness of solutions of our system (1.1). For this purpose, we need to assume the following assumptions.

$(\mathcal{A}_1)$ There exist $\mathbf{l}_i > 0$, $i = 1, 2, 3$ with $\mathbf{l}_3 < 1$, such that

$$\|g_1(t, x, \bar{x}, \bar{X}) - g_1(t, y, \bar{y}, \bar{Y})\| \leq \mathbf{l}_1\|x - y\| + \mathbf{l}_2\|\bar{x} - \bar{y}\| + \mathbf{l}_3\|\bar{X} - \bar{Y}\|,$$

$$\text{for all } (x, y), (\bar{x}, \bar{y}), (\bar{X}, \bar{Y}) \in \mathcal{E} \text{ and } t \in \mathcal{J}.$$

$(\mathcal{A}_2)$ There exist $\mathbf{l}'_i > 0$, $i = 1, 2, 3$ with $\mathbf{l}'_3 < 1$, such that

$$\|g_2(t, x, \bar{x}, \bar{X}) - g_2(t, y, \bar{y}, \bar{Y})\| \leq \mathbf{l}'_1\|x - y\| + \mathbf{l}'_2\|\bar{x} - \bar{y}\| + \mathbf{l}'_3\|\bar{X} - \bar{Y}\|,$$

$$\text{for all } (x, y), (\bar{x}, \bar{y}), (\bar{X}, \bar{Y}) \in \mathcal{E} \text{ and } t \in \mathcal{J}.$$

$(\mathcal{A}_3)$ There exist $\mathbf{k}_i > 0$, $i = 1, 2, 3, 4$ with $\mathbf{k}_4 < 1$, such that

$$\|g_1(t, x, \bar{x}, \bar{X})\| \leq \mathbf{k}_1\|x\| + \mathbf{k}_2\|\bar{x}\| + \mathbf{k}_3\|\bar{X}\| + \mathbf{k}_4, \quad \text{for all } x, \bar{x}, \bar{X} \in \mathcal{C}(\mathcal{J}, \mathcal{X}) \text{ and } t \in \mathcal{J}.$$

$(\mathcal{A}_4)$ There exist $\mathbf{k}'_i > 0,\ i = 1,2,3,4$ with $\mathbf{k}'_4 < 1$, such that

$$\|g_2(t, y, \bar{y}, \bar{Y})\| \le \mathbf{k}'_1\|y\| + \mathbf{k}'_2\|\bar{y}\| + \mathbf{k}'_3\|\bar{Y}\| + \mathbf{k}'_4, \quad \text{for all } y, \bar{y}, \bar{Y} \in \mathcal{C}(\mathcal{J}, \mathcal{X}) \text{ and } t \in \mathcal{J}.$$

For brevity, we will use these notations below

$$\mathfrak{g}^1_{x,y}(t) = g_1(t, x(t), y(t), {}^c\mathcal{D}^{\beta_1,\xi}_{0^+}\phi_p({}^c\mathcal{D}^{\alpha_1,\xi}_{0^+}x(t))), \quad \mathfrak{g}^2_{x,y}(t) = g_2(t, x(t), y(t), {}^c\mathcal{D}^{\beta_2,\xi}_{0^+}\phi_p({}^c\mathcal{D}^{\alpha_2,\xi}_{0^+}y(t))),$$

$$\varpi_{\alpha_i,\beta_i} = \frac{(q-1)M_i^{q-2}(\xi(T) - \xi(0))^{\alpha_i+\beta_i}}{2\Gamma(\alpha_i + \beta_i + 1)}, \ i = 1,2.$$

$$\mathbf{L} = \max\Big\{\frac{\mathbf{l}_1\,\varpi_{\alpha_1,\beta_1}}{1-\mathbf{l}_3} + \frac{\mathbf{l}'_1\,\varpi_{\alpha_2,\beta_2}}{1-\mathbf{l}'_3}, \frac{\mathbf{l}_2\,\varpi_{\alpha_1,\beta_1}}{1-\mathbf{l}_3} + \frac{\mathbf{l}'_2\,\varpi_{\alpha_2,\beta_2}}{1-\mathbf{l}'_3}\Big\},$$

$$\mathbf{K} = \max\Big\{\frac{\mathbf{k}_1\,\varpi_{\alpha_1,\beta_1}}{1-\mathbf{k}_3} + \frac{\mathbf{k}'_1\,\varpi_{\alpha_2,\beta_2}}{1-\mathbf{k}'_3}, \frac{\mathbf{k}_2\,\varpi_{\alpha_1,\beta_1}}{1-\mathbf{k}_3} + \frac{\mathbf{k}'_2\,\varpi_{\alpha_2,\beta_2}}{1-\mathbf{k}'_3}\Big\},$$

$$\mathbf{D} = \frac{\mathbf{k}_4}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1} + \frac{\mathbf{k}'_4}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}.$$

Lemma 3.1 *Let $\beta, \alpha \in (0,1)$ and $g,\ x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$. The function x is a solution of the following problem*

$$\begin{cases} {}^c\mathcal{D}^{\beta,\xi}_{0^+}\phi_p({}^c\mathcal{D}^{\alpha,\xi}_{0^+}x(t)) = g(t), \quad t \in \mathcal{J}, \\ x(0) = -x(T), \end{cases} \tag{3.1}$$

if and only if it satisfies the following integral equation

$$x(t) = \int_0^t \frac{\xi'(s)(\xi(t) - \xi(s))^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s \xi'(r)(\xi(s) - \xi(r))^{\beta-1}g(r)dr\Big)ds$$
$$- \int_0^T \frac{\xi'(s)(\xi(T) - \xi(s))^{\alpha-1}}{2\Gamma(\alpha)}\phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s) - \xi(r))^{\beta-1}}{\Gamma(\beta)}g(r)dr\Big)ds. \tag{3.2}$$

Proof: Suppose that $x(t) \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ is a solution of the problem (3.1). Then, by applying $\mathcal{I}^{\beta;\xi}_{0^+}$ on both sides of (3.1), we get

$$\phi_p\left({}^c\mathcal{D}^{\alpha,\xi}_{0^+}x(t)\right) = c_0 + \frac{1}{\Gamma(\beta)}\int_0^t \xi'(s)(\xi(t) - \xi(s))^{\beta-1}g(s)ds, \quad \text{where } c_0 \in \mathbb{R}. \tag{3.3}$$

Since ${}^c\mathcal{D}^{\alpha;\xi}_{0^+}x(0) = 0$ it follows that $c_0 = 0$.
Now, we apply ϕ_q on both sides of the equation (3.3), we obtain

$${}^c\mathcal{D}^{\alpha,\xi}_{0^+}x(t) = \phi_q\left(\mathcal{I}^{\beta,\xi}_{0^+}g(t)\right). \tag{3.4}$$

Next, we apply the operator $\mathcal{I}^{\alpha,\xi}_{0^+}$ on both sides of (3.4), then we obtain

$$x(t) = c_1 + \frac{1}{\Gamma(\alpha)}\int_0^t \xi'(s)(\xi(t) - \xi(s))^{\alpha-1}\phi_q\left(\mathcal{I}^{\beta,\xi}_{0^+}g(s)\right)ds, \quad \text{where } c_1 \in \mathbb{R} \tag{3.5}$$

By using the antiperiodic condition $x(0) = -x(T)$, we get

$$c_1 = -\frac{1}{2\Gamma(\alpha)}\int_0^T \xi'(s)(\xi(T) - \xi(s))^{\alpha-1}\phi_q\Big(\mathcal{I}^{\beta,\xi}_{0^+}g(s)\Big)ds.$$

Substituting c_1 in (3.5), we get

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \xi'(s)(\xi(t) - \xi(s))^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s \xi'(r)(\xi(s) - \xi(r))^{\beta-1}g(r)dr\Big)ds$$
$$- \frac{1}{2\Gamma(\alpha)}\int_0^T \xi'(s)(\xi(T) - \xi(s))^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^s \xi'(r)(\xi(s) - \xi(r))^{\beta-1}g(r)dr\Big)ds.$$

Conversely, by direct computation, it is clear that if $x(t)$ satisfies the integral equation (3.2), then the equation (3.1) holds which completes the proof. □

We define the operator

$$\begin{aligned} &\mathcal{F} : \mathcal{E} \to \mathcal{E} \\ &\quad \mathcal{F} = \mathcal{G} + \mathcal{Q} = (\mathcal{G}_1, \mathcal{G}_2) + (\mathcal{Q}_1, \mathcal{Q}_2), \end{aligned}$$

such that

$$\begin{aligned} &\mathcal{G}_i : \mathcal{E} \to \mathcal{C}(\mathcal{J}, \mathcal{X}) \\ &\quad \mathcal{G}_i(x,y)(t) = -\int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_i-1}}{2\Gamma(\alpha_i)} \phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_i-1}}{\Gamma(\beta_i)} \mathfrak{g}^i_{x,y}(r)dr\Big)ds, \quad i=1,2 \end{aligned}$$

and

$$\begin{aligned} &\mathcal{Q}_i : \mathcal{E} \to \mathcal{C}(\mathcal{J}, \mathcal{X}) \\ &\quad \mathcal{Q}_i(x,y)(t) = \int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_i-1}}{\Gamma(\beta_i)} \mathfrak{g}^i_{x,y}(r)dr\Big)ds, \quad i=1,2. \end{aligned}$$

By using Lemma 3.1, we can conclude that $(x,y) \in \mathcal{E}$ is a solution of (1.1) if and only if it satisfies

$$(x,y) = \mathcal{F}(x,y). \tag{3.6}$$

Therefore, it is easy to see that proving the problem (1.1) has at least one solution is equivalent to proving that the problem (3.6) has at least one solution $(x,y) \in \mathcal{E}$.

Lemma 3.2 *The operator $\mathcal{G}$ is λ-Lipschitz with constant $\mathbf{L}$. Additionally, $\mathcal{G}$ satisfies the following growth condition.*

$$\|\mathcal{G}(x,y)\|_{\mathcal{E}} \leq \mathbf{K}\|(x,y)\|_{\mathcal{E}} + \mathbf{D}. \tag{3.7}$$

Proof: Let (x,y), $(\bar{x},\bar{y}) \in \mathcal{E}$. Then, we have

$$\begin{aligned} \|\mathcal{G}(x,y) - \mathcal{G}(\bar{x},\bar{y})\|_{\mathcal{E}} &= \|\mathcal{G}_1(x,y) - \mathcal{G}_1(\bar{x},\bar{y})\| + \|\mathcal{G}_2(x,y) - \mathcal{G}_2(\bar{x},\bar{y})\| \\ &\leq \int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_1-1}}{2\Gamma(\alpha_1)} \phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_1-1}}{\Gamma(\beta_1)} \|\mathfrak{g}^1_{x,y}(r) - \mathfrak{g}^1_{\bar{x},\bar{y}}(r)\|dr\Big)ds \\ &+ \int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_2-1}}{2\Gamma(\alpha_2)} \phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_2-1}}{\Gamma(\beta_2)} \|\mathfrak{g}^2_{x,y}(r) - \mathfrak{g}^2_{\bar{x},\bar{y}}(r)\|dr\Big)ds. \end{aligned}$$

By Lemma 2.3 and using the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we obtain

$$\begin{aligned} \|\mathcal{G}(x,y) - \mathcal{G}(\bar{x},\bar{y})\|_{\mathcal{E}} &\leq (q-1)M_1^{q-2}\Big(\frac{\mathbf{l}_1}{1-\mathbf{l}_3}\|x-\bar{x}\| + \frac{\mathbf{l}_2}{1-\mathbf{l}_3}\|y-\bar{y}\|\Big) \\ &\times \Big(\int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_1-1}}{2\Gamma(\alpha_1)} \Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_1-1}}{\Gamma(\beta_1)} dr\Big)ds\Big) \\ &+ (q-1)M_2^{q-2}\Big(\frac{\mathbf{l}'_1}{1-\mathbf{l}'_3}\|x-\bar{x}\| + \frac{\mathbf{l}'_2}{1-\mathbf{l}'_3}\|y-\bar{y}\|\Big) \\ &\times \Big(\int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_2-1}}{2\Gamma(\alpha_2)} (\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_2-1}}{\Gamma(\beta_2)} dr)ds\Big) \\ &\leq (\frac{\mathbf{l}_1\,\varpi_{\alpha_1,\beta_1}}{1-\mathbf{l}_3} + \frac{\mathbf{l}'_1\,\varpi_{\alpha_2,\beta_2}}{1-\mathbf{l}'_3})\|x-\bar{x}\| + (\frac{\mathbf{l}_2\,\varpi_{\alpha_1,\beta_1}}{1-\mathbf{l}_3} + \frac{\mathbf{l}'_2\,\varpi_{\alpha_2,\beta_2}}{1-\mathbf{l}'_3})\|y-\bar{y}\|. \end{aligned}$$

Therefore

$$\|\mathcal{G}(x,y) - \mathcal{G}(\bar{x},\bar{y})\|_{\mathcal{E}} \leq \mathbf{L}\|(x,y) - (\bar{x},\bar{y})\|_{\mathcal{E}}.$$

Then $\mathcal{G}$ is lipshitz having constant $\mathbf{L}$.
Now, let us show the growth condition 3.7, by Lemma 2.3 and $(\mathcal{A}_3)$, we have

$$\begin{aligned}
\|\mathcal{G}_1(x,y)\| &\leq (q-1)M_1^{q-2}\Big(\frac{\mathbf{k}_1}{1-\mathbf{k}_3}\|x\| + \frac{\mathbf{k}_2}{1-\mathbf{k}_3}\|y\| + \frac{\mathbf{k}_4}{1-\mathbf{k}_3}\Big) \\
&\quad \times \left(\int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_1-1}}{2\Gamma(\alpha_1)}\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_1-1}}{\Gamma(\beta_1)}dr\Big)ds\right) \\
&\leq \Big(\frac{\mathbf{k}_1}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1}\Big)\|x\| + \Big(\frac{\mathbf{k}_2}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1}\Big)\|y\| + \Big(\frac{\mathbf{k}_4}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1}\Big).
\end{aligned} \tag{3.8}$$

Similarly, By using Lemma 2.3 and $(\mathcal{A}_4)$, we obtain

$$\begin{aligned}
\|\mathcal{G}_2(x,y)\| &\leq (q-1)M_2^{q-2}\Big(\frac{\mathbf{k}'_1}{1-\mathbf{k}'_3}\|x\| + \frac{\mathbf{k}'_2}{1-\mathbf{k}'_3}\|y\| + \frac{\mathbf{k}'_4}{1-\mathbf{k}'_3}\Big) \\
&\quad \times \left(\int_0^T \frac{\xi'(s)(\xi(T)-\xi(s))^{\alpha_2-1}}{2\Gamma(\alpha_2)}\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_2-1}}{\Gamma(\beta_2)}dr\Big)ds\right) \\
&\leq \Big(\frac{\mathbf{k}'_1}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}\Big)\|x\| + \Big(\frac{\mathbf{k}'_2}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}\Big)\|y\| + \frac{\mathbf{k}'_4}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}.
\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\|\mathcal{G}(x,y)\|_{\mathcal{E}} = \|\mathcal{G}_1(x,y)\| + \|\mathcal{G}_2(x,y)\| \leq \mathbf{K}\|(x,y)\|_{\mathcal{E}} + \mathbf{D}.$$

□

Lemma 3.3 *The operator $\mathcal{Q}$ is continuous. Furthermore, $\mathcal{Q}$ satisfies the following growth condition*

$$\|\mathcal{Q}(x,y)\|_{\mathcal{E}} \leq 2(\mathbf{K}\|(x,y)\|_{\mathcal{E}} + \mathbf{D}). \tag{3.10}$$

Proof: Consider a sequence $(x_n,y_n) \in B_{\mathcal{R}} := \{(x,y) \in \mathcal{E} : \|(x,y)\|_{\mathcal{E}} \leq \mathcal{R}\}$ that converges to $(x,y) \in B_{\mathcal{R}}$.
Then, we get $\|\mathfrak{g}^1_{x_n,y_n}(t) - \mathfrak{g}^1_{x,y}(t)\|_{\mathcal{E}} \to 0$ as $n \to \infty$.
By using Lemma 2.3, we have

$$\begin{aligned}
\|\mathcal{Q}_1(x_n,y_n)(t) - \mathcal{Q}_1(x,y)(t)\| &= \left\|\mathcal{I}_{0^+}^{\alpha_1,\xi}\phi_q\big(\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x_n,y_n}(t)\big) - \mathcal{I}_{0^+}^{\alpha_1,\xi}\phi_q\big(\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(t)\big)\right\| \\
&\leq (q-1)M_1^{q-2}\mathcal{I}_{0^+}^{\alpha_1,\xi}\Big(\mathcal{I}_{0^+}^{\beta_1,\xi}\|\mathfrak{g}^1_{x_n,y_n}(t) - \mathfrak{g}^1_{x,y}(t)\|\Big).
\end{aligned}$$

By the assumption $(\mathcal{A}_3)$, we get

$$\begin{aligned}
\frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_1-1}(\xi(s)-\xi(0))^{\beta_1}}{\Gamma(\alpha_1)\Gamma(\beta_1+1)}\|\mathfrak{g}^1_{x_n,y_n}(t) - \mathfrak{g}^1_{x,y}(t)\| &\leq \left(2\Big(\frac{\mathbf{k}_1+\mathbf{k}_2}{1-\mathbf{k}_3}\Big)\mathcal{R} + \frac{\mathbf{k}_4}{1-\mathbf{k}_3}\right) \\
&\quad \times \frac{\xi'(s)\Big(\xi(t)-\xi(s))^{\alpha_1-1}(\xi(s)-\xi(0)\Big)^{\beta_1}}{\Gamma(\beta_1+1)\Gamma(\alpha_1)}.
\end{aligned}$$

Then the function $s \mapsto \dfrac{\xi'(s)\Big(\xi(t)-\xi(s))^{\alpha_1-1}(\xi(s)-\xi(0)\Big)^{\beta_1}}{\Gamma(\alpha_1)\Gamma(\beta_1+1)}$ is Lebesgue integrable over $[0,t]$.
Since ξ is continuous, using the Lebesgue Dominated Convergence Theorem, we deduce that.

$$\left\|\mathcal{I}_{0^+}^{\alpha_1,\xi}\phi_q\big(\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x_n,y_n}(t)\big) - \mathcal{I}_{0^+}^{\alpha_1,\xi}\phi_q\big(\mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(t)\big)\right\| \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\|\mathcal{Q}_1(x_n, y_n) - \mathcal{Q}_1(x, y)\|_{\mathcal{E}} \to 0 \quad \text{as } n \to \infty, \tag{3.11}$$

Similarly, by using Lemma 2.3 and $(\mathcal{A}_4)$, we get

$$\|\mathcal{Q}_2(x_n, y_n) - \mathcal{Q}_2(x, y)\|_{\mathcal{E}} \to 0 \quad \text{as } n \to \infty, \tag{3.12}$$

by combing (3.11) and (3.12), we have

$$\|\mathcal{Q}(x_n, y_n) - \mathcal{Q}(x, y)\|_{\mathcal{E}} = \|\mathcal{Q}_1(x_n, y_n) - \mathcal{Q}_1(x, y)\| + \|\mathcal{Q}_2(x_n, y_n) - \mathcal{Q}_2(x, y)\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, the operator $\mathcal{Q}$ is a continuous.
Now let us show the growth condition 3.10, by using the lemma 2.3 and $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}
\|\mathcal{Q}_1(x, y)\| &\leq (q-1)M_1^{q-2}\Big(\frac{\mathbf{k}_1}{1-\mathbf{k}_3}\|x\| + \frac{\mathbf{k}_2}{1-\mathbf{k}_3}\|y\| + \frac{\mathbf{k}_4}{1-\mathbf{k}_3}\Big) \\
&\quad \times \left(\int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_1-1}}{\Gamma(\beta_1)} dr\Big) ds\right) \\
&\leq \Big(\frac{2\mathbf{k}_1}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1}\Big)\|x\| + \Big(\frac{2\mathbf{k}_2}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1}\Big)\|y\| + \Big(\frac{2\mathbf{k}_4}{1-\mathbf{k}_3}\varpi_{\alpha_1,\beta_1}\Big).
\end{aligned}$$

In the same way, using lemma 2.3 and $(\mathcal{A}_4)$, we conclude that

$$\begin{aligned}
\|\mathcal{Q}_2(x, y)\| &\leq (q-1)M_2^{q-2}\Big(\frac{\mathbf{k}'_1}{1-\mathbf{k}'_3}\|x\| + \frac{\mathbf{k}'_2}{1-\mathbf{k}'_3}\|y\| + \frac{\mathbf{k}'_4}{1-\mathbf{k}'_3}\Big) \\
&\quad \times \left(\int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_2-1}}{\Gamma(\beta_2)} dr\Big) ds\right) \\
&\leq \Big(\frac{2\mathbf{k}'_1}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}\Big)\|x\| + \Big(\frac{2\mathbf{k}'_2}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}\Big)\|y\| + \Big(\frac{2\mathbf{k}'_4}{1-\mathbf{k}'_3}\varpi_{\alpha_2,\beta_2}\Big).
\end{aligned}$$

Hence, we have

$$\|\mathcal{Q}(x, y)\|_{\mathcal{E}} = \|\mathcal{Q}_1(x, y)\| + \|\mathcal{Q}_2(x, y)\| \leq 2(\mathbf{K}\|(x, y)\|_{\mathcal{E}} + \mathbf{D}).$$

□

Lemma 3.4 *The operator $\mathcal{Q}$ is compact. Consequently $\mathcal{Q}$ is λ-Lipschitz with zero constant.*

Let $(x, y) \in B_{\mathcal{R}}$ and $0 \leqslant t_1 < t_2 \leqslant T$, then we have

$$\begin{aligned}
\|\mathcal{Q}_1(x, y)(t_2) - \mathcal{Q}_1(x, y)(t_1)\| &= \int_0^{t_2} \frac{\xi'(s)(\xi(t_2)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\phi_q \mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)ds \\
&\quad - \int_0^{t_1} \frac{\xi'(s)(\xi(t_1)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\phi_q \mathcal{I}_{0^+}^{\beta_1,\xi_1}\mathfrak{g}^1_{x,y}(s)ds \\
&= \int_0^{t_1} \frac{\xi'(s)\Big[(\xi(t_2)-\xi(s))^{\alpha_1-1} - (\xi(t_1)-\xi(s))^{\alpha_1-1}\Big]}{\Gamma(\beta_1)}\phi_q \mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)ds \\
&\quad + \int_{t_1}^{t_2} \frac{\xi'(s)(\xi(t_2)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\phi_q \mathcal{I}_{0^+}^{\beta_1,\xi}\mathfrak{g}^1_{x,y}(s)ds.
\end{aligned}$$

With Lemma 2.3 and $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}\|\mathcal{Q}_1(x,y)(t_2)-\mathcal{Q}_1(x,y)(t_1)\| &\leq 2(q-1)M_1^{q-2}\Big(\big(\frac{\mathbf{k}_1+\mathbf{k}_2}{1-\mathbf{k}_3}\big)\mathcal{R}+\frac{\mathbf{k}_4}{1-\mathbf{k}_3}\Big)\frac{(\xi(T)-\xi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\\ &\times\Big(\int_0^{t_1}\frac{\xi'(s)\Big((\xi(t_2)-\xi(s))^{\alpha_1-1}-(\xi(t_1)-\xi(s))^{\alpha_1-1}\Big)}{\Gamma(\alpha_1)}+\int_{t_1}^{t_2}\frac{\xi'(s)(\xi(t_2)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\Big)\\ &\leq 2(q-1)M_1^{q-2}\Big(\Big(\frac{\mathbf{k}_1+\mathbf{k}_2}{1-\mathbf{k}_3}\Big)\mathcal{R}+\frac{\mathbf{k}_4}{1-\mathbf{k}_3}\Big)\frac{(\xi(T)-\xi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\beta_1+1)}\\ &\times\Big((\xi(t_1)-\xi(0))^{\beta_1}+2(\xi(t_2)-\xi(t_1))^{\beta_1}-(\xi(t_2)-\xi(0))^{\beta_1}\Big).\end{aligned}$$

Since, ξ is a continuous function, we get

$$\lim_{t_1\to t_2}\|\mathcal{Q}_1(x,y)(t_2)-\mathcal{Q}_1(x,y)(t_1)\|=0. \tag{3.13}$$

Similarly, by using lemma 2.3 and $(\mathcal{A}_4)$, we obtain

$$\lim_{t_1\to t_2}\|\mathcal{Q}_2(x,y)(t_2)-\mathcal{Q}_2(x,y)(t_1)\|=0. \tag{3.14}$$

From (3.13) and (3.14), we can infer

$$\|\mathcal{Q}(x,y)(t_2)-\mathcal{Q}(x,y)(t_1)\|_{\mathcal{E}}\to 0 \text{ as } t_1\to t_2.$$

Hence, $\mathcal{Q}(B_{\mathcal{R}})$ is equicontinuous.

As a result of the Arzelà-Ascoli Theorem [13], we conclude that $\mathcal{Q}(\mathcal{B}_{\mathcal{R}})$ is relatively compact. Therefore, $\mathcal{Q}$ is compact. By lemma 2.2 we deduce that $\mathcal{Q}$ is λ-Lipschitz with constant zero.

Theorem 3.1 *Suppose that the assumptions* $(\mathcal{A}_1)-(\mathcal{A}_4)$ *are satisfied, then the problem (1.1) has at least a solution, provided that* $\max(3\mathbf{K},\mathbf{L})$. *Furthermore, the solution set of (1.1) is bounded in* $\mathcal{E}$.

Proof: We know that the operators $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{F}$ are continuous as well as bounded, Furthermore $\mathcal{G}$ is λ-Lipschitz having constant $\mathbf{L}$ and $\mathcal{Q}$ is λ-Lipschitz having zero constant. By using Lemmas 2.1 and 2.2 we can deduce that the operator $\mathcal{F}$ is λ-condensing.
Now, consider the set

$$\mathcal{S}_\epsilon=\Big\{(x,y)\in\mathcal{E}:\ \text{there exist } 0\leq\epsilon\leq 1 \text{ such that } (x,y)=\epsilon\mathcal{F}(x,y)\Big\}.$$

Consider $(x,y)\in\mathcal{S}_\epsilon$, then we have

$$\|(x,y)\|_{\mathcal{E}}=\|\epsilon\mathcal{F}(x,y)\|_{\mathcal{E}}\leq\|\mathcal{G}(x,y)\|_{\mathcal{E}}+\|\mathcal{Q}(x,y)\|_{\mathcal{E}}.$$

By Lemmas 3.2 and 3.3, we get

$$\|(x,y)\|_{\mathcal{E}}\leq 3(\mathbf{K}\|(x,y)\|_{\mathcal{E}}+\mathbf{D}).$$

Therefore, $\mathcal{S}_\epsilon$ is bounded in $\mathcal{E}$. As a result of Theorem 2.1 we conclude that the operator $\mathcal{F}$ has at least one fixed point, which is a solution of the problem (1.1). Furthermore, the set of solutions is bounded in $\mathcal{E}$. □

Theorem 3.2 *Suppose that* $(\mathcal{A}_1)-(\mathcal{A}_4)$ *are satisfied, then the problem* (1.1) *has a unique solution provided that*

$$\mathbf{L}<\frac{1}{3}. \tag{3.15}$$

Proof: Let (x,y), $(\bar{x},\bar{y})\in\mathcal{E}$, then we have

$$\begin{aligned}\|\mathcal{Q}(x,y)-\mathcal{Q}(\bar{x},\bar{y})\|_{\mathcal{E}} &= \|\mathcal{Q}_1(x,y)-\mathcal{Q}_1(\bar{x},\bar{y})\|+\|\mathcal{Q}_2(x,y)-\mathcal{Q}_2(\bar{x},\bar{y})\| \\ &\le \int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_1-1}}{\Gamma(\beta_1)}\|\mathfrak{g}^1_{x,y}(r)-\mathfrak{g}^1_{\bar{x},\bar{y}}(r)\|dr\Big)ds \\ &+ \int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\phi_q\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_2-1}}{\Gamma(\beta_2)}\|\mathfrak{g}^2_{x,y}(r)-\mathfrak{g}^2_{\bar{x},\bar{y}}(r)\|dr\Big)ds.\end{aligned}$$

Then by using Lemma 2.3 with $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, we get

$$\begin{aligned}\|\mathcal{Q}(x,y)-\mathcal{Q}(\bar{x},\bar{y})\|_{\mathcal{E}} &\le (q-1)M_1^{q-2}\Big(\frac{\mathbf{l}_1}{1-\mathbf{l}_3}\|x-\bar{x}\|+\frac{\mathbf{l}_2}{1-\mathbf{l}_3}\|y-\bar{y}\|\Big) \\ &\times\left(\int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_1-1}}{\Gamma(\alpha_1)}\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_1-1}}{\Gamma(\beta_1)}dr\Big)ds\right) \\ &+(q-1)M_2^{q-2}\Big(\frac{\mathbf{l}'_1}{1-\mathbf{l}'_3}\|x-\bar{x}\|+\frac{\mathbf{l}'_2}{1-\mathbf{l}'_3}\|y-\bar{y}\|\Big) \\ &\times\left(\int_0^t \frac{\xi'(s)(\xi(t)-\xi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\Big(\int_0^s \frac{\xi'(r)(\xi(s)-\xi(r))^{\beta_2-1}}{\Gamma(\beta_2)}dr\Big)ds\right) \\ &\le 2\Big(\frac{\mathbf{l}_1\varpi_{\alpha_1,\beta_1}}{1-\mathbf{l}_3}+\frac{\mathbf{l}'_1\varpi_{\alpha_2,\beta_2}}{1-\mathbf{l}'_3}\Big)\|x-\bar{x}\|+2\Big(\frac{\mathbf{l}_2\varpi_{\alpha_1,\beta_1}}{1-\mathbf{l}_3}+\frac{\mathbf{l}'_2\varpi_{\alpha_2,\beta_2}}{1-\mathbf{l}'_3}\Big)\|y-\bar{y}\|.\end{aligned}$$

It follows that

$$\|\mathcal{Q}(x,y)-\mathcal{Q}(\bar{x},\bar{y})\|_{\mathcal{E}}\le 2\mathbf{L}\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{E}}.$$

Hence, we get

$$\begin{aligned}\|\mathcal{F}(x,y)-\mathcal{F}(\bar{x},\bar{y})\|_{\mathcal{E}} &= \|\mathcal{G}(x,y)-\mathcal{G}(\bar{x},\bar{y})\|+\|\mathcal{Q}(x,y)-\mathcal{Q}(\bar{x},\bar{y})\|_{\mathcal{E}} \\ &\le \mathbf{L}\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{E}}+2\mathbf{L}\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{E}} \\ &\le 3\mathbf{L}\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{E}}.\end{aligned}$$

As a result of the Banach fixed point theorem, $\mathcal{F}$ has a unique fixed point, which is the unique solution to the problem (1.1). □

4. An illustrative example

In this section, we present an illustrative example, we consider the following coupled system of ξ-Caputo fractional differential equations

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\frac{1}{2},\sqrt{t+1}}\phi_2\big({}^c\mathcal{D}_{0^+}^{\frac{1}{3},\sqrt{t+1}}x(t)\big)=g_1\big(t,x(t),y(t),{}^c\mathcal{D}_{0^+}^{\frac{1}{2},\sqrt{t+1}}\phi_2({}^c\mathcal{D}_{0^+}^{\frac{1}{3},\sqrt{t+1}}x(t))\big), & t\in[0,1], \\ {}^c\mathcal{D}_{0^+}^{\frac{1}{4},\sqrt{t+1}}\phi_2\big({}^c\mathcal{D}_{0^+}^{\frac{2}{3},\sqrt{t+1}}y(t)\big)=g_2\big(t,x(t),y(t),{}^c\mathcal{D}_{0^+}^{\frac{1}{4},\sqrt{t+1}}\phi_2({}^c\mathcal{D}_{0^+}^{\frac{2}{3},\sqrt{t+1}}y(t))\big), & t\in[0,1], \\ x(0)=-x(1), \\ y(0)=-y(1), \end{cases} \tag{4.1}$$

such that

$$g_1\big(t,x,y,{}^c\mathcal{D}_{0^+}^{\frac{1}{2},\sqrt{t+1}}\phi_2({}^c\mathcal{D}_{0^+}^{\frac{1}{3},\sqrt{t+1}}x(t))\big)=\frac{e^{-t}|x(t)|}{19+e^t}+(10+t^2)^{-1}\Big(|y(t)|+|{}^c\mathcal{D}_{0^+}^{\frac{1}{2},\sqrt{t+1}}\phi_2({}^c\mathcal{D}_{0^+}^{\frac{1}{3},\sqrt{t+1}}x(t))|\Big),$$

and

$$g_2\big(t,x,y,{}^c\mathcal{D}_{0^+}^{\frac{1}{4},\sqrt{t+1}}\phi_2({}^c\mathcal{D}_{0^+}^{\frac{2}{3},\sqrt{t+1}}y(t))\big)=\frac{e^{-t}|x(t)|}{13e^{-t}+1}+(7+t^2)^{-1}|y(t)|+\frac{3}{22-t^2}|{}^c\mathcal{D}_{0^+}^{\frac{1}{4},\sqrt{t+1}}\phi_2({}^c\mathcal{D}_{0^+}^{\frac{2}{3},\sqrt{t+1}}y(t))|.$$

It is easy to see that the system (4.1) is a special case of the problem (1.1) when $T = 1$, $\alpha_1 = \dfrac{1}{2}$, $\alpha_2 = \dfrac{1}{4}$, $\beta_1 = \dfrac{1}{3}$, $\beta_2 = \dfrac{2}{3}$, $\xi(t) = \sqrt{t+1}$ and $p = q = 2$. In what follows, we can easily verify that

$$\mathbf{l}_1 = \mathbf{k}_1 = \frac{1}{20},\ \mathbf{l}_2 = \mathbf{l}_3 = \mathbf{k}_2 = \mathbf{k}_3 = \frac{1}{10},\ \mathbf{l}'_1 = \mathbf{k}'_1 = \frac{1}{14},\ \mathbf{l}'_2 = \mathbf{l}'_3 = \mathbf{k}'_2 = \mathbf{k}'_3 = \frac{1}{7} \text{ and } \mathbf{k}_4 = \mathbf{k}'_4 = 0.$$

It follows that

$$\varpi_{\alpha_1,\beta_1} = \frac{(\sqrt{2}-1)^{\frac{5}{6}}}{2\Gamma(\frac{11}{6})} \text{ and } \varpi_{\alpha_2,\beta_2} = \frac{(\sqrt{2}-1)^{\frac{11}{12}}}{2\Gamma(\frac{23}{12})}.$$

Therefore

$$\mathbf{L} = 0.0295 < \frac{1}{3}.$$

By virtue of Theorem 3.2, the system (4.1) has a unique solution.

5. Conclusion

In this study, we investigated the existence and uniqueness of solutions for an implicit coupled system of ξ-Caputo fractional differential equations involving the p-Laplacian operator under anti-periodic boundary conditions in a general Banach space. The existence results were derived using a fixed point theorem introduced by Isaia [15], which combines the coincidence degree theory for condensing maps with the Banach contraction principle. A detailed example is provided to demonstrate the practical relevance and validity of our theoretical results. These results contribute to the growing body of research on fractional differential equations and their applications to complex nonlinear systems.

Acknowledgments

The authors are grateful to the referee for his/her valuable remarks that led to the improvement of the original manuscript.

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