



New Congruences for 2-Color Partitions and t -Core Partitions

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ABSTRACT: Let $c_N(n)$ denote the number of 2-color partition of n subject to the restriction that one of the colors appears only in parts that are divisible by N . If t is a positive integer, then a partition of a nonnegative integer n is a t -core if none of the hook numbers of the associated Ferrers-Young diagram is a multiple of t . Let $a_t(n)$ denote the number of t -core partitions of n . In this paper, we obtain new congruences modulo 3 for the 2-color partition function $c_{11}(n)$, t -core partition functions $a_5(n)$ and $a_{11}(n)$.

Key Words: Partitions, 2-Color Partitions, t -Core Partitions, Congruences.

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1. Introduction

A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$. The partition function $p(n)$ is the number of partitions of n . The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}} = \frac{1}{f_1},$$

where $(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k)$, in short we use,

$$f_k := (q^k; q^k)_{\infty} = \prod_{m=1}^{\infty} (1 - q^{mk}), \quad k \in \mathbb{N}.$$

Let $c_N(n)$ counts the number of 2-color partitions of n with colors r and b subject to the restriction that the color b appears only in parts that are divisible by N . For example, $c_2(2) = 3$, the three partitions of 2 it enumerates are $2_r, 2_b, 1_r + 1_r$. In [1], Chan studied the partition function $c_2(n)$,

$$\sum_{n=0}^{\infty} c_2(n)q^n := \frac{1}{f_1 f_2}, \tag{1.1}$$

Further, Chan [1] showed that $c_2(n)$ satisfies

$$\sum_{n=0}^{\infty} c_2(3n+2)q^n = 3 \frac{f_3^3 f_6^3}{f_1^4 f_2^4}, \tag{1.2}$$

which is analogues to Ramanujan's identity [7],

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \frac{f_5^5}{f_1^6}. \quad (1.3)$$

The identity (1.2) immediately leads to

$$c_2(3n+2) \equiv 0 \pmod{3}. \quad (1.4)$$

In [8], Sinick defined the generating function of $c_N(n)$,

$$\sum_{n=0}^{\infty} c_N(n)q^n = \frac{1}{f_1 f_N}, \quad N > 1 \quad (1.5)$$

and proved that $c_N(n)$ obey no Ramanujan-type congruences except (1.4). Recently, arithmetic properties of the partition function $c_N(n)$ have been studied by many authors (See [3,6]).

If $\Lambda = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ is a partition of n , then the corresponding Ferrers-Young diagram of Λ is the following staircase arrangement of nodes with λ_k nodes in the k -th row:

$$\begin{array}{ccccccc} \bullet & \bullet & \dots & \bullet & \bullet & & \lambda_1 \text{ nodes} \\ \bullet & \bullet & \dots & \bullet & & & \lambda_2 \text{ nodes} \\ \vdots & & & & & & \\ \bullet & \dots & \bullet & & & & \lambda_k \text{ nodes} \end{array}$$

Label the nodes (i, j) , as if they are matrix entries. The (i, j) hook is the set of nodes directly below, together with the set of nodes directly to the right of the (i, j) nodes, as well as the (i, j) node itself. If λ'_j denotes the number of nodes in column j , the hook number $H(i, j)$ of the (i, j) node is defined by $H(i, j) := \lambda_i + \lambda'_j - i - j + 1$.

For $t > 0$, a partition of a nonnegative integer n is a t -core if none of the hook numbers of the associated Ferrers-Young diagram is a multiple of t . The function $a_t(n)$ is the number of t -core partitions of n . Garvan et al. [4] showed,

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_{\infty}^t}{(q; q)_{\infty}} = \frac{f_t^t}{f_1}. \quad (1.6)$$

These partitions have been studied extensively by many mathematicians and play important roles in the study of irreducible representations of the symmetric group S_n . In this paper, employing q -series identities we obtain the following congruences:

Theorem 1.1 *If $\ell \in \{3, 6, 8, 9, 10\}$ then for all $\alpha, n \geq 0$,*

$$c_{11} \left(3 \cdot 11^{\alpha} (11n + \ell) + \frac{3 \cdot 11^{\alpha} + 1}{2} \right) \equiv 0 \pmod{3}. \quad (1.7)$$

Theorem 1.2 *Let $p \geq 5$ be prime, $\left(\frac{-11}{p}\right) = -1$, then for all $\alpha, n \geq 0$,*

$$c_{11} \left(3p^{2\alpha+1}(pn + j) + \frac{3p^{2\alpha+2} + 1}{2} \right) \equiv 0 \pmod{3}, \quad j = 1, 2, \dots, p-1. \quad (1.8)$$

Theorem 1.3 *For all $n \geq 0, k \geq 1$,*

$$a_5(15^k n + 15^k - 1) \equiv a_5(n) \pmod{3}. \quad (1.9)$$

Theorem 1.4 For all $k, n \geq 0$,

$$a_5 (15^{k+1}n + 7 \cdot 15^k - 1) \equiv 0 \pmod{3}, \quad (1.10)$$

$$a_5 (15^{k+1}n + 13 \cdot 15^k - 1) \equiv 0 \pmod{3}, \quad (1.11)$$

$$a_5 (15^{k+1}n + 11 \cdot 15^k - 1) \equiv 0 \pmod{3}, \quad (1.12)$$

$$a_5 (15^{k+1}n + 14 \cdot 15^k - 1) \equiv 0 \pmod{3}. \quad (1.13)$$

Theorem 1.5 If $\ell \in \{3, 6, 8, 9, 10\}$ then for all $n \geq 0$,

$$a_{11}(3(11n + \ell) + 2) \equiv 0 \pmod{3}. \quad (1.14)$$

In Section 3, we prove Theorem 1.1 and Theorem 1.2. In Section 4, we prove Theorems 1.3 to 1.5.

2. Preliminaries

We require the following lemmas to prove the main results in the next two sections.

Lemma 2.1 (H.H. Chan and Toh [2, Theorem 1, Eqn. (1.19)])

We have,

$$\sum_{n=0}^{\infty} p_{[1^1 11^1]}(3n+2)q^n = 2 \frac{f_3 f_{33}}{f_1^2 f_{11}^2} + 3q \frac{f_3^2 f_{33}}{f_1^3 f_{11}^3} + 3q^2 \frac{f_3^3 f_{33}}{f_1^4 f_{11}^4}, \quad (2.1)$$

$$\text{where } \sum_{n=0}^{\infty} p_{[1^1 11^1]}(n)q^n = \frac{1}{f_1 f_{11}}.$$

Lemma 2.2 (Hirschhorn [5, p. 103, Eqn. 10.6.1])

We have the following 11-dissection of f_1 ,

$$f_1 = f_{121} (A(q^{11}) - qB(q^{11}) - q^2C(q^{11}) + q^5 + q^7D(q^{11}) - q^4E(q^{11})), \quad (2.2)$$

where

$$\begin{aligned} A(q) &= \frac{(q^4; q^{11})_{\infty} (q^7; q^{11})_{\infty}}{(q^2; q^{11})_{\infty} (q^9; q^{11})_{\infty}}, & B(q) &= \frac{(q^2; q^{11})_{\infty} (q^9; q^{11})_{\infty}}{(q; q^{11})_{\infty} (q^{10}; q^{11})_{\infty}}, \\ C(q) &= \frac{(q^5; q^{11})_{\infty} (q^6; q^{11})_{\infty}}{(q^3; q^{11})_{\infty} (q^8; q^{11})_{\infty}}, & D(q) &= \frac{(q^3; q^{11})_{\infty} (q^8; q^{11})_{\infty}}{(q^4; q^{11})_{\infty} (q^7; q^{11})_{\infty}}, \\ E(q) &= q \frac{(q; q^{11})_{\infty} (q^{10}; q^{11})_{\infty}}{(q^5; q^{11})_{\infty} (q^6; q^{11})_{\infty}}. \end{aligned}$$

Lemma 2.3 (Liu and Wang [6, Theorem 5])

We have,

$$\sum_{n=0}^{\infty} p_{[1^1 5^1]}(3n)q^n \equiv \frac{f_3}{f_5} \pmod{3}, \quad (2.3)$$

$$\sum_{n=0}^{\infty} p_{[1^1 5^1]}(3n+1)q^n \equiv f_1 f_5 \pmod{3}, \quad (2.4)$$

$$\sum_{n=0}^{\infty} p_{[1^1 5^1]}(3n+2)q^n \equiv -\frac{f_{15}}{f_1} \pmod{3}, \quad (2.5)$$

$$\text{where } \sum_{n=0}^{\infty} p_{[1^1 5^1]}(n)q^n = \frac{1}{f_1 f_5}.$$

From the Binomial Theorem, for any positive integer, k ,

$$f_k^3 \equiv f_{3k} \pmod{3}. \quad (2.6)$$

3. Congruences modulo 3 for $c_{11}(n)$

In this section, we prove Theorem 1.1 and Theorem 1.2.

Theorem 3.1 For all $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} c_{11} \left(3 \cdot 11^\alpha n + \frac{3 \cdot 11^\alpha + 1}{2} \right) q^n \equiv f_1 f_{11} \pmod{3}. \quad (3.1)$$

Proof: From (1.5), we have

$$\sum_{n=0}^{\infty} c_{11}(n)q^n = \frac{1}{f_1 f_{11}}. \quad (3.2)$$

Using (2.1), we have

$$\sum_{n=0}^{\infty} c_{11}(3n+2)q^n = \sum_{n=0}^{\infty} p_{[1^{1+11^1}]}(3n+2)q^n = 2 \frac{f_3 f_{33}}{f_1^2 f_{11}^2} + 3q \frac{f_3^2 f_{33}^2}{f_1^3 f_{11}^3} + 3q^2 \frac{f_3^3 f_{33}^3}{f_1^4 f_{11}^4}. \quad (3.3)$$

Using (2.6) in (3.3), we obtain

$$\sum_{n=0}^{\infty} c_{11}(3n+2)q^n \equiv f_1 f_{11} \pmod{3}. \quad (3.4)$$

which is the case $\alpha = 0$ of (3.1). Now suppose (3.1) holds for some $\alpha \geq 0$.

Substituting (2.2) in (3.1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} c_{11} \left(3 \cdot 11^\alpha n + \frac{3 \cdot 11^\alpha + 1}{2} \right) q^n \\ & \equiv f_{11} f_{121} (A(q^{11}) - qB(q^{11}) - q^2 C(q^{11}) + q^5 + q^7 D(q^{11}) - q^4 E(q^{11})) \pmod{3}. \end{aligned} \quad (3.5)$$

Extracting the terms containing q^{11n+5} from both sides of (3.5) and then replacing q^{11} by q , we obtain

$$\sum_{n=0}^{\infty} c_{11} \left(3 \cdot 11^{\alpha+1} n + \frac{3 \cdot 11^{\alpha+1} + 1}{2} \right) q^n \equiv f_1 f_{11} \pmod{3}, \quad (3.6)$$

which is (3.1) with $\alpha + 1$ for α . This completes the proof of (3.1) by induction. \square

Proof of Theorem 1.1. Comparing the coefficients of $q^{11n+\ell}$, for $\ell \in \{3, 6, 8, 9, 10\}$, from both sides of (3.5), we obtain (1.7).

Theorem 3.2 Let $p \geq 5$ be prime, $\left(\frac{-11}{p}\right) = -1$, then for all $\alpha, n \geq 0$,

$$\sum_{n=0}^{\infty} c_{11} \left(3p^{2\alpha} n + \frac{3p^{2\alpha} + 1}{2} \right) q^n \equiv f_1 f_{11} \pmod{3}. \quad (3.7)$$

Proof: We proceed by induction on α . The $\alpha = 0$ case of (3.7) is (3.4). Suppose that (3.7) is true for some $\alpha \geq 0$. Applying Euler's pentagonal number theorem [5, p. 10] in (3.7), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} c_{11} \left(3p^{2\alpha} n + \frac{3p^{2\alpha} + 1}{2} \right) q^n \equiv f_1 f_{11} \\ & = \sum_{k,m=-\infty}^{\infty} (-1)^{k+m} q^{k(3k+1)/2 + 11m(3m+1)/2} \pmod{3}. \end{aligned} \quad (3.8)$$

For a prime p with $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, let us consider

$$\frac{3k^2 + k}{2} + 11 \cdot \frac{3m^2 + m}{2} \equiv \frac{12p^2 - 12}{24} \pmod{p},$$

which is equivalent to

$$(6k + 1)^2 + 11(6m + 1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-11}{p}\right) = -1$, the only solution of the above condition is $k, m = \frac{\pm p - 1}{6}$.

If $p \equiv 1 \pmod{6}$, then $k \equiv m \equiv \frac{p-1}{6} \pmod{p}$. Let $k = rp + \frac{p-1}{6}$ and $m = sp + \frac{p-1}{6}$, for some integers r and s , we have

$$k(3k + 1)/2 + 11m(3m + 1)/2 = (p^2 - 1)/2 + p^2(3r^2 + r)/2 + 11p^2(3s^2 + s)/2.$$

If $p \equiv -1 \pmod{6}$, then $k \equiv m \equiv \frac{-p-1}{6} \pmod{p}$. Let $k = -rp - \frac{p+1}{6}$ and $m = -sp - \frac{p+1}{6}$, we also have

$$k(3k + 1)/2 + 11m(3m + 1)/2 = (p^2 - 1)/2 + p^2(3r^2 + r)/2 + 11p^2(3s^2 + s)/2.$$

Extracting the terms containing $q^{pn + \frac{p^2-1}{2}}$ from both sides of (3.8) and employing the above analysis, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} c_{11} \left(3p^{2\alpha} \left(pn + \frac{p^2 - 1}{2} \right) + \frac{3p^{2\alpha} + 1}{2} \right) q^{pn + \frac{p^2-1}{2}} \\ & \equiv \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{(p^2-1)/2 + p^2(3r^2+r)/2 + 11p^2(3s^2+s)/2} \pmod{3}, \end{aligned} \quad (3.9)$$

dividing both sides by $q^{\frac{p^2-1}{2}}$ and then replacing q^p by q , we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} c_{11} \left(3p^{2\alpha+1}n + \frac{3p^{2\alpha+2} + 1}{2} \right) q^n \equiv \\ & \sum_{r,s=-\infty}^{\infty} (-1)^{r+s} q^{p(3r^2+r)/2 + 11p(3s^2+s)/2} \pmod{3}. \end{aligned} \quad (3.10)$$

Again, applying Euler's pentagonal number theorem again, we derive that

$$\sum_{n=0}^{\infty} c_{11} \left(3p^{2\alpha+1}n + \frac{3p^{2\alpha+2} + 1}{2} \right) q^n \equiv f_p f_{11p} \pmod{3}, \quad (3.11)$$

extracting the terms containing q^{pn} from both sides and then replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} c_{11} \left(3p^{2\alpha+2}n + \frac{3p^{2\alpha+2} + 1}{2} \right) q^n \equiv f_1 f_{11} \pmod{3}, \quad (3.12)$$

which is (3.7) with $\alpha + 1$ for α . This completes the proof of (3.7) by induction. \square

Proof of Theorem 1.2. Comparing the coefficients of q^{pn+j} , for $1 \leq j \leq p-1$, from both sides of (3.11), we arrive at (1.8).

4. Congruences modulo 3 for $a_5(n)$ and $a_{11}(n)$

In this section, we prove some infinite families of congruences modulo 3 for $a_5(n)$ and $a_{11}(n)$.

Theorem 4.1 *We have,*

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv f_3 f_5 \pmod{3}, \quad (4.1)$$

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv f_1 f_{15} \pmod{3}, \quad (4.2)$$

$$\sum_{n=0}^{\infty} a_5(3n+2)q^n \equiv 2 \frac{f_5^5}{f_1} \pmod{3}. \quad (4.3)$$

Proof: From (1.6), we have

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{f_5^5}{f_1}. \quad (4.4)$$

Using (2.6), we obtain

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{f_5^6}{f_1 f_5} \equiv \frac{f_{15}^2}{f_1 f_5} = f_{15}^2 \sum_{n=0}^{\infty} p_{[1^5]}(n)q^n \pmod{3}. \quad (4.5)$$

Comparing the coefficients of q^{3n} , from both sides of (4.5) and replacing q^3 by q , we obtain

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv f_5^2 \sum_{n=0}^{\infty} p_{[1^5]}(3n)q^n \pmod{3}, \quad (4.6)$$

substituting (2.3) in (4.6), we arrive at (4.1). Similar proof follows for (4.2) and (4.3) using (2.4) and (2.5). \square

Proof of Theorem 1.3. From (4.3), we have

$$\sum_{n=0}^{\infty} a_5(3n+2)q^n \equiv 2f_5^5 \sum_{n=0}^{\infty} p(n)q^n \pmod{3}. \quad (4.7)$$

Equating the coefficients of q^{5n+4} , from both sides of (4.7), dividing both sides by q^4 and then replacing q^5 by q , we have

$$\sum_{n=0}^{\infty} a_5(15n+14)q^n \equiv 2f_1^5 \sum_{n=0}^{\infty} p(5n+4)q^n \pmod{3}. \quad (4.8)$$

Using (1.3), we obtain

$$\sum_{n=0}^{\infty} a_5(15n+14)q^n \equiv \frac{f_5^5}{f_1} = \sum_{n=0}^{\infty} a_5(n)q^n \pmod{3}, \quad (4.9)$$

which yields,

$$a_5(15n+14) \equiv a_5(n) \pmod{3}, \quad (4.10)$$

applying mathematical induction, we deduce that

$$a_5(15^k n + 15^k - 1) \equiv a_5(n) \pmod{3}, \text{ for all } k \geq 1,$$

which is our (1.9).

Proof of Theorem 1.4. Applying Jacobi's identity [5, p. 11] in (4.1), we have

$$\sum_{n=0}^{\infty} a_5(3n)q^n \equiv f_5 \sum_{k=0}^{\infty} (-1)^k (2k+1)q^{(k^2+k)/2} \pmod{3}. \quad (4.11)$$

It is easily checked that for any integer k , $\frac{k^2+k}{2} \equiv 0, 1, 3 \pmod{5}$, comparing the coefficients of q^{5n+2} and q^{5n+4} from both sides of (4.11), we have

$$a_5(15n+6) \equiv 0 \pmod{3}, \quad (4.12)$$

$$a_5(15n+12) \equiv 0 \pmod{3}, \quad (4.13)$$

which are the $k=0$ case of (1.10) and (1.11). Further on substituting (4.12) and (4.13) in (1.9), we obtain (1.10) and (1.11). Applying Euler's pentagonal number theorem in (4.2), we have

$$\sum_{n=0}^{\infty} a_5(3n+1)q^n \equiv f_{15} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2+k}{2}} \pmod{3}, \quad (4.14)$$

from the fact that there is no integer k such that $\frac{3k^2+k}{2}$ is congruent to 3 and 4 modulo 5, we have

$$a_5(15n+10) \equiv 0 \pmod{3}, \quad (4.15)$$

$$a_5(15n+13) \equiv 0 \pmod{3}, \quad (4.16)$$

which are $k=0$ case of (1.12) and (1.13). Again on Substituting (4.15) and (4.16) in (1.9), we obtain (1.12) and (1.13).

Theorem 4.2 We have,

$$\sum_{n=0}^{\infty} a_{11}(33n+17)q^n \equiv f_1^5 f_{11} \pmod{3}. \quad (4.17)$$

Proof:

From (1.6), we have

$$\sum_{n=0}^{\infty} a_{11}(n)q^n = \frac{f_{11}^{11}}{f_1}. \quad (4.18)$$

Using (2.6), we obtain

$$\sum_{n=0}^{\infty} a_{11}(n)q^n = \frac{f_{11}^{12}}{f_1 f_{11}} \equiv \frac{f_{33}^4}{f_1 f_{11}} = f_{33}^4 \sum_{n=0}^{\infty} p_{[1^+11^+]}(n)q^n \pmod{3}, \quad (4.19)$$

which yields

$$\sum_{n=0}^{\infty} a_{11}(3n+2)q^n \equiv f_{11}^4 \sum_{n=0}^{\infty} p_{[1^+11^+]}(3n+2)q^n \pmod{3}. \quad (4.20)$$

Substituting (2.1) in (4.20), we have

$$\sum_{n=0}^{\infty} a_{11}(3n+2)q^n \equiv f_{11}^4 \left(2 \frac{f_3 f_{33}}{f_1^2 f_{11}^2} + 3q \frac{f_3^2 f_{33}^2}{f_1^3 f_{11}^3} + 3q^2 \frac{f_3^3 f_{33}^3}{f_1^4 f_{11}^4} \right) \pmod{3}, \quad (4.21)$$

using (2.6), we obtain

$$\sum_{n=0}^{\infty} a_{11}(3n+2)q^n \equiv f_{11}^5 f_1 \pmod{3}. \quad (4.22)$$

Substituting (2.2) in (4.22), we obtain

$$\sum_{n=0}^{\infty} a_{11}(3n+2)q^n \equiv f_{11}^5 f_{121} (A(q^{11}) - qB(q^{11}) - q^2C(q^{11}) + q^5 + q^7D(q^{11}) - q^4E(q^{11})) \pmod{3}, \quad (4.23)$$

Extracting the terms containing q^{11n+5} from both sides of (4.23), dividing both sides by q^5 and then replacing q^{11} by q , we obtain (4.17). □

Proof of Theorem 1.5. Comparing the coefficients of $q^{11n+\ell}$, for $\ell \in \{3, 6, 8, 9, 10\}$, from both sides of (4.23), we obtain (1.14).

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