Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712 in press
doi:10.5269/bspm.70929

Upper triangular matrices property on unbounded Hilbert spaces: Different Weyl Type findings

Dalia S. Ali, Mayada T. Wazi, Rana A. Y. Almuttalibi and Nadia M. G. Al-Saidi*

abstract: The usage of unbounded Hilbert spaces, in particular the Hilbert spaces of analytically functions, is emerging in applications. Many areas of mathematics and physics, including quantum physics and control theory, have applications for them. In this attempt, we recommend these space types and a self-adjoint extension choice. The analysis of the unbounded upper triangular operator matrix with diagonal domain is one of the key characteristics of such spaces. The essential spectrum, the Weyl spectrum, and the Browder spectrum of this operator matrix must coincide with the union of the essential spectrum, the Weyl spectrum, and the Browder spectra of the diagonal entries in order for this to happen.

Key Words: Unbounded operator, Browder's theorem, Weyl' s theorem, unbounded Hilbert space, self-adjoint operator, positive measurement, upper triangle matrix.

Contents

1. Introduction

When realized in quantifier physical systems, such as Hamiltonians, momentum operators, etc., they take the form of self-adjoint operators. Positive measures impose an unconstrained and self-adjoint realization. The selection of suitable Hilbert spaces and the selection of the self-adjoint extension are two prerequisites that must be met for this to operate. We suggest the class of analytic functions of a complex variable. This class has many applications in mathematics, such as geometric function theory [1,2,3], physics [4,5,6] and computer sciences, specially the field of image processing [7,8,9] . Moreover, some applications of signal processing are given in [10]. Based on the above applications, beneficial and advantageous of Hilbert spaces of analytic functions (bounded and unbounded), we proceed to discover more properties of these spaces acting on the class of analytic functions of a complex variables. These spaces are considered as unbounded of infinity many power series of analytic functions.

A triangular operator matrix (TOM) is a particular variety of square matrix in the field of linear algebra and mathematical analysis. If all the entries above the primary diagonal are zero, a square matrix is referred to as upper triangular if every entry below the primary diagonal is zero. This property is discussed by many researchers for bounded spaces. For unbounded Hilbert spaces, Bai et al. [11,12,13] presented different studies on this direction including some spectra properties of unbounded upper triangular operator matrices (UUTOM) and upper semi-Weyl and upper semi-Browder spectra and essential Weyl and Browder spectra. Qi et al. [14] considered the closedness of ranges of UUTOM. Dong and Wu [15] suggested the kachurovskij spectrum of lipschitz continuous nonlinear UUTOM. Abu-Janah [16] studied the spectra of UUTOM as operators on some sequence spaces. Ellouz et al. [17] investigated

* Corresponding author

Submitted January 08, 2024. Published June 24, 2024
2010 *Mathematics Subject Classification:* 35B40, 35L70.

the closedness and self-adjointness of 3 ×3-UUTOM with application to a non-relativistic three-channel potential scattering system. Rashid [18] documented the variations of Weyl class methods (see [19,20,21]) for UUTOM. For 2D-upper triangular operator matrices on the Banach space, some research has been done on the survival of generalized Weyl's, generalized Browder's, generalized a-Weyl's, and generalized a-theorems Browder's [22,23,24].
In this effort, we suggest various Hilbert spaces of analytic functions for a complex variable as well as a self-adjoint extension option. One of the essential features of such spaces is the study of the diagonal domain of the unbounded upper triangular operator matrix. The union of the essential spectrum, the Weyl spectrum, and the Browder spectra of the diagonal entries is required for the essential spectrum, the Weyl spectrum, and the Browder spectrum of this operator matrix to coincide.

2. Preliminaries

Examples of Banach spaces are Hilbert spaces with their norm determined by the inner product. A Hilbert space is usually a Banach space, although the opposite is not required. Therefore, a norm provided by an inner product need not exist in a Banach space. The following concepts can be located in [25,26,27,28,29]
.

2.1. Concepts

Definition 2.1

1. Assume that $\mathbb{H}$ is a Hilbert space and $\triangle_{\Omega} \subset \mathbb{H}$ is a subset of $\mathbb{H}$ combining the operator Ω. Then the operator $[\Omega, \triangle_{\Omega}]$ is known as bounded operator if
$$\| \Omega \| = \sup_{\psi \in \triangle_{\Omega}, \psi \neq 0} = \frac{\| \Omega\psi \|}{\| \psi \|} = \sup_{\psi \in \triangle_{\Sigma}, \|\psi\|=1} \| \Omega\psi \| < \infty.$$
2. Elsewhere, it is known as an unbounded operator.
3. Assume that $[\Omega, \Delta_{\Omega}]$ is an unbounded operator on $\mathbb{H}$. The adjoint of $[\Omega, \Delta_{\Omega}]$ is the operator $[\Omega^*, \Delta^*_{\Omega}]$ achieving the map
$$\Omega^* : \varphi \in \Delta^*_{\Omega} \to \psi, \qquad (\varphi, \Omega\phi) = (\psi, \phi), \forall \phi \in \Delta_{\Omega},$$
4. where the operator $[\Omega^*, \Delta^*_{\Omega}]$ is linear in $\mathbb{H}$.
5. A self-adjoint operator Ω is positive if $< \Omega\psi, \psi > \geq 0$ for every $\psi \in \Delta(\Omega) \subseteq \mathbb{H}$.
6. A densely defined operator $[\Omega, \Delta_{\Omega}] \in \mathbb{H}$ with $\Delta(\Omega) \subset \Delta(\Omega^*)$ is called posinormal (positive-normal) if there occurs a positive operator, called interrupter Π, achieving $\Omega\Omega^* = \Omega^* \Pi \Omega$.
7. A densely close linear operator $[\Omega, \Delta\Omega]$ with $\Delta\Omega \subset \Delta\Omega^*$ is called totally posinormal if $\Omega - \lambda I$ is posinormal for every $\lambda \in \mathbb{C}$.
8. A closed linear operator $[\Omega, \Delta\Omega]$ is called a semi-Fredholm if the range of the operator Ω symbolizing by $\mathfrak{R}(\Omega)$ is closed and null of Ω and the dimension of the null space of Ω symbolizing by $\varsigma(\Omega) < \infty$.
9. Suppose that two operators $[\Omega_1, \Delta_{\Omega_1}], [\Omega_2, \Delta_{\Omega_2}] \in \mathbb{H}$. Then the upper triangular operator is structured by
$$U_{\Omega} = \begin{pmatrix} \Omega_1 & \Omega \\ 0 & \Omega_2 \end{pmatrix},$$
10. where $U_0 = [\Omega_1, \Delta_{\Omega_1}] \oplus [\Omega_2, \Delta_{\Omega_2}]$. For more details, one can see [30,31] .

Definition 2.2. Assume that $\sigma(\Omega)$ is the spectrum of the operator Ω. Then $\sigma_A(\Omega)$ presents the approximate spectrum in $\sigma(\Omega)$; $\sigma_W(\Omega)$ presents the Weyl-spectrum (i.e $\sigma(\Omega - \lambda I)$ is not Weyl). The operator $[\Omega, \Delta\Omega]$ is identified to satisfy the property

1. if $\sigma_A(\Omega)\backslash\sigma_W(\Omega)$
2. if $\sigma(\Omega)\backslash\sigma_W(\Omega)$.

Definition 2.3. [32] Operators of the polaroid category are those for which the isolated points of the spectrum, or also the isolated points of the approximation point spectrum, are, correspondingly, the right poles or poles of the resolvent. These modules of operators, which are quite extensive and include a number of significant classes of operators acting on Hilbert spaces that generalize the characteristics of normal operators in various ways, such as hyponormal and paranormal operators Other polaroid operator examples include isometries, group algebra convolution operators, analytic Toeplitz operators constructed on Hardy spaces, semi transference operators, and weighted one-sided transferals.
Definition 3.4. The single valued extension possession (SVEP) of closed linear operators is significant in Fredholm theory. In this effort, we are primarily interested in the SVEP at a point, a localized variant of SVEP first proposed by J. Finch , and we refer it to the finiteness of the descent of a closed linear operator. The link between the Browder and Weyl spectra and the Browder and Weyl theorems is crucially determined by SVEP . We denoted the set of $\lambda \in \mathbb{C}$ where $[\Omega, \Delta\Omega]$ does not have SVEP by $\Sigma(\Omega)$. Or, $[\Omega, \Delta\Omega]$ is SVEP if it satisfies $\Sigma(\Omega) = \emptyset$.

2.2. Unbounded Hilbert spaces (UHSs)

In this section, we shall introduce two different types of UHSs, the first one is regarding the class of analytic functions and the second one is the class of ψ analytic functions. This class is the most important one to determine analytic solutions of many classes of differential equations in a complex domain [35,36] .

2.3. UHS of analytic functions

In this part, the unbounded Hilbert spaces $\mathbb{H}_m(\Phi),\ m = 1, ...,$ of analytic functions

$$\Phi(z) = \Phi(z_1, ..., z_m),$$

where $z = (z_1, ..., z_m)$ is an element of the m-dimensional complex Euclidean space Ξ_m. The inner product of two analytic functions of $\Phi, \Phi' \in \mathbb{H}_m, m \to \infty$ is presented by the structure [29]

$$< \Phi, \Phi' >= \int_{\Xi_m} \overline{\Phi}\Phi' d\mu_m(z),$$

where $z_n = x_n + iy_n$

$$d\mu_m(z) = \frac{\exp\left(-\sum^m{}_{n=1}\right)|z_n|^2 \prod_{n=1}^m dx_n dy_n}{\pi^m}.$$

Then $\Phi \in \mathbb{H}_m$ if and only if

$$< \Phi, \Phi > = \int_{\Xi_m} \overline{\Phi}\Phi d\mu_m(z) < \infty.$$

With the help of the measurable field κ_m of nonzero positive self-adjoint operators, the following definition is intended to be generalized in this investigation:The inner product of two analytic functions of $\Phi, \Phi' \in \mathbb{H}_m^\kappa$ is given by the equality

$$< \Phi, \Phi' >= \int_{\Xi_m} \overline{\Phi}\Phi' d\nu_m(z),$$

where $z_n = x_n + iy_n$ and

$$d\nu_m := \kappa_m \left(\frac{\exp\left(-\sum^m{}_{n=1}\right)|z_n|^2 \prod_{n=1}^m dx_n dy_n}{\pi^m} \right).$$

Then $\Phi \in \mathbb{H}_m^\kappa$ if and only if

$$< \Phi, \Phi > = \int_{\Xi_m} \overline{\Phi}\Phi d\nu_m(z) < \infty.$$

Obviously, when $\kappa_m = 1$ for all m, we get the measure $d\mu_m(z)$. In this investigation, we shall present some useful properties of the space $\mathbb{H}_m^\kappa$. Consequently, every function $\Phi \in \mathbb{H}_m^\kappa$ can be written by the formula

$$\Phi(z) = \sum_\iota \alpha[\iota]\upsilon[\iota](z),$$

where

$$\upsilon[\iota] = \prod_{n=1}^{m} \frac{(z_n^{\iota_n})}{\sqrt{\iota_n!}},$$

and $\iota = (\iota_1, ..., \iota_m)$ gives an m-tuple of non-negative integers. Moreover, we have the norm

$$\| \Phi\|^2 = \sum_\iota |\alpha[\iota]|^2, \qquad |z_m| \leq 1.$$

The definition of the class of analytic functions is formulated by the series

$$\Psi(z) = \sum_{i=0}^{\infty} c_i(z),$$

where c_i are the coefficients of the series (constants) such that

$$|\Psi(z)| < r := \frac{1}{\lim_{i\to\infty} \sqrt[i]{|c_i|}}, \qquad c_i \neq 0.$$

Under the same steps in the above section, we formulate the unbounded Hilbert spaces $\mathbb{H}_m(\Psi),\ m = 1, ...,$ of analytic functions

$$\Psi(z) = \Psi(z_1, ..., z_m),$$

where $z = (z_1, ..., z_m)$ is an element of the m-dimensional complex Euclidean space Ξ_m. In addition, we have the norm

$$\| \Psi\|^2 = \sum_i |c_i|^2, \qquad |z_m| \leq 1,\ \forall m.$$

3. Results

This area is where we prepare our findings. Our findings are based on a number of requirements that the operators in the UHSs must meet in order for the upper triangular operator to be implied by the generalized Weyl's theorem. Our method for this request is by using the concept of polaroid [18]. We shall follow the strategies in [25] to show that the suggested operators are totally posinormal.

Proposition 3.1 Let $\mathbb{H}_m^\kappa[\Psi]$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Upsilon, \Delta\Upsilon]$ be formulated as follows:

$$\Upsilon(z) = \Upsilon(z_1, ..., z_m) = \sum_i \eta[c_i](z), \qquad (z_m = x_m + iy_m, \qquad \eta_m[c_i] \in \mathbb{C})$$

where

$$\Delta\Upsilon = \{(z_1, ..., z_m) \in \Xi_m : \sum_i |\eta[c_j]|^2 < \infty\}.$$

Then the operator $[\Upsilon, \Delta\Upsilon]$ is an unbounded posinormal operator.

Proof. Define a set

$$s_0 := \{z = (z_1, .., z_m) : \{z\}_n \neq 0, n \in \mathbb{N}, |z_j| \leq 1, j = 1, ..., m\},$$

where $\{z\}_n$ grants the sequence produced by the element $(z_1, .., z_m)$. That is there is at least one component of the subsequent sequence which is dissimilar from zero component

$$\{z\}_1 = (z_1, .., z_m)_1, ..., \{z\}_n = (z_1, .., z_m)_n.$$

Therefore, the set s_0 is dense in $\mathbb{H}^{\kappa}_m[\Psi]$. Accordingly, we get $s_0 \subseteq \Delta\Upsilon$ and then $\Delta\Upsilon$ is dense in $\mathbb{H}^{\kappa}_m[\Psi]$. The adjoint operator of Υ is given by $[\Upsilon^*(z), \Delta\Upsilon^*]$:

$$\Upsilon^*(z) = (\overline{\eta_1 z_1}, ..., \overline{\eta_m z_m}) = \sum_i \overline{\eta[c_i]}(\overline{z}),$$

where

$$\Delta\Upsilon^* = \{(z_1, ..., z_m) \in \Xi_m : \sum_i |\overline{\eta[c_i]}|^2 < \infty\}.$$

Consequently, we obtain

$$\begin{aligned} \Delta\Upsilon &= \{(z_1, ..., z_m) \in \Xi_m : \textstyle\sum_i |\eta[c_i]|^2 < \infty\} \\ &= \{(z_1, ..., z_m) \in \Xi_m : \textstyle\sum_i |\overline{\eta[c_i]}|^2 < \infty\} = \Delta\Upsilon^*. \end{aligned}$$

Moreover, a computation yields

$$\begin{aligned} & \Upsilon(z) \cdot \Upsilon^*(z) \\ &= \textstyle\sum_i \eta[c_i](z) \cdot \sum_i \overline{\eta[c_i]}(\overline{z}) \\ &= \textstyle\sum_i \eta[c_i](z_1, ..., z_m) \cdot \sum_i \overline{\eta[c_i]}(\overline{z_1}, ..., \overline{z_m}) \\ &= (\textstyle\sum_i \eta[c_i](z_1, ..., \sum_i \eta[c_i](z_m) \cdot (\sum_i \overline{\eta[c_i](z_1)}, ..., \sum_i \overline{\eta[c_i](z_m)}) \\ &= \left(\textstyle\sum_i \eta[c_i] \sum_i \overline{\eta[c_i]}\right)(z_1)\overline{(z_1)} \\ &+ ... + \left(\textstyle\sum_i \eta[c_i] \sum_i \overline{\eta[c_i]}\right)(z_m)\overline{(z_m)} \\ &= \left(\textstyle\sum_i \overline{\eta[c_i]} \sum_i \eta[c_i]\right)\overline{(z_1)}(z_1) \\ &+ ... + \left(\textstyle\sum_i \overline{\eta[c_i]} \sum_i \eta[c_i]\right)(\overline{z_m})(z_m) \\ &= (\textstyle\sum_i \overline{\eta[c_i](z_1)}, ..., \sum_i \overline{\eta[c_i](z_m)} \cdot (\sum_i \eta[c_i](z_1), ..., \sum_i \eta[c_i](z_m) \\ &= (\textstyle\sum_i \overline{\eta[c_i](z_1)}, ..., \sum_i \overline{\eta[c_i](z_m)}) \cdot 1_m \cdot (\sum_i \eta[c_i](z_1), ..., \sum_i \eta[c_i]\ (z_m) \\ &= \Upsilon^*(z) \cdot 1_m \cdot \Upsilon(z), \end{aligned}$$

where $1_m = (\underbrace{1, ..., 1}_{m-times})$. Thus, the operator $[\Upsilon, \Delta\Upsilon]$ is an unbounded posinormal operator over the set s_0. This ends the proof.

We proceed to prove that the operator $[\Upsilon, \Delta\Upsilon]$ is totally posinormal. That is to prove the next inclusion property

Proposition 3.2 Let $\mathbb{H}^{\kappa}_m[\Psi]$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[\Upsilon, \Delta\Upsilon]$ be formulated as follows:

$$\Upsilon(z) = \Upsilon(z_1, ..., z_m) = \sum_i \eta[c_i](z),$$

$$(z_m = x_m + iy_m, \qquad \eta_m[c_i] \in \mathbb{C})$$

where

$$\Delta\Upsilon = \{(z_1, ..., z_m) \in \Xi_m : \sum_i |\frac{\eta[c_i]|^2}{2} < \infty\}.$$

Then the operator $[\Upsilon, \Delta\Upsilon]$ is totally posinormal operator.

Proof. To show that $[\Upsilon, \Delta\Upsilon]$ is totally posinormal operator, we must follow the following steps:

Step 1. Posinormality of $[\Upsilon, \Delta\Upsilon]$.By Proposition [pro-1], clearly that the operator $[\Upsilon, \Delta\Upsilon]$ is unbounded posinormal operator on the dense set in $\mathbb{H}^{\kappa}_m[\Psi]$

$$s_0 = \{z = (z_1, .., z_m) : \{z\}_n \neq 0, n \in \mathbb{N}, |z_j| \leq 1, j = 1, ..., m\}.$$

Step 2. Inclusion property.The adjoint operator of Υ is given by $[\Upsilon^*(z), \Delta\Upsilon^*]$:

$$\Upsilon^*(z) = (\overline{\eta_1 z_1}, ..., \overline{\eta_m z_m}) = \sum_i \overline{\eta[c_i]}(\overline{z}),$$

where

$$\Delta\Upsilon^* = \{(z_1, ..., z_m) \in \Xi_m : \sum_i |\overline{\eta[c_i]}|^2 < \infty\}.$$

Now, we aim to show that $\Delta\Upsilon \subset \Delta\Upsilon^*$, as follows:

$$\begin{aligned}\Delta\Upsilon &= \{(z_1, ..., z_m) \in \Xi_m : \textstyle\sum_i |\frac{\eta[c_i]|^2}{2} < \infty\}\\ &= \{(z_1, ..., z_m) \in \Xi_m : \textstyle\sum_i |\frac{\overline{\eta[c_i]}|^2}{2} < \infty\}\\ &\subset \{(z_1, ..., z_m) \in \Xi_m : \textstyle\sum_i |\overline{\eta[c_i]}|^2 < \infty\}\\ &= \Delta\Upsilon^*.\end{aligned}$$

Step 3. Posinormality of $[\Upsilon - \lambda I, \Delta(\Upsilon - \lambda I)]$.It remains to prove that the operator $[\Upsilon - \lambda I, \Delta(\Upsilon - \lambda I)]$ is unbounded posinormal operator.

$$\begin{aligned}(\Upsilon - \lambda I)(z) \cdot (\Upsilon^* - \lambda^* I)(z) &= \textstyle\sum_i (\eta[c_i] - \lambda_i)(z) \cdot \sum_i \overline{(\eta[c_i] - \lambda_i)}(\overline{z})\\ &= \textstyle\sum_i (\eta[c_i] - \lambda_i)(z_1, ..., z_m) \cdot \sum_i \overline{(\eta[c_i] - \lambda_i)(\overline{z_1}, ..., \overline{z_m})}\\ &= (\textstyle\sum_i (\eta[c_i] - \lambda_i)(z_1, ..., \sum_i (\eta[c_i] - \lambda_i)(z_m) \cdot\\ &\qquad (\textstyle\sum_i \overline{(\eta[c_i] - \lambda_i)(z_1)}, ..., \sum_i \overline{(\eta[c_i] - \lambda_i)(z_m)})\\ &= \left(\textstyle\sum_i (\eta[c_i] - \lambda_i) \sum_i \overline{(\eta[c_i] - \lambda_i)}\right) (z_1)\overline{(z_1)}\\ &+ ... + \left(\textstyle\sum_i (\eta[c_i] - \lambda_i) \sum_i \overline{(\eta[c_i] - \lambda_i)}\right) (z_m)\overline{(z_m)}\\ &= \left(\textstyle\sum_i \overline{(\eta[c_i] - \lambda_i)} \sum_i (\eta[c_i] - \lambda_i)\right) \overline{(z_1)}(z_1)\\ &+ ... + \left(\textstyle\sum_i \overline{(\eta[c_i] - \lambda_i)} \sum_i (\eta[c_i] - \lambda_i)\right) (\overline{z_m})(z_m)\\ &= (\textstyle\sum_i \overline{(\eta[c_i] - \lambda_i)(z_1)}, ..., \sum_i \overline{(\eta[c_i] - \lambda_i)(z_m)}\\ &\qquad \cdot(\textstyle\sum_i (\eta[c_i] - \lambda_i)(z_1), ..., \sum_i (\eta[c_i] - \lambda_i)(z_m)\\ &= (\textstyle\sum_i \overline{(\eta[c_i] - \lambda_i)(z_1)}, ..., \sum_i \overline{(\eta[c_i] - \lambda_i)(z_m)})\\ &\qquad \cdot 1_m \cdot (\textstyle\sum_i (\eta[c_i] - \lambda_i)(z_1), ..., \sum_i (\eta[c_i] - \lambda_i)\ (z_m)\\ &= (\Upsilon^* - \lambda^* I)(z) \cdot 1_m \cdot (\Upsilon - \lambda I)(z),\end{aligned}$$

where $1_m = (\underbrace{1, ..., 1}_{m-times})$. Thus, the operator $[\Upsilon - \lambda I, \Delta(\Upsilon - \lambda I)]$ is an unbounded posinormal operator over the set s_0.

Step 4. Linearity of $[\Upsilon, \Delta\Upsilon]$.Let $z, w \in s_0$ and a, b be fixed constants. Then

$$\begin{aligned}\Upsilon(az + bw) &= \textstyle\sum_i \eta[c_i](az + bw)\\ &= a \textstyle\sum_i \eta[c_i](z) + b \sum_i \eta[c_i](w)\\ &= a\Upsilon(z) + b\Upsilon(w).\end{aligned}$$

Therefore, the operator $[\Upsilon, \Delta\Upsilon]$ is closed linear operator in the space $\mathbb{H}_m^\kappa$. Hence, the operator $[\Upsilon, \Delta\Upsilon]$ is totally posinormal.

Next result shows the sufficient and necessary conditions on the operator $[\Upsilon, \Delta\Upsilon]$ to be polaroid.

Proposition 3.3 Let $\mathbb{H}_m^\kappa[\Psi]$ be the collections of Hilbert spaces of analytic functions of complex variables. Assume that all the conditions of Proposition [pro-2] are validated. Moreover, the following assumptions are satisfied

1. $\rho(\Upsilon) \neq \emptyset$ (the resolvent set of an operator of Υ)
2. $\sigma(\Upsilon - \lambda I) = \{0\}$ (the spectrum of the operator $\Upsilon - \lambda I$) on some invariant subspace.

If one of the following hypotheses is achieved

1. λ is an isolated point in $\sigma(\Upsilon)$
2. λ is a simple pole in $\rho(\Upsilon)$

then the operator $[\varUpsilon, \varDelta\varUpsilon]$ is polaroid.
Proof. By Proposition 3.2, the operator $[\varUpsilon, \varDelta\varUpsilon]$ is totally posinormal. Suppose that λ is an isolated point in $\sigma(\varUpsilon)$. In view of [25]-Lemma 2.1, λ is an isolated point in $\sigma(\varUpsilon) \Leftrightarrow \lambda$ is a simple pole in $\rho(\varUpsilon)$, then by the definition of polaroid, we have that the operator $[\varUpsilon, \varDelta\varUpsilon]$ is polaroid.
Proposition 3.4 Let the assumptions of Proposition 3.2 be valid. If the following assumptions are satisfied

1. $\rho(\varUpsilon) \neq \emptyset$ (the resolvent set of an operator of $\varUpsilon$)
2. $\sigma(\varUpsilon - \lambda I) = \{0\}$ (the spectrum of the operator $\varUpsilon - \lambda I$) on some invariant subspace,

then

1. the operators $[\varUpsilon, \varDelta\varUpsilon]$ and $[\varUpsilon^*, \varDelta\varUpsilon^*]$ achieve Weyl's Theorem;
2. the operator $[\varUpsilon^*, \varDelta\varUpsilon^*]$ achieves $a-$Weyl's Theorem;
3. $[\varUpsilon, \varDelta\varUpsilon]$ achieves a-Weyl's Theorem, whenever $[\varUpsilon^*, \varDelta\varUpsilon^*]$ has SVEP.
4. the operators $[\varUpsilon, \varDelta\varUpsilon]$ and $[\varUpsilon^*, \varDelta\varUpsilon^*]$ achieve Browder' s Theorem.
5. the operator $[\varUpsilon^*, \varDelta\varUpsilon^*]$ achieves $a-$Browder's Theorem.
6. $[\varUpsilon, \varDelta\varUpsilon]$ achieves a-Browder's Theorem and the properties (A) and (W), whenever $[\varUpsilon^*, \varDelta\varUpsilon^*]$ has SVEP.

Proof. A straight application of [25]-Theorem 3.1 indicates the first three substances. The fourth one arises by consuming [25]-Theorem 3.2. Whereas, the last two substances occur by employing [25]-Theorem 3.3 and [25]-Theorem 4.1 respectively.
Proposition 3.5. Let the assumptions of Proposition 3.4 be satisfied for the operator $[\varUpsilon, \varDelta\varUpsilon]$. Then for all polaroid operators $[G, \varDelta G]$ such that $\varSigma(\varUpsilon)\bigcup \varSigma(G) = \emptyset$ then the upper triangular operator

$$U_\varOmega = \begin{pmatrix} \varUpsilon & \varOmega \\ 0 & G \end{pmatrix},$$

achieves the a-Weyl's theorem, where $\varOmega : \varDelta G \to \varDelta\varUpsilon$ ($\varOmega \in \mathbb{H}^\kappa_m[\varPsi]$). Moreover, if $[\varUpsilon, \varDelta\varUpsilon]$ and $[G, \varDelta G]$ have SVEP, then $U_\varOmega$ have SVEP, while $U^*_\varOmega$ achieves the a-Weyl's theorem and Browder's theorem.
Proof. The first part comes from [18]-Theorem 9, while the last two assertions come from [18]-Remark 3(P15) and [18] -Theorem 10 respectively.
Example 3.6 Let $\mathbb{H}^\kappa_m[\varPsi]$ be the collections of Hilbert spaces of analytic functions of complex variables. And let the operator $[G, \varDelta G]_\lambda$ be defined with state with eigenvalue $\lambda \in \mathbb{C}$, as follows:

$$G(z) = G(\lambda_1 z_1, ..., \lambda_m z_m) = \sum_i \lambda[i](z),$$

where $\lambda = (\lambda_1, ..., \lambda_m)$

$$\varDelta G = \{(z_1, ..., z_m) \in \varXi_m : \sum_i \frac{|\lambda[i]|^2}{2} < \infty\}.$$

Then the operator $[G, \varDelta G]_\lambda$ is polarid operator satisfying $\sigma(G - \lambda I) = \{0\}$ and $\rho(G) \neq \emptyset$ (following the same steps in Proposition 3.3) on the set

$$E_0 := \{z = (z_1, .., z_m) : \{z\}_n \neq 0, n \in \mathbb{N}, |z_j| \leq \frac{1}{1-q}, q > 1, j = 1, ..., m\}.$$

The adjoint operator of $[G, \varDelta G]_\lambda$ is defined as follows $[G^*(z), \varDelta G^*]_\lambda$:

$$G^*(z) = (\overline{\lambda_1 z_1}, ..., \overline{\lambda_m z_m}) = \sum_i \overline{\lambda[i]}(\overline{z}),$$

where

$$\varDelta G^{*} = \{(z_1, ..., z_m) \in \varXi_m : \sum_i |\overline{\lambda[i]}|^2 < \infty\}.$$

Then in view of Proposition 3.5, we obtain that the UTO

$$U_I = \begin{pmatrix} \varUpsilon & I \\ 0 & G \end{pmatrix},$$

achieves the a-Weyl's theorem.

4. Conclusion

In this paper, we suggested the Hilbert space of analytic functions in the complex plane, with an extension norm. We have received different possess of the suggested operator acting on the Hilbert space of analytic functions, such as the posinormal (see Proposition 3.1, inclusion (Proposition 3.2) and polarid property (Proposition 3.5). We presented the main conditions on the operator $[\varUpsilon, \varDelta\varUpsilon]$ to satisfy the Weyl's, a-Weyl's and Browder's theorems (see [34]). We have proved that the suggested operator formulating the UTO have the same facilities of the Weyl's, a-Weyl's and Browder's theorems.

Acknowledgment

This is to knowledge the university of Baghdad and university of Technology for providing a suitable research environment and cooperation.

References

1. Cowen Jr, Carl C. Composition operators on spaces of analytic functions. Routledge, 2019.
2. Penner, Alexander-Georg, Felix von Oppen, Gergely Zarand, and Martin R. Zirnbauer. "Hilbert Space Geometry of Random Matrix Eigenstates." Physical Review Letters 126, no. 20 (2021): 200604.
3. Han, Chen. "Engineering Management Model Analysis Using Partial Differential Equation Hilbert Space." Security and Communication Networks 2022 (2022).
4. Antoine, Jean-Pierre. "Partial inner product spaces of analytic functions." In Generalized functions, operator theory, and dynamical systems, pp. 26-47. Chapman and Hall/CRC, 2021.
5. Carroll, Sean M. "Reality as a vector in Hilbert space." In Quantum Mechanics and Fundamentality, pp. 211-224. Springer, Cham, 2022.
6. Griffiths, Robert B. "Reply to "Comment on Nonlocality claims are inconsistent with Hilbert-space quantum mechanics." Physical Review A 104, no. 6 (2021): 066202.
7. Chen, Jiafeng, Lu Yu, and Wenyi Wang. "Hilbert Space Filling Curve Based Scan-Order for Point Cloud Attribute Compression." IEEE Transactions on Image Processing (2022).
8. Ramos-Llorden, Gabriel, Gonzalo Vegas-Sanchez-Ferrero, Congyu Liao, Carl-Fredrik Westin, Kawin Setsompop, and Yogesh Rathi. "SNR-enhanced diffusion MRI with structure-preserving low-rank denoising in reproducing kernel Hilbert spaces." Magnetic resonance in medicine 86, no. 3 (2021): 1614-1632.
9. Arfat, Yasir, Poom Kumam, Muhammad Aqeel Ahmad Khan, Parinya Sa Ngiamsunthorn, and Attapol Kaewkhao. "An inertially constructed forward-backward splitting algorithm in Hilbert spaces." Advances in Difference Equations 2021, no. 1 (2021): 1-23.
10. Li, Xinxin, Cheng Xu, and Hongyan Shen. "Rayleigh Wave Reconstruction from Ambient Noise Cross-Correlation Function by Radon-Wigner Transform." In IOP Conference Series: Earth and Environmental Science, vol. 660, no. 1, p. 012065. IOP Publishing, 2021.
11. Bai, Qingmei, Junjie Huang, and Alatancang Chen. "Essential, Weyl and Browder spectra of unbounded upper triangular operator matrices." Linear and Multilinear Algebra 64, no. 8 (2016): 1583-1594.
12. Bai, Wurichaihu, Qingmei Bai, and Alatancang Chen. "Upper semi-Weyl and upper semi-Browder spectra of unbounded upper triangular operator matrices." Journal of Function Spaces 2018 (2018).
13. Bai, Qingmei, Alatancang Chen, and Junjie Huang. "Some spectra properties of unbounded 2× 2 upper triangular operator matrices." Annals of Functional Analysis 10, no. 3 (2019): 412-424.
14. Qi, Yaru, Junjie Huang, and Alatancang Chen. "Closedness of ranges of unbounded upper triangular operator matrices." Operators and Matrices 15, no. 2 (2021): 413-422.
15. Dong, Xiaomei, and Deyu Wu. "The kachurovskij spectrum of lipschitz continuous nonlinear block operator matrices." Advances in Operator Theory 6, no. 3 (2021): 1-18.

16. Abu-Janah, Suad H. "Spectra of upper triangular double-band matrices as operators on some sequence spaces." 17 th Issue DEC 2021 (2021): 28.

17. Ellouz, Hanen, Ines Feki, and Aref Jeribi. "On the closedness and self-adjointness of 3× 3 operator matrices and application to a non-relativistic three-channel potential scattering model." Asymptotic Analysis Preprint (2022): 1-18.

18. Rashid, Mohammad HM. "Variations of Weyl Type Theorems for Upper Triangular Operator Matrices." Acta Mathematica Vietnamica 46, no. 4 (2021): 719-735.

19. Weyl, H.: Uber beschrankte quadatische Formen, deren Differenz Vollstetig ist. Rend. Circ.Mat. Palermo 27, 373-392 (1909)

20. Coburn, L.A.: Weyl's theorem for nonnormal operators. Mich. Math. J. 13, 285-288 (1966)

21. Berkani, M., Arroud, A.: Generalized Weyl's theorem and hyponormal operators. J. Aust. Math. Soc. 76, 291-302 (2004)

22. Aiena, Pietro, and Salvatore Triolo. "Weyl-type theorems on Banach spaces under compact perturbations." Mediterranean journal of mathematics 15, no. 3 (2018): 1-18.

23. Jia, Boting, and Youling Feng. "Weyl type theorems under compact perturbations." Mediterranean Journal of Mathematics 15, no. 1 (2018): 1-13.

24. Carpintero, C., A. Malaver, E. Rosas, and J. Sanabria. "On the hereditary character of new strong variations of Weyl type Theorems." (2019).

25. Gupta, A., and K. Mamtani. "Weyl-type theorems for unbounded posinormal operators." Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences) 52, no. 4 (2017): 191-197.

26. Gupta, Anuradha, and Karuna Mamtani. "Weyl type theorems for unbounded class-$\mathcal{A}$ operators." Afrika Matematika 28, no. 5 (2017): 745-754.

27. Teta, Alessandro. "Linear Operators in Hilbert Spaces." In A Mathematical Primer on Quantum Mechanics, pp. 79-122. Springer, Cham, 2018.

28. Gupta, Anuradha, and Karuna Mamtani. "Weyl-type theorems for restrictions of closed linear unbounded operators." Acta Univ. M. Belii Ser. Math (2015): 72-79.

29. Bargmann, Valentine. "Remarks on a Hilbert space of analytic functions." Proceedings of the National Academy of Sciences 48, no. 2 (1962): 199-204.

30. Rashid, Mohammad Hussein Mohammad. "Generalized Weyl's theorem and property (gw) for upper triangular operator matrices." Arabian Journal of Mathematics 9, no. 1 (2020): 167-179.

31. Rashid, M.H.M." Property (w) of upper triangular operator matrices". Tamkang J. Math. 51 (2020): 81-99

32. Aiena, P. Polaroid-Type Operators. In: Fredholm and Local Spectral Theory II . Lecture Notes in Mathematics, vol 2235(2018). Springer, Cham.

33. Finch, James. "The single valued extension property on a Banach space." Pacific journal of mathematics 58, no. 1 (1975): 61-69.

34. Berkani, M. "Index of B-Fredholm operators and generalization of a Weyl theorem." Proceedings of the American Mathematical Society 130, no. 6 (2002): 1717-1723.

35. Zhabborov, Nasriddin Mirzoodilovich, Tolib Urolovich Otaboev, and Sh Ya Khursanov. "The Schwarz Inequality and the Schwarz Formula for A-Analytic Functions." Contemporary Mathematics. Fundamental Directions 64, no. 4 (2018): 637-649.

36. Tishabaev, Z.K., Otaboev, T.U. and Khursanov, S., 2020. Residues and Argument Principle for A (z)-Analytic Functions. Journal of Mathematical Sciences, 245.

Mayada T. Wazi,

Department of Electromechanical Engineering, University of Technology, Iraq.

E-mail address: `Mayada.t.wazi@uotechnology.edu.iq`

and

Dalia S. Ali,

Department of Mathematics, College of Science for Women,

University of Baghdad, Baghdad, Iraq.

E-mail address: `dalya.s@csw.uobaghdad.edu.iq`

and

Rana A. Y. Almuttalibi,
Department of the office of university president,
University of Information Technology and Communications, Baghdad, Iraq.
E-mail address: rana.yousif1982@gmail.com

and

Nadia M. G. Al-Saidi,
Department of applied Sciences,
University of Technology, Baghdad, Iraq.
E-mail address: nadia.m.ghanim@uotechnology.edu.iq