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# Degree of Convergence of Functions of Several Variables in Generalized Hölder Spaces

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ABSTRACT: In this paper, degree of convergence of a function of double conjugate Fourier series in two dimensional generalized Hölder spaces using double Hausdorff means is obtained. The degree of convergence of a function of $N$-multiple conjugate Fourier series in $N$-dimensional generalized Hölder spaces using $N$-dimensional Hausdorff means is also obtained. Some important corollaries are also deduced form our main results.

Key Words: Degree of convergence, generalized Hölder space, $N$-dimensional Hausdorff matrices, multiple conjugate Fourier series, modulus of continuity.

## Contents



## 1. Introduction

Degree of approximation of a function of one dimensional variable in Lipschitz, Besov, Hölder, generalized Hölder spaces has been studied by the authors [4,5,11,12,16,13,17,14] etc. using different single and product summability means of conjugate Fourier series.
Different double summability means of double conjugate Fourier series has been investigated by the authors [9] and [15].
The double Hausdorff matrices of double sequences was first studied by [7]. Later, [10] and [18] have also studied double Hausdorff matrices of double sequences.
Degree of approximation of a function of two variables has been studied by [8] using rectangular partial sums of double conjugate Fourier series.
Since the degree of approximation of a function of conjugate Fourier series in above mentioned spaces only gives the degree of the polynomials with respect to the function but the degree of convergence of a function of conjugate Fourier series gives the convergence of the polynomial with respect to the function.
In this paper, we study the degree of convergence of a function of two dimensional variable in generalized Hölder space by double Hausdorff matrices of double conjugate Fourier series. We also study the degree of convergence of a function of $N$-dimensional variable in generalized Hölder space by $N$-dimensional Hausdorff matrices of $N$-multiple conjugate Fourier series ($N \geq 3$).
The organization of the paper is as follows: In section 1, we give definitions and notations related to the present work of the paper. In section 2, we propose our main result for obtaining best approximation of

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conjugate of a function $\tilde{g}(x,y)$ of two dimensional variable in generalized Hölder spaces $(H_r^{(\xi_1,\xi_2)}; r \geq 1)$ using double Hausdorff matrices of double conjugate Fourier series. In subsection 2.1, we prove five lemmas which are used in obtaining our main result. In subsection 2.2, we establish our main result and in subsection 2.3, four corollaries are deduced from our main result. In section 3, we propose another result to obtain the best approximation of conjugate of a function $\tilde{g}(x_1, x_2, \cdots, x_N)$ of $N$-dimensional variable in generalized Hölder spaces $(H_r^{(\xi_1,\xi_2,\cdots,\xi_N)}; r \geq 1)$ using $N$-dimensional Hausdorff matrices of $N$-multiple conjugate Fourier series and in subsection 3.1, two corollaries are deduced from our second result.

The Fourier series of a periodic function $g \in L^1(\mathbb{T}), \mathbb{T} = [-\pi, \pi]$ is given by

$$\sum_{m\in\mathbb{Z}} \tilde{g}(m) e^{imx}. \tag{1.1}$$

The conjugate series of (1) is defined by

$$\sum_{m\in\mathbb{Z}} (-i \text{ sign } m) \tilde{g}(m) e^{imx}, \tag{1.2}$$

where

$$\text{sign } m := \begin{cases} 1 & \text{if } \; m > 0, \\ 0 & \text{if } \; m = 0, \\ -1 & \text{if } \; m < 0. \end{cases}$$

and

$$\mathbb{Z} := \{\cdots - 2, -1, 0, 1, 2, \cdots\}.$$

Series (2) is said to be conjugate Fourier series. It is known that the corresponding conjugate function of (2) is defined as

$$\tilde{g}(x) = -\frac{1}{\pi} \int_0^{\pi} \frac{g(x+l) - g(x-l)}{2\tan(\frac{l}{2})} dl.$$

Consider $g(x,y)$ be measurable and bounded function on the two dimensional torus $\mathbb{T}^2$. Given a function $g \in L^1(\mathbb{T}^2)$, $2\pi$-periodic in each variable. The double Fourier series of a function $g(x,y)$ is given by

$$\sum\sum_{(n,m)\in\mathbb{Z}^2} c_{n,m}(g) e^{i(nx+my)}, \tag{1.3}$$

where

$$c_{n,m}(g) := \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} g(u,v) e^{-i(nx+my)} du dv.$$

One can associate three conjugate series to the double Fourier series (3) in the following ways:

$$\sum\sum_{(n,m)\in\mathbb{Z}^2} (-i \text{ sign } n) c_{n,m}(g) e^{i(nx+my)} \tag{1.4}$$

(conjugate taken in first variable),

$$\sum\sum_{(n,m)\in\mathbb{Z}^2} (-i \text{ sign } m) c_{n,m}(g) e^{i(nx+my)} \tag{1.5}$$

(conjugate taken in second variable) and

$$\sum\sum_{(n,m)\in\mathbb{Z}^2} (-i \text{ sign } n)(-i \text{ sign } m) c_{n,m}(g) e^{i(nx+my)} \tag{1.6}$$

(conjugate taken in both the variables).
The corresponding conjugate functions are defined as follows:

$$\tilde{g}^{10}(x,y) = \lim_{h\to 0} \tilde{g}^{10}(h;x,y),$$

(conjugate taken in the first variable), where

$$\tilde{g}^{10}(x,y) = \frac{1}{\pi}\int_h^{\pi} \frac{\psi_x(g(\cdot,y);s)}{2\tan(\frac{s}{2})}ds,\ \ h>0 \tag{1.7}$$

or

$$\tilde{g}^{10}(x,y) = -\frac{1}{\pi}\int_0^{\pi} \frac{g(x+s,y)-g(x-s,y)}{2\tan(\frac{s}{2})}ds. \tag{1.8}$$

Similarly,

$$\tilde{g}^{01}(x,y) = -\frac{1}{\pi}\int_0^{\pi} \frac{g(x,y+l)-g(x,y-l)}{2\tan(\frac{l}{2})}dl, \tag{1.9}$$

(conjugate taken in the second variable) and

$$\begin{aligned}&\tilde{g}^{11}(x,y)\\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \frac{g(x+s,y+l)-g(x-s,y+l)-g(x+s,y-l)+g(x-s,y-l)}{2\tan(\frac{s}{2})\cdot 2\tan(\frac{l}{2})}dsdl\end{aligned} \tag{1.10}$$

(conjugate taken in both the variables).
The integrals (8),(9) and (10) are taken in the sense of the "Cauchy principal value" at the points $x=0$, or $y=0$, or $x=y=0$, respectively.
In this paper, we shall consider the symmetric square partial sums of series (10).
The double Hausdorff matrix $H$ has the entries

$$h_{n,p;m,q} = \begin{cases} \binom{n}{p}\binom{m}{q}\Delta^{n-p}\Delta^{m-q}\mu_{p,q} & 0\le p\le n, 0\le q\le m; \\ 0 & p>n, q>m, \end{cases} \tag{1.11}$$

where $\{\mu_{p,q}\}$ is any real or complex sequence and for any sequence $\mu_{p,q}$, the forward difference operator $\Delta$ is defined by

$$\Delta\Delta\mu_{p,q} = \mu_{p,q} - \mu_{p+1,q} - \mu_{p,q+1} + \mu_{p+1,q+1}$$

and

$$\Delta^{n-p}\Delta^{m-q}\mu_{p,q} = \sum_{u=0}^{n-p}\sum_{v=0}^{m-q}(-1)^{p+q}\binom{n-p}{u}\binom{m-q}{v}\mu_{p+u,q+v}.$$

The necessary and sufficient condition for double Hausdorff matrix $(H)$ to be conservative is the existence of a mass function $\chi(u,v)\in BV[0,1]\times[0,1]$ such that

$$\int_0^1\int_0^1 |d\chi(u,v)| < \infty$$

and

$$\mu_{n,m} = \int_0^1\int_0^1 u^n v^m d\chi(u,v).$$

Without loss of generality, we may assume that $\chi(0,0)=0$. In addition, if we have $\chi(1,1)=1$, and the continuity conditions

$$\chi(u,+0)=\chi(u,0), \qquad \chi(u,+0) = \lim_{v\to 0}\chi(u,v),$$

$$\chi(+0,v)=\chi(0,v), \qquad \chi(+0,v) = \lim_{u\to 0}\chi(u,v)$$

are satisfied, then $\mu_{0,0}=1$.
We say that $\mu_{n,m}$ is a regular moment constant ( [7,18]).
Now, re-writing double Hausdorff matrix defined in (11), as

$$h_{n,p;m,q} = \begin{cases} \binom{n}{p}\binom{m}{q}\int_0^1\int_0^1 u^p(1-u)^{n-p}v^q(1-v)^{m-q}d\chi(u,v), & p=0,1,2,\cdots,n;\ q=0,1,\cdots,m; \\ 0, & p>n, q>m, \end{cases} \tag{1.12}$$

where single Hausdorff matrix $h_{n,p}$ or $h_{m,q}$ is given by

$$h_{n,p} = \begin{cases} \binom{n}{p}\int_0^1 u^p(1-u)^{n-p}d\chi(u), & p=0,1,2,\cdots,n; \\ 0, & p>n. \end{cases}$$

Let $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_{n,m}$ be a double conjugate Fourier series with $\tilde{s}_{n,m}=\sum_{p=0}^{n}\sum_{q=0}^{m} d_{p,q}$ as its $(n,m)^{th}$ partial sums.
The double Hausdorff means $\tilde{t}_{n,m}^{H}$ is given by

$$\tilde{t}_{n,m}^{H} = \sum_{p=0}^{n}\sum_{q=0}^{m} h_{n,p;m,q}\tilde{s}_{p,q}.$$

If $\tilde{t}_{n,m}^{H}\to c$ as $n,m\to\infty$, then the double conjugate Fourier series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} d_{n,m}$ with the sequence of $(n,m)^{th}$ partial sums $(\tilde{s}_{n,m})$ is summable to some finite value $c$ by the $H$ method.

**Remark: 1.** A double Hausdorff matrix method reduces to

(i) double Cesàro means $(C,\lambda,\sigma)$ if mass function $\chi(u,v)=\lambda\sigma\int_0^u\int_0^v(1-u)^{\lambda-1}(1-v)^{\sigma-1}dudv$.

(ii) double Euler's means $(E,\rho_n,\rho_m)$ if mass function

$$\chi(u,v) = \begin{cases} 0, & if\ \ u\in[0,a] \text{ and } v\in[0,b], \\ 1, & if\ \ u\in[a,1] \text{ and } v\in[b,1] \end{cases}$$

where $a=\frac{1}{1+\rho_n}$, $\rho_n>0$ and $b=\frac{1}{1+\rho_m}$, $\rho_m>0$.

**Note: 1.**

(i) Putting $\lambda=\sigma=1$ in Remark 1(i), $(C,\lambda,\sigma)$ means reduces to $(C,1,1)$ means.

(ii) Putting $\rho_n=1\ \forall\ n$ and $\rho_m=1\ \forall\ m$ in Remark 1(ii), $(E,\rho_n,\rho_m)$ means reduces to $(E,1,1)$ means.

The Hölder class for continuous periodic function $g(x,y)$ of period $2\pi$ in each variables $x$ and $y$ is defined as

$$H_{(\alpha,\beta)} = \left\{g : |g(x,y;s,l)| := |g(x+s,y+l)-g(x,y)| \le C_1(|s|^{\alpha}+|l|^{\beta})\right\}$$

for some $0<\alpha,\beta\le 1$ and for all $x,y,s,l$, where $C_1$ is a positive constant may depend on $f$, but not on $x,y,s,l$. This class of functions is also known as Lipschitz class and denoted by $Lip(\alpha,\beta)$. It can be easily varified that $H_{(\alpha,\beta)}$ is a Banach space with respect to the norm $\|\cdot\|_{(\alpha,\beta)}$ is defined by

$$\|g\|_{\alpha,\beta} = \|g\|_C + \sup_{x\neq s, y\neq l}\Delta^{\alpha,\beta}g(x,y;s,l),$$

where

$$\Delta^{\alpha,\beta}g(x,y;s,l) = \frac{|g(x+s,y+l)-g(x,y)|}{|x-s|^{\alpha}+|y-l|^{\beta}} \quad (x\neq s, y\neq l).$$

By convention $\Delta^{0,0}g(x,y;s,l)=0$ and

$$\|g\|_C=\sup_{(x,y)\in\mathbb{T}^2}|g(x,y)|.$$

The function space $L^r[[0,2\pi]\times[0,2\pi]]$ is given by

$$\begin{aligned}&L^r[[0,2\pi]\times[0,2\pi]]\\&=\left\{g:[[0,2\pi]\times[0,2\pi]]\to\mathbb{R}\times\mathbb{R}:\int_0^{2\pi}\int_0^{2\pi}|g(x,y)|^r dxdy<\infty, r\geq 1\right\}.\end{aligned}\tag{1.13}$$

The norm of (13) is defined by

$$\|f\|_r:=\begin{cases}\left\{\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|g(x,y)|^r\,dxdy\right\}^{\frac{1}{r}}, & r\in[1,\infty);\\ \operatorname{ess\,sup}_{x,y\in[0,2\pi]}|g(x,y)|, & r=\infty.\end{cases}$$

Let $\xi_1,\xi_2:[-\pi,\pi;-\pi,\pi]\to\mathbb{R}\times\mathbb{R}$ be a real valued arbitrary function. The class of function $H_r^{(\xi_1,\xi_2)}$ is defined by

$$H_r^{(\xi_1,\xi_2)}=\left\{g\in L^r[[0,2\pi]\times[0,2\pi]]:\sup_{s\neq 0,l\neq 0}\frac{\|g(x+s,y+l)-g(x,y)\|_r}{\xi_1(s)+\xi_2(l)}\right\},$$

where $\xi_1$ and $\xi_2$ are the moduli of continuity that is $\xi_1$ and $\xi_2$ are positive non-decreasing continuous function with the properties:

$$\lim_{s\to 0^+}\xi_1(s)=\xi_1(0)=0;\ \lim_{l\to 0^+}\xi_2(l)=\xi_2(0)=0$$

and

$$\xi_1(s_1+s_2)\leq\xi_1(s_1)+\xi_1(s_2);\quad \xi_2(l_1+l_2)\leq\xi_2(l_1)+\xi_2(l_2).$$

We define

$$\|g\|_r^{(\xi_1,\xi_2)}=\|g\|_r+\sup_{s\neq 0,l\neq 0}\frac{\|g(x+s,y+l)-g(x,y)\|_r}{\xi_1(s)+\xi_2(l)}.$$

Clearly, $\|\cdot\|_r^{(\xi_1,\xi_2)}$ is a norm on $H_r^{(\xi_1,\xi_2)}$.
It can be varified that the completeness of the space $L^r[[0,2\pi]\times[0,2\pi]]$ implies the completeness of the space $H_r^{(\xi_1,\xi_2)}$.
Let $\left(\frac{\xi_1(s)}{\eta_1(s)}\right)$ and $\left(\frac{\xi_2(l)}{\eta_2(l)}\right)$ both be non-decreasing and positive. Then

$$\|g\|_r^{(\eta_1,\eta_2)}\leq\max\left(1,\frac{\xi_1(2\pi)}{\eta_1(2\pi)},\frac{\xi_2(2\pi)}{\eta_2(2\pi)}\right)\|g\|_r^{(\xi_1,\xi_2)}<\infty.$$

Thus,

$$H_r^{(\xi_1,\xi_2)}\subseteq H_r^{(\eta_1,\eta_2)}\subseteq L^r[[0,2\pi]\times[0,2\pi]].$$

**Note: 2.** If $\xi_1(s)=|s|^\alpha,\xi_2(l)=|l|^\beta$ and $r\to\infty$, then $H_r^{(\xi_1,\xi_2)}$ class reduces to $H_{\alpha,\beta}$ class.
The degree of convergence of a summation method to a given function $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a measure that how fast $t_{n,m}$ converges to $g$, which is given by

$$\|g-t_{n,m}\|=O\left(\frac{1}{\nu_{n,m}}\right)\quad([2]),$$

where $\nu_{n,m} \to \infty$ as $n, m \to \infty$.
We write

$$\begin{aligned}\Psi(s,l) &= \psi(x,s;y,l)\\ &= \frac{1}{4}\left[g(x+s,y+l) - g(x+s,y-l) - g(x-s,y+l) + g(x-s,y-l)\right];\end{aligned}$$

$$\Psi(x,y) = \int_0^x \int_0^y |\psi(u,v)| du dv;$$

$$\tilde{K}_{n,m}(s,l) = \frac{1}{4\pi^2} \sum_{p=0}^{n} h_{n,p} \frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})} \cdot \sum_{q=0}^{m} h_{m,q} \frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}. \tag{1.14}$$

## 2. Degree of Convergence of a Function of Two Dimensional Variable of Double Conjugate Fourier Series

Degree of convergence of a function of two dimensional variable in generalized Hölder space of two dimensional Hausdorff matrices of double conjugate Fourier series is obtained in the following theorem:

### 2.1. Main Theorem

**Theorem 2.1** *Let $\tilde{g}(x,y)$ be a function, conjugate to a function $g(x,y)$(periodic with period $2\pi$ in both $x$ and $y$), Lebesgue integrable on $\mathbb{T}(-\pi,\pi;-\pi,\pi)$. Then, the error estimation of $\tilde{g}(x,y)$ in the space $(H_r^{(\xi_1,\xi_2)}, r \geq 1)$ using double Hausdorff matrix $(H)$, is given by*

$$\begin{aligned}\left\|\tilde{t}^{H}_{n,m}(x,y) - \tilde{g}(x,y)\right\|_r^{(\eta)} &= O\Bigg[4\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\log(m+1)\\ &+ \log(n+1)(m+1) + 2\Bigg) \cdot \left(\int_{\frac{1}{n+1}}^{\pi} \int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} \left(\frac{1}{s^2 l^2}\right) ds dl\right)\Bigg],\end{aligned} \tag{2.1}$$

*where $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ denote the moduli of continuity such that $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing.*

### 2.2. Lemmas

Following Lemmas are required for the proof of our main result:

**Lemma 2.1** $\tilde{K}_{n,m}(s,l) = O\left(\frac{1}{sl}\right)$ *for* $0 < s \leq \frac{1}{n+1}$ *and* $0 < l \leq \frac{1}{m+1}$.

**Proof:** For $0 < s \leq \frac{1}{n+1}, \sin(\frac{s}{2}) \geq \frac{s}{\pi}, |\cos ns| \leq 1$ and for $0 < l \leq \frac{1}{m+1}, \sin(\frac{l}{2}) \geq \frac{l}{\pi}, |\cos ml| \leq 1$.

$$\begin{aligned}&\left|\tilde{K}_{n,m}(s,l)\right|\\ &= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p} \frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})} \cdot \sum_{q=0}^{m} h_{m,q} \frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right|\\ &= \frac{1}{4\pi^2}\left\{\sum_{p=0}^{n} h_{n,p} \frac{\left|\cos(p+\frac{1}{2})s\right|}{\left|\sin(\frac{s}{2})\right|}\right\} \cdot \left\{\sum_{q=0}^{m} h_{m,q} \frac{\left|\cos(q+\frac{1}{2})l\right|}{\left|\sin(\frac{l}{2})\right|}\right\}\\ &\leq \frac{1}{4\pi^2}\left\{\sum_{p=0}^{n} h_{n,p} \cdot \frac{1}{\frac{s}{\pi}}\right\} \cdot \left\{\sum_{q=0}^{m} h_{m,q} \cdot \frac{1}{\frac{l}{\pi}}\right\}\\ &\leq \frac{\pi^2}{4\pi^2 sl} \sum_{p=0}^{n} h_{n,p} \cdot \sum_{q=0}^{m} h_{m,q}.\end{aligned}$$

We have $\sum_{p=0}^{n} h_{n,p} = 1$ and $\sum_{q=0}^{m} h_{m,q} = 1$ ([3, p.397]), then

$$\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{1}{sl}\right).$$

□

**Lemma 2.2** *$\tilde{K}_{n,m}(s,l) = O\left(\frac{1}{(m+1)sl^2}\right)$ for $0 < s \leq \frac{1}{n+1}$ and $\frac{1}{m+1} < l \leq \pi$.*

**Proof:** *For $0 < s \leq \frac{1}{n+1}, \sin(\frac{s}{2}) \geq \frac{s}{\pi}, |\cos(ns)| \leq 1$ and for $\frac{1}{m+1} < l \leq \pi, \sin(\frac{l}{2}) \geq \frac{l}{\pi}, |\sin(m+1)l| \leq 1,\ \sup_{0\leq v\leq 1} |\chi'(v)| = M$.*

$$\begin{aligned}
&\left|\tilde{K}_{n,m}(s,l)\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\frac{|\cos\left(p+\frac{1}{2}\right)s|}{|\sin(\frac{s}{2})|}\right. \\
&\quad \left.\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\frac{\cos\left(q+\frac{1}{2}\right)l}{|\sin(\frac{l}{2})|}\right| \\
&\leq \frac{1}{4\pi^2}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\frac{1}{\frac{s}{\pi}}\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\frac{\cos\left(q+\frac{1}{2}\right)l}{\frac{l}{\pi}}\right| \\
&\leq \frac{1}{4sl}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\cos\left(q+\frac{1}{2}\right)l\right| \\
&\leq \frac{M}{4sl}\left|\left[\int_0^1\left\{\binom{n}{0}u^0(1-u)^n+\binom{n}{1}u(1-u)^{n-1}+\cdots+\binom{n}{n}u^n(1-u)^{n-n}\right\}du\right]\right. \\
&\quad \left.\cdot(1-v)^m Re\left[\sum_{q=0}^{m}\int_0^1\binom{m}{q}\left(\frac{v}{1-v}\right)^q e^{i(q+\frac{1}{2})l}dv\right]\right| \\
&\leq \frac{M}{4sl}\left|\left(\int_0^1(1-u+u)^n du\right)\cdot(1-v)^m Re\left[\sum_{q=0}^{m}\int_0^1\binom{m}{q}\left(\frac{v}{1-v}\right)^q e^{iql}e^{\frac{il}{2}}dv\right]\right| \\
&\leq \frac{M}{4sl}\left|\left(\int_0^1 1du\right)\cdot(1-v)^m Re\left[e^{\frac{il}{2}}\sum_{q=0}^{m}\int_0^1\binom{m}{q}\left(\frac{ve^{il}}{1-v}\right)^q dv\right]\right| \\
&\leq \frac{M}{4sl}\left|(1-v)^m Re\left[e^{\frac{il}{2}}\int_0^1\left\{\binom{m}{0}\left(\frac{ve^{it}}{1-v}\right)^0+\binom{m}{1}\left(\frac{ve^{il}}{1-v}\right)^1+\cdots+\binom{m}{m}\left(\frac{ve^{il}}{1-v}\right)^m\right\}dv\right]\right| \\
&\leq \frac{M}{4sl}\left|Re\left[e^{\frac{il}{2}}\int_0^1\left\{\binom{m}{0}(1-v)^m+\binom{m}{1}ve^{il}(1-v)^{m-1}+\cdots+\binom{m}{m}(ve^{il})^m(1-v)^0\right\}dv\right]\right| \\
&\leq \frac{M}{4sl}\left|Re\left[e^{\frac{il}{2}}\int_0^1(1-v+ve^{il})^m dv\right]\right| \\
&\leq \frac{M}{4sl}\left|Re\left[e^{\frac{il}{2}}\int_0^1\left\{1+v(e^{il}-1)\right\}^m dv\right]\right|
\end{aligned}$$

*Now, solving integration*

$$\int_0^1\left\{1+v(e^{il}-1)\right\}^m dv$$

*Put*

$$\{1+v(e^{il}-1)\}=t$$
$$(e^{il}-1)dv=dt$$
$$dv=\frac{dt}{(e^{il}-1)}$$

$$\int_0^1 \frac{t^m}{(e^{il}-1)}dt=\frac{1}{(e^{il}-1)(m+1)}\left[\{1+v(e^{il}-1)\}^{m+1}\right]_0^1=\left[\frac{e^{i(m+1)l}-1}{(m+1)(e^{il}-1)}\right]$$

$$\begin{aligned}
&\leq \frac{M}{4sl}\left|Re\left[e^{\frac{il}{2}}\cdot\frac{e^{i(m+1)l}-1}{(m+1)(e^{il}-1)}\right]\right|\\
&\leq \frac{M}{4sl}\left|Re\left[\frac{e^{i(m+1)l}-1}{(m+1)(e^{\frac{il}{2}}-e^{-\frac{il}{2}})}\right]\right|\\
&\leq \frac{M}{4sl}\left|Re\left[\frac{\cos(m+1)l+i\sin(m+1)l-1}{(m+1)2i\sin(\frac{l}{2})}\right]\right|\\
&\leq \frac{M}{4sl}\left|\left(\frac{\sin(m+1)l}{2(m+1)\sin(\frac{l}{2})}\right)\right|\\
&\leq \frac{M}{8sl}\cdot\frac{\pi}{(m+1)l}.
\end{aligned}$$

*Hence,*

$$\left|\tilde{K}_{n,m}(s,l)\right|=O\left(\frac{1}{(m+1)sl^2}\right).$$

□

**Lemma 2.3** $\tilde{K}_{n,m}(s,l)=O\left(\frac{1}{(n+1)s^2l}\right)$ *for* $\frac{1}{n+1}<s\leq\pi$ *and* $0<l\leq\frac{1}{m+1}$.

**Proof:** *For* $\frac{1}{n+1}<s\leq\pi,\ \sin(\frac{s}{2})\geq\frac{s}{\pi},\ |\sin(n+1)s|\leq 1,\ \sup_{0\leq u\leq 1}|\chi'(u)|=F$ *and for* $0<l\leq\frac{1}{m+1},\ \sin\left(\frac{l}{2}\right)\geq\frac{l}{\pi},\ |\cos(ml)|\leq 1.$

$$\begin{aligned}
&\left|\tilde{K}_{n,m}(s,l)\right|\\
&=\frac{1}{4\pi^2}\left|\sum_{p=0}^{n}h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\sum_{q=0}^{m}h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right|\\
&=\frac{1}{4\pi^2}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\frac{\cos(p+\frac{1}{2})s}{|\sin(\frac{s}{2})|}\right.\\
&\quad\left.\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\frac{|\cos(q+\frac{1}{2})l|}{|\sin(\frac{l}{2})|}\right|\\
&\leq\frac{1}{4\pi^2}\left|\sum_{p=0}^{p}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\frac{\cos(p+\frac{1}{2})s}{\frac{s}{\pi}}\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\frac{1}{\frac{l}{\pi}}\right|\\
&\leq\frac{1}{4sl}\left|\sum_{p=0}^{p}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\cos\left(p+\frac{1}{2}\right)s\right.\\
&\quad\left.\cdot\left[\int_0^1\left\{\binom{m}{0}v^0(1-v)^m+\binom{m}{1}v(1-v)^{m-1}+\cdots+\binom{m}{m}v^m(1-v)^{m-m}\right\}d\chi(v)\right]\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{F}{4sl}\left|(1-u)^n Re\left[\sum_{p=0}^{n}\int_0^1\binom{n}{p}\left(\frac{u}{1-u}\right)^p e^{i(p+\frac{1}{2})s}du\right]\cdot\left[\int_0^1(1-v+v)^m dv\right]\right| \\
&\leq \frac{F}{4sl}\left|(1-u)^n Re\left[\sum_{p=0}^{n}\int_0^1\binom{n}{p}\left(\frac{u}{1-u}\right)^p e^{ips}e^{\frac{is}{2}}du\right]\right| \\
&\leq \frac{F}{4sl}\left|(1-u)^n Re\left[e^{\frac{is}{2}}\int_0^1\sum_{p=0}^{n}\binom{n}{p}\left(\frac{ue^{is}}{1-u}\right)^p du\right]\right| \\
&\leq \frac{F}{4sl}\left|(1-u)^n Re\left[e^{\frac{is}{2}}\int_0^1\left\{\binom{n}{p}\left(\frac{ue^{is}}{1-u}\right)^0+\binom{n}{1}\left(\frac{ue^{is}}{1-u}\right)+\cdots+\binom{n}{n}\left(\frac{ue^{is}}{1-u}\right)^n\right\}du\right]\right| \\
&\leq \frac{F}{4sl}\left|Re\left[e^{\frac{is}{2}}\int_0^1\left\{\binom{n}{0}(1-u)^n+\binom{n}{1}(ue^{is})(1-u)^{n-1}+\cdots+\binom{n}{n}(ue^{is})^n(1-u)^0\right\}du\right]\right| \\
&\leq \frac{F}{4sl}\left|Re\left[e^{\frac{is}{2}}\int_0^1(1-u+ue^{is})^n du\right]\right| \\
&\leq \frac{F}{4sl}\left|Re\left[e^{\frac{is}{2}}\int_0^1\left\{1+u(e^{is}-1)\right\}^n du\right]\right| \\
&\leq \frac{F}{4sl}\left|Re\left[\frac{e^{i(n+1)s}-1}{(n+1)(e^{\frac{is}{2}}-e^{-\frac{is}{2}})}\right]\right| \\
&\leq \frac{F}{4sl}\left|Re\left[\frac{\cos(n+1)s+i\sin(n+1)s-1}{(n+1)2i\sin(\frac{s}{2})}\right]\right| \\
&\leq \frac{F}{4sl}\left|\frac{\sin(n+1)s}{2(n+1)\sin(\frac{s}{2})}\right| \\
&\leq \frac{F}{8sl}\cdot\frac{\pi}{(n+1)s}
\end{aligned}$$

*Hence,*

$$\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{1}{(n+1)s^2 l}\right).$$

□

**Lemma 2.4** $\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{1}{(n+1)(m+1)s^2l^2}\right)$ *for* $\frac{1}{n+1} < s \leq \pi$ *and* $\frac{1}{m+1} < l \leq \pi$.

**Proof:** *For* $\frac{1}{n+1} < s \leq \pi,\ \sin(\frac{s}{2}) \geq \frac{s}{\pi},\ |\sin(n+1)s| \leq 1,\ \sup_{0\leq u\leq 1}|\chi'(u)| = F$ *and for* $0 < l \leq \frac{1}{m+1},\ \sin(\frac{l}{2}) \geq \frac{l}{\pi},\ |\sin(m+1)l| \leq 1,\ \sup_{0\leq v\leq 1}|\chi'(v)| = M$.

$$
\begin{aligned}
&\left|\tilde{K}_{n,m}(s,l)\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\frac{\cos(p+\frac{1}{2})s}{|\sin(\frac{s}{2})|}\right. \\
&\qquad\left.\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\frac{\cos(q+\frac{1}{2})l}{|\sin(\frac{l}{2})|}\right| \\
&\le \frac{1}{4\pi^2}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\frac{\cos(p+\frac{1}{2})s}{\frac{s}{\pi}}\right. \\
&\qquad\left.\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\frac{\cos(q+\frac{1}{2})l}{\frac{l}{\pi}}\right| \\
&\le \frac{1}{4sl}\left|\sum_{p=0}^{n}\int_0^1\binom{n}{p}u^p(1-u)^{n-p}d\chi(u)\cos\left(p+\frac{1}{2}\right)s\right. \\
&\qquad\left.\cdot\sum_{q=0}^{m}\int_0^1\binom{m}{q}v^q(1-v)^{m-q}d\chi(v)\cos\left(q+\frac{1}{2}\right)l\right| \\
&\le \frac{FM}{4sl}\left|(1-u)^n Re\left[\sum_{p=0}^{n}\int_0^1\binom{n}{p}\left(\frac{u}{1-u}\right)^p e^{i(p+\frac{1}{2})s}du\right]\right. \\
&\qquad\left.\cdot(1-v)^m Re\left[\sum_{q=0}^{m}\int_0^1\binom{m}{q}\left(\frac{v}{1-v}\right)^q e^{i(q+\frac{1}{2})l}dv\right]\right| \\
&\le \frac{FM}{4sl}\left|(1-u)^n Re\left[e^{\frac{is}{2}}\int_0^1\left\{\sum_{p=0}^{n}\binom{n}{p}\left(\frac{ue^{is}}{1-u}\right)^p\right\}du\right]\right. \\
&\qquad\left.\cdot(1-v)^m Re\left[e^{\frac{il}{2}}\int_0^1\left\{\sum_{q=0}^{m}\binom{m}{q}\left(\frac{ve^{il}}{1-v}\right)^q\right\}dv\right]\right| \\
&\le \frac{FM}{4sl}\left|(1-u)^n Re\left[e^{\frac{is}{2}}\int_0^1\left\{\binom{n}{0}\left(\frac{ue^{is}}{1-u}\right)^0+\binom{n}{1}\left(\frac{ue^{is}}{1-u}\right)+\cdots+\binom{n}{n}\left(\frac{ue^{is}}{1-u}\right)^n\right\}du\right]\right. \\
&\qquad\left.\cdot(1-v)^m Re\left[e^{\frac{il}{2}}\int_0^1\left\{\binom{m}{0}\left(\frac{ve^{il}}{1-v}\right)^0+\binom{m}{1}\left(\frac{ve^{il}}{1-v}\right)+\cdots+\binom{m}{m}\left(\frac{ve^{il}}{1-v}\right)^m\right\}dv\right]\right| \\
&\le \frac{FM}{4sl}\left|Re\left[e^{\frac{is}{2}}\int_0^1\left\{\binom{n}{0}(1-u)^n+\binom{n}{1}(ue^{is})(1-u)^{n-1}+\cdots+\binom{n}{n}(ue^{is})^n(1-u)^0\right\}du\right]\right. \\
&\quad\left.\cdot Re\left[e^{\frac{il}{2}}\int_0^1\left\{\binom{m}{0}(1-v)^m+\binom{m}{1}(ve^{il})(1-v)^{m-1}+\cdots+\binom{m}{m}(ve^{il})^m(1-v)^0\right\}dv\right]\right| \\
&\le \frac{FM}{4sl}\left|Re\left[e^{\frac{is}{2}}\int_0^1(1-u+ue^{is})^n du\right]\cdot Re\left[e^{\frac{il}{2}}\int_0^1(1-v+ve^{il})^m dv\right]\right| \\
&\le \frac{FM}{4sl}\left|Re\left[\frac{e^{i(n+1)s}-1}{(n+1)(e^{\frac{is}{2}}-e^{-\frac{is}{2}})}\right]\cdot Re\left[\frac{e^{i(m+1)l}-1}{(m+1)(e^{\frac{il}{2}}-e^{-\frac{il}{2}})}\right]\right|
\end{aligned}
$$

$$\leq \frac{FM}{4sl}\left|Re\left[\frac{\cos(n+1)+i\sin(n+1)-1}{(n+1)2i\sin(\frac{s}{2})}\right]\cdot Re\left[\frac{\cos(m+1)+i\sin(m+1)-1}{(m+1)2i\sin(\frac{l}{2})}\right]\right|$$
$$\leq \frac{FM}{4sl}\left|\left(\frac{\sin(n+1)s}{2(n+1)\sin(\frac{s}{2})}\right)\cdot\left(\frac{\sin(m+1)l}{2(m+1)\sin(\frac{l}{2})}\right)\right|$$
$$\leq \frac{FM}{16sl}\cdot\frac{\pi}{(n+1)s}\cdot\frac{\pi}{(m+1)l}$$
$$\leq \frac{FM\pi^2}{16(n+1)(m+1)s^2l^2}$$

*Hence,*

$$\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{1}{(n+1)(m+1)s^2l^2}\right).$$

□

**Lemma 2.5** *Let* $\tilde{g}(x,y)\in H_r^{(\xi_1,\xi_2)}(r\geq 1)$*. Then for* $0<s\leq\pi$ *and* $0<l\leq\pi$*,*

*(i)* $\|\psi(\cdot,s;\cdot,l)\|_r = O\{\xi_1(s)+\xi_2(l)\}$;

*(ii) For positive, non-decreasing* $\eta_1$ *and* $\eta_2$*; and for all possible* $s\neq 0$ *and* $l\neq 0$*, we obtain*

$$\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r$$
$$=\begin{cases} O\left((\eta_1(s)+\eta_2(l))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right) & \text{for } s<|u|, l<|v|;\\ O\left((\eta_1(|u|)+\eta_2(l))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right) & \text{for } s>|u|, l<|v|;\\ O\left((\eta_1(s)+\eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right) & \text{for } s<|u|, l>|v|;\\ O\left((\eta_1(|u|)+\eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right) & \text{for } s>|u|, l>|v|. \end{cases}$$

**Proof:** *(i)*

$$\begin{aligned}
|\psi(x,s;y,l)| &= \frac{1}{4}|\,[\tilde{g}(x+s,y+l)-\tilde{g}(x+s,y-l)-\tilde{g}(x-s,y+l)+\tilde{g}(x-s,y-l)]\,|\\
&\leq \frac{1}{4}|\tilde{g}(x+s,y+l)-\tilde{g}(x+s,y-l)-\tilde{g}(x-s,y+l)+\tilde{g}(x+s,y+l)\\
&\quad+2\tilde{g}(x,y)-2\tilde{g}(x,y)|\\
&\leq \frac{1}{4}[|\tilde{g}(x+s,y+l)-\tilde{g}(x,y)|+|\tilde{g}(x+s,y-l)-\tilde{g}(x,y)|\\
&\quad+|\tilde{g}(x-s,y+l)-\tilde{g}(x,y)|+|\tilde{g}(x-s,y-l)-\tilde{g}(x,y)|]\\
\|\psi(\cdot,s;\cdot,l)\|_r &\leq \frac{1}{4}[\|\tilde{g}(x+s,y+l)-\tilde{g}(x,y)\|_r+\|\tilde{g}(x+s,y-l)-\tilde{g}(x,y)\|_r\\
&\quad+\|\tilde{g}(x-s,y+l)-\tilde{g}(x,y)\|_r+\|\tilde{g}(x-s,y-l)-\tilde{g}(x,y)\|_r]\\
&= O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}\\
&\quad+O\{\xi_1(s)+\xi_2(l)\}\\
\|\psi(\cdot,s;\cdot,l)\|_r &= O\left\{\xi_1(s)+\xi_2(l)\right\}.
\end{aligned}$$

$$\begin{aligned}
&(ii)\quad |\psi(x+u,s;y+v,l)-\psi(x,s;y,l)| \\
&=\frac{1}{4}[|\{\tilde{g}(x+u+s,y+v+l)-\tilde{g}(x+u+s,y+v-l)-\tilde{g}(x+u-s,y+v+l) \\
&\quad+\tilde{g}(x+u-s,y+v-l)\}-\{\tilde{g}(x+s,y+l)-\tilde{g}(x+s,y-l)-\tilde{g}(x-s,y+l) \\
&\quad+\tilde{g}(x-s,y-l)\}|] \\
&\le\frac{1}{4}[|\tilde{g}(x+u+s,y+v+l)-\tilde{g}(x+s,y+l)|+|\tilde{g}(x+u+s,y+v-l)-\tilde{g}(x+s,y-l)| \\
&\quad+|\tilde{g}(x+u-s,y+v+l)-\tilde{g}(x-s,y+l)|+|\tilde{g}(x+u-s,y+v-l)-\tilde{g}(x-s,y-l)|]
\end{aligned}$$

*Applying generalized Minkowski's inequality, we have*

$$\begin{aligned}
&\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \\
&\le\frac{1}{4}[\|\tilde{g}(\cdot+u+s,\cdot+v+l)-\tilde{g}(\cdot+s,\cdot+l)\|_r+\|\tilde{g}(\cdot+u+s,\cdot+v-l)-\tilde{g}(\cdot+s,\cdot-l)\|_r \\
&\quad+\|\tilde{g}(\cdot+u-s,\cdot+v+l)-\tilde{g}(\cdot-s;\cdot+l)\|_r+\|\tilde{g}(\cdot+u-s,\cdot+v-l)-\tilde{g}(\cdot-s,\cdot-l)\|_r] \\
&=[O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}] \\
&=O\{\xi_1(s)+\xi_2(l)\}. \qquad (2.2)
\end{aligned}$$

*Since* $\eta_1$ *and* $\eta_2$ *are non-decreasing and positive,* $s<|u|, l<|v|$*, then using (16), we get*

$$\begin{aligned}
\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r&=O[\xi_1(s)+\xi_2(l)] \\
&=O\left[(\eta_1(s)+\eta_2(l))\cdot\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right]
\end{aligned}$$

*Similarly,*

$$\begin{aligned}
&\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \\
&\le\frac{1}{4}[\|\tilde{g}(\cdot+s+u,\cdot+v+l)-\tilde{g}(\cdot+s,\cdot+l)\|_r+\|\tilde{g}(\cdot+s+u,\cdot+v-l)-\tilde{g}(\cdot+s,\cdot-l)\|_r \\
&\quad+\|\tilde{g}(\cdot+s-u,\cdot+v+l)-\tilde{g}(\cdot-s,\cdot+l)\|_r+\|\tilde{g}(\cdot+s-u,\cdot+v-l)-\tilde{g}(\cdot-s,\cdot-l)\|_r] \\
&=[O\{\xi_1(|u|)+\xi_2(l)\}+O\{\xi_1(|u|)+\xi_2(l)\}+O\{\xi_1(|u|)+\xi_2(l)\}+O\{\xi_1(|u|)+\xi_2(l)\}] \\
&=O\{\xi_1(|u|)+\xi_2(l)\}. \qquad (2.3)
\end{aligned}$$

*Since* $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ *is non-decreasing and positive,* $s>|u|, l<|v|$*, then* $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\ge\frac{\xi_1(|u|)+\xi_2(l)}{\eta_1(|u|)+\eta_2(l)}$*. Thus, using (17), we get*

$$\begin{aligned}
\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r&=O[\xi_1(|u|)+\xi_2(l)] \\
&=O\left[(\eta_1(|u|)+\eta_2(l))\cdot\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right].
\end{aligned}$$

*Similarly,*

$$\begin{aligned}
&\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \\
&\le\frac{1}{4}[\|\tilde{g}(\cdot+u+s,\cdot+l+v)-\tilde{g}(\cdot+s,\cdot+l)\|_r+\|\tilde{g}(\cdot+u+s,\cdot+l-v)-\tilde{g}(\cdot+s,\cdot-l)\|_r \\
&\quad+\|\tilde{g}(\cdot+u-s,\cdot+l+v)-\tilde{g}(\cdot-s,\cdot+l)\|_r+\|\tilde{g}(\cdot+u-s,\cdot+l-v)-\tilde{g}(\cdot-s,\cdot-l)\|_r] \\
&=[O\{\xi_1(s)+\xi_2(|v|)\}+O\{\xi_1(s)+\xi_2(|v|)\}+O\{\xi_1(s)+\xi_2(|v|)\}+O\{\xi_1(s)+\xi_2(|v|)\}] \\
&=O\{\xi_1(s)+\xi_2(|v|)\}. \qquad (2.4)
\end{aligned}$$

*Since* $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ *is non-decreasing and positive,* $s < |u|, l > |v|$, *then* $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} \geq \frac{\xi_1(s)+\xi_2(|v|)}{\eta_1(s)+\eta_2(|v|)}$. *Thus, using (18), we get*

$$\begin{aligned}\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r &= O[\xi_1(s)+\xi_2(|v|)]\\ &= O\left[(\eta_1(s)+\eta_2(|v|))\cdot\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right].\end{aligned}$$

*Similarly,*

$$\begin{aligned}&\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r\\ &\leq \frac{1}{4}[\|\tilde{g}(\cdot+s+u,\cdot+l+v)-\tilde{g}(\cdot+s,\cdot+l)\|_r+\|\tilde{g}(\cdot+s+u,\cdot+l-v)-\tilde{g}(\cdot+s,\cdot-l)\|_r\\ &\quad+\|\tilde{g}(\cdot+s-u,\cdot+l+v)-\tilde{g}(\cdot-s,\cdot+l)\|_r+\|\tilde{g}(\cdot+s-u,\cdot+l-v)-\tilde{g}(\cdot-s,\cdot-l)\|_r]\\ &=[O\{\xi_1(|u|)+\xi_2(|v|)\}+O\{\xi_1(|u|)+\xi_2(|v|)\}+O\{\xi_1(|u|)+\xi_2(|v|)\}+O\{\xi_1(|u|)+\xi_2(|v|)\}]\\ &=O\{\xi_1(|u|)+\xi_2(|v|)\}. \end{aligned}\tag{2.5}$$

*Since* $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ *is non-decreasing and positive,* $s > |u|, l > |v|$, *then* $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} \geq \frac{\xi_1(|u|)+\xi_2(|v|)}{\eta_1(|u|)+\eta_2(|v|)}$. *Thus, using (19), we get*

$$\begin{aligned}\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r &= O[\xi_1(|u|)+\xi_2(|v|)]\\ &= O\left[(\eta_1(|u|)+\eta_2(|v|))\cdot\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right].\end{aligned}$$

□

### 2.3. Proof of the main Theorem 2.1

**Proof:** The $(p,q)$th partial sums $\tilde{s}_{p,q}(x,y)$ of the series (10) is given by

$$\begin{aligned}\tilde{s}_{p,q}(x,y)-\tilde{g}(x,y) &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \psi(x,s;y,l)\frac{\cos(p+\frac{1}{2})s}{2\sin(\frac{s}{2})}\cdot\frac{\cos(q+\frac{1}{2})l}{2\sin(\frac{l}{2})}dsdl\\ &= \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi \psi(x,s;y,l)\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}dsdl.\end{aligned}\tag{2.6}$$

The $\tilde{t}^H_{n,m}(x,y)$ is double Hausdorff matrix mean of $\tilde{s}_{n,m}(x,y)$ and taking in view (20), we write

$$\begin{aligned}\tilde{t}^H_{n,m}(x,y)-\tilde{g}(x,y) &= \sum_{p=0}^{n}\sum_{q=0}^{m} h_{n,p;m,q}\left\{\tilde{s}_{p,q}(x,y)-\tilde{g}(x,y)\right\}\\ &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \psi(x,s;y,l)\sum_{p=0}^{n}\sum_{q=0}^{m} h_{n,p;m,q}\frac{\cos(p+\frac{1}{2})s}{2\sin(\frac{s}{2})}\frac{\cos(q+\frac{1}{2})l}{2\sin(\frac{l}{2})}dsdl\\ &= \int_0^\pi\int_0^\pi \psi(x,s;y,l)\tilde{K}_{n,m}(s,l)dsdl.\end{aligned}$$

Let

$$\begin{aligned}\tilde{l}_{n,m}(x,y) &= \tilde{t}^H_{n,m}(x,y)-\tilde{g}(x,y)\\ &= \int_0^\pi\int_0^\pi \psi(x,s;y,l)\tilde{K}_{n,m}(s,l)dsdl.\end{aligned}\tag{2.7}$$

Let us estimate the following:

$$\begin{aligned}\|\tilde{l}_{n,m}(\cdot,\cdot)\|_r^{(\eta)} &= \|\tilde{l}_{n,m}(x,y)\|_r + \sup_{s\neq 0, l\neq 0} \frac{\|\tilde{l}_{n,m}(x+s,y+l)-\tilde{l}_{n,m}(x,y)\|_r}{\eta_1(s)+\eta_2(l)} \\ &= \|\tilde{l}_{n,m}(x,y)\|_r + \sup_{u\neq 0, v\neq 0} \frac{\|\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y)\|_r}{\eta_1(|u|)+\eta_2(|v|)} \\ &\quad + \sup_{s\neq 0, v\neq 0} \frac{\|\tilde{l}_{n,m}(x+s,y+v)-\tilde{l}_{n,m}(x,y)\|_r}{\eta_1(s)+\eta_2(|v|)} \\ &\quad + \sup_{u\neq 0, l\neq 0} \frac{\|\tilde{l}_{n,m}(x+u,y+l)-\tilde{l}_{n,m}(x,y)\|_r}{\eta_1(|u|)+\eta_2(l)}. \end{aligned} \tag{2.8}$$

From (21), we write

$$\begin{aligned}&\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y) \\ &\quad = \int_0^\pi \int_0^\pi \left(\psi(x+u,s;y+v,l)-\psi(x,s;y,l)\right)\tilde{K}_{n,m}(s,l)dsdl.\end{aligned}$$

Using generalized Minkowski's inequality ( [6]), we have

$$\begin{aligned}&\left\|\tilde{l}_{n,m}(\cdot+u,\cdot+v)-\tilde{l}_{n,m}(\cdot,\cdot)\right\|_r \\ &\leq \int_0^\pi \int_0^\pi \|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \tilde{K}_{n,m}(s,l)dsdl \\ &= \left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}} + \int_0^{\frac{1}{n+1}}\int_{\frac{1}{m+1}}^{\pi} + \int_{\frac{1}{n+1}}^{\pi}\int_0^{\frac{1}{m+1}} + \int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\right)\|\psi(\cdot+u,s;\cdot+v,l) \\ &\quad -\psi(\cdot,s;\cdot,l)\|_r\cdot\tilde{K}_{n,m}(s,l)dsdl \\ &= A+B+C+D.\end{aligned} \tag{2.9}$$

Using Lemmas 2.2 and 2.6(ii), we get

$$\begin{aligned}A &= \int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}} \|(\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r\tilde{K}_{n,m}(s,l)dsdl \\ &= O\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}} (\eta_1(|u|)+\eta_2(|v|))\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\cdot\left(\frac{1}{sl}\right)dsdl\right) \\ &= O\left((\eta_1(|u|)+\eta_2(|v|))\left\{\int_0^{\frac{1}{m+1}}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\cdot\frac{1}{l}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right) \\ &= O\left(\log(n+1)(\eta_1(|u|)+\eta_2(|v|))\cdot\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\int_0^{\frac{1}{m+1}}\frac{1}{l}dl\right) \\ &= O\left(\log(n+1)\cdot\log(m+1)(\eta_1(|u|)+\eta_2(|v|))\cdot\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right).\end{aligned} \tag{2.10}$$

Using Lemmas 2.3 and 2.6(ii), we get

$$
\begin{aligned}
B &= \int_0^{\frac{1}{n+1}} \int_{\frac{1}{m+1}}^{\pi} \|(\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l))\|_r \tilde{K}_{n,m}(s,l)dsdl \\
&= O\left(\int_0^{\frac{1}{n+1}} \int_{\frac{1}{m+1}}^{\pi} (\eta_1(|u|)+\eta_2(|v|))\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} \cdot \frac{1}{(m+1)sl^2}dsdl\right) \\
&= O\left(\frac{1}{(m+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right) \\
&= O\left(\frac{\log(n+1)}{(m+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right).
\end{aligned}
\tag{2.11}
$$

Using Lemmas 2.4 and 2.6(ii), we get

$$
\begin{aligned}
C &= \int_{\frac{1}{n+1}}^{\pi} \int_0^{\frac{1}{m+1}} \|(\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l))\|_r \tilde{K}_{n,m}(s,l)dsdl \\
&= O\left(\int_{\frac{1}{n+1}}^{\pi} \int_0^{\frac{1}{m+1}} (\eta_1(|u|)+\eta_2(|v|))\cdot\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} \cdot \frac{1}{(n+1)s^2l}dsdl\right) \\
&= O\left(\frac{1}{(n+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi} \frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\int_0^{\frac{1}{m+1}}\frac{1}{l}dl\right) \\
&= O\left(\frac{\log(m+1)}{(n+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi} \frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right).
\end{aligned}
\tag{2.12}
$$

Using Lemmas 2.5 and 2.6(ii), we get

$$
\begin{aligned}
D &= \int_{\frac{1}{n+1}}^{\pi} \int_{\frac{1}{n+1}}^{\pi} \|(\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l))\|_r \tilde{K}_{n,m}(s,l)dsdl \\
&= O\left(\int_{\frac{1}{n+1}}^{\pi} \int_{\frac{1}{m+1}}^{\pi} (\eta_1(|u|)+\eta_2(|v|))\cdot\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} \cdot \frac{1}{(n+1)(m+1)s^2l^2}dsdl\right) \\
&= O\left(\frac{1}{(n+1)(m+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi} \int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right).
\end{aligned}
\tag{2.13}
$$

Combining (23)-(27), we have

$$
\begin{aligned}
&\left\|\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y)\right\|_r \\
&= O\left(\log(n+1)\cdot\log(m+1)(\eta_1(|u|)+\eta_2(|v|))\cdot\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right) \\
&+O\left(\frac{\log(n+1)}{(m+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right) \\
&+O\left(\frac{\log(m+1)}{(n+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right) \\
&+O\left(\frac{1}{(n+1)(m+1)}(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sup_{u\neq0,v\neq0}\frac{\left\|\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y)\right\|_r}{(\eta_1(|u|)+\eta_2(|v|))} \\
&= O\left(\log(n+1)\cdot\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right) \\
&+O\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{m+1}\right)+\xi_2(l)}{\eta_2\left(\frac{1}{m+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right) \\
&+O\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right) \\
&+O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right).
\end{aligned}
\tag{2.14}
$$

Similarly, we can have

$$
\begin{aligned}
&\sup_{s\neq0,v\neq0}\frac{\left\|\tilde{l}_{n,m}(x+s,y+v)-\tilde{l}_{n,m}(x,y)\right\|_r}{(\eta_1(s)+\eta_2(|v|))} \\
&= O\left(\log(n+1)\cdot\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)
\end{aligned}
$$

$$+O\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{m+1}\right)+\xi_2(l)}{\eta_2\left(\frac{1}{m+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right)$$
$$+O\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right)$$
$$+O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right). \tag{2.15}$$

Similarly, we can have

$$\sup_{u\neq 0,l\neq 0}\frac{\left\|\tilde{l}_{n,m}(x+u,y+l)-\tilde{l}_{n,m}(x,y)\right\|_r}{(\eta_1(|u|)+\eta_2(l))}$$
$$=O\left(\log(n+1)\cdot\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)$$
$$+O\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{m+1}\right)+\xi_2(l)}{\eta_2\left(\frac{1}{m+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right)$$
$$+O\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right)$$
$$+O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right). \tag{2.16}$$

Now, we solve

$$\left\|\tilde{l}_{n,m}(x,y)\right\|_r=\left\|\left(\tilde{t}^H_{n,m}(x,y)-\tilde{g}(x,y)\right)\right\|_r$$
$$=\left(\int_0^{\pi}\int_0^{\pi}\|\psi(x,s;y,l)\|_r\,\tilde{K}_{n,m}(s,l)dsdl\right)$$
$$\leq\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}+\int_0^{\frac{1}{n+1}}\int_{\frac{1}{m+1}}^{\pi}+\int_{\frac{1}{n+1}}^{\pi}\int_0^{\frac{1}{m+1}}+\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\right)\cdot\|\psi(x,s;y,l)\|_r$$
$$\cdot\left|\tilde{K}_{n,m}(s,l)\right|dsdl.$$

Using Lemmas 2.2 to 2.5 and 2.6(i), we have

$$\left\|\tilde{l}_{n,m}(x,y)\right\|_r$$
$$=O\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}\frac{\xi_1(s)+\xi_2(l)}{sl}dsdl\right)+O\left(\frac{1}{(m+1)}\int_0^{\frac{1}{n+1}}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{sl^2}dsdl\right)$$
$$+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\int_0^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l}dsdl\right)$$
$$+O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l^2}dsdl\right)$$

$$
\begin{aligned}
&= O\left(\left\{\int_0^{\frac{1}{m+1}} \frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right)\\
&\quad + O\left(\frac{1}{(m+1)}\left\{\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l^2}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right)\\
&\quad + O\left(\frac{1}{(n+1)}\left\{\int_{\frac{1}{n+1}}^{\pi} \frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{s^2}ds\right\}\int_0^{\frac{1}{m+1}}\frac{1}{l}dl\right)\\
&\quad + O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l^2}dsdl\right)\\
&= O\left(\log(n+1)\cdot\log(m+1)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}\right)\\
&\quad + O\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l^2}dl\right)\\
&\quad + O\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{s^2}ds\right)\\
&\quad + O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l^2}dsdl\right). \qquad (2.17)
\end{aligned}
$$

Using (28)-(31) in (22), we get

$$
\begin{aligned}
\|\tilde{l}_{n,m}(\cdot,\cdot)\|_r^{(\eta)} &= O\left(\log(n+1)\cdot\log(m+1)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}\right)\\
&\quad + O\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l^2}dl\right)\\
&\quad + O\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{s^2}ds\right)\\
&\quad + O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l^2}dsdl\right)\\
&\quad + O\left[3\left(\log(n+1)\cdot\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\right]\\
&\quad + O\left[3\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\right)\left(\frac{1}{l^2}\right)dl\right)\right]\\
&\quad + O\left[3\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right)\right]\\
&\quad + O\left[3\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\right]. \qquad (2.18)
\end{aligned}
$$

Since $\xi_1(s)+\xi_2(l) = \frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ and $(\eta_1(s)+\eta_2(l)) \leq (\eta_1(\pi)+\eta_2(\pi))\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$, $0 < s \leq \pi, 0 < l \leq \pi$, we get

$$\begin{aligned}\|\tilde{l}_{n,m}(\cdot,\cdot)\|_r^{(\eta)} &= O\left[4\left(\log(n+1)\log(m+1)\left\{\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right\}\right)\right]\\
&+O\left[4\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\right)\left(\frac{1}{l^2}\right)dl\right)\right]\\
&+O\left[4\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right)\right]\\
&+O\left[4\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\right].\end{aligned} \tag{2.19}$$

Since $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ are moduli of continuity, $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive, non-decreasing and

$$\begin{aligned}\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right) &\geq \frac{\log(n+1)}{(m+1)}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{l^2}dl\\
&\geq \frac{\log(n+1)}{2}\cdot\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\end{aligned}$$

Then,

$$O\left(\frac{\log(n+1)}{(m+1)}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right) = O\left(\frac{\log(n+1)}{2}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right). \tag{2.20}$$

Since $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ are moduli of continuity, $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive, non-decreasing and

$$\begin{aligned}&\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\right)\left(\frac{1}{s^2}\right)ds\right)\\
&\qquad\geq\left(\frac{\log(m+1)}{(n+1)}\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{s^2}\right)ds\right)\\
&\qquad\geq\left(\frac{\log(m+1)}{2}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right).\end{aligned}$$

Then,

$$O\left(\frac{\log(m+1)}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right) = O\left(\frac{\log(m+1)}{2}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right). \tag{2.21}$$

Since $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ are moduli of continuity, $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive, non-decreasing and

$$\begin{aligned}
&\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\left(\frac{1}{s^2l^2}\right)dsdl\right)\\
&\quad\geq\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{s^2l^2}dsdl\right)\\
&\quad\geq\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{4\left(\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)\right)}\left(\frac{(n+1)(m+1)}{(n+1)(m+1)}\right).
\end{aligned}$$

Then,

$$O\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)=O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right). \tag{2.22}$$

Combining (33)-(36), we get

$$\begin{aligned}
&\|\tilde{l}_{n,m}(\cdot,\cdot)\|_r^{(\eta)}\\
&=O\Bigg[\left(\log(n+1)\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)+\left(\frac{\log(n+1)}{2}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\\
&\qquad+\left(\frac{\log(m+1)}{2}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\Bigg]\\
&\qquad+O\left[\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\left(\frac{1}{s^2l^2}\right)dsdl\right]\\
&=O\Bigg[\left(\log(n+1)\cdot\log(m+1)+\frac{\log(n+1)}{2}+\frac{\log(m+1)}{2}\right)\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\Bigg]\\
&\qquad+O\left[\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\left(\frac{1}{s^2l^2}\right)dsdl\right]\\
&=O\Bigg[\left(\log(n+1)\cdot\log(m+1)+\frac{\log(n+1)}{2}+\frac{\log(m+1)}{2}\right)\\
&\qquad\cdot O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\Bigg]\\
&\qquad+O\left[\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right]\\
&=O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Big(2\log(n+1)\log(m+1)+\log(n+1)(m+1)+2\Big)\\
&\qquad\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\Bigg].
\end{aligned}$$

□

### 2.4. Corollaries

Following corollaries are deduced from Theorem 2.1.

**Corollary 2.1** *Following Note 1(i), we obtain*

$$\left\|\tilde{t}_{n,m}^{(C,1,1)} - \tilde{g}\right\|_r^{(\eta)} = O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\cdot\log(m+1) + \log(n+1)(m+1) + 2\Bigg) \int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\Bigg],$$

*where $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ denote the moduli of continuity such that $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing.*

**Corollary 2.2** *Following Note 1(ii), we obtain*

$$\left\|\tilde{t}_{n,m}^{(E,1,1)} - \tilde{g}\right\|_r^{(\eta)} = O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\cdot\log(m+1) + \log(n+1)(m+1) + 2\Bigg) \int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi} \frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\Bigg],$$

*where $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ denote the moduli of continuity such that $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing.*

**Corollary 2.3** *Let $\tilde{g} \in H_{(\alpha,\beta),r}; r \geq 1$ and suppose $\xi_1(s)+\xi_2(l) = (sl)^\alpha$, $\eta_1(s)+\eta_2(l) = (sl)^\beta$, $0 \leq \beta < \alpha \leq 1$, then*

$$\left\|\tilde{t}_{n,m}^{H}(x,y) - \tilde{g}(x,y)\right\|_r^{(\eta)} = \begin{cases} O\Bigg[\Bigg(2\log(n+1)\log(m+1) + \log(n+1)(m+1) + 2\Bigg) \cdot((n+1)(m+1))^{\beta-\alpha}\Bigg] & if\ 0 \leq \beta < \alpha < 1, \\ O\Bigg[\Bigg(\frac{2\log(n+1)\log(m+1)+\log(n+1)(m+1)+2}{(n+1)(m+1)}\Bigg)\log(n+1)\pi\cdot\log(m+1)\pi\Bigg] & if\ \beta = 0,\quad \alpha = 1. \end{cases}$$

**Proof:** Putting $\xi_1(s)+\xi_2(l) = (sl)^\alpha$, $\eta_1(s)+\eta_2(l) = (sl)^\beta$, $0 \leq \beta < \alpha \leq 1$, in Theorem 2.1, we get

$$\left\|\tilde{t}_{n,m}^{H}(x,y) - \tilde{g}(x,y)\right\|_r^{(\eta)} = O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1) \cdot\log(m+1) + \log(n+1)(m+1) + 2\Bigg)\cdot\Bigg(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi} s^{\alpha-\beta-2}\cdot l^{\alpha-\beta-2}dlds\Bigg)\Bigg],$$

$$\implies \left\|\tilde{t}^{H}_{n,m}(x,y)-\tilde{g}(x,y)\right\|_r^{(\eta)}$$

$$=\begin{cases} O\Bigg[\Bigg(2\log(n+1)\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad\cdot\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}(sl)^{\alpha-\beta-2}dlds\Bigg] & if\ 0\leq\beta<\alpha<1, \\ O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad\cdot\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{sl}dlds\Bigg] & if\ \beta=0,\ \alpha=1, \end{cases}$$

$$\therefore\quad \left\|\tilde{t}^{H}_{n,m}(x,y)-\tilde{g}(x,y)\right\|_r^{(\eta)}$$

$$=\begin{cases} O\Bigg[\Bigg(2\log(n+1)\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad\cdot((n+1)(m+1))^{\beta-\alpha}\Bigg] & if\ 0\leq\beta<\alpha<1, \\ O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\cdot\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad\cdot\{\log(n+1)\pi\}\cdot\{\log(m+1)\pi\}\Bigg] & if\ \beta=0,\ \alpha=1. \end{cases}$$

□

**Corollary 2.4** *Let* $\tilde{g}\in H_{(\alpha,\beta),r}; r\geq 1,\ a,b\in\mathbb{R}$ *and suppose* $\xi_1(s)+\xi_2(l)=\frac{(sl)^{\alpha}}{(\log\frac{1}{s})^{a}\cdot(\log\frac{1}{l})^{a}},\ \eta_1(s)+\eta_2(l)=\frac{(sl)^{\beta}}{(\log\frac{1}{s})^{b}\cdot(\log\frac{1}{l})^{b}}$ *and* $0\leq\beta<\alpha\leq 1,\ 0<s,l\leq\pi,$ *then*

$$\left\|\tilde{t}^{H}_{n,m}(x,y)-\tilde{g}(x,y)\right\|_r^{(\eta)}=\begin{cases} O\Bigg[\Bigg(2\log(n+1)\cdot\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad\cdot\{\log(n+1)\log(m+1)\}^{b-a}\Bigg] & if \alpha=\beta, a-b>-1, \\ O\Bigg[\Bigg(2\log(n+1)\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad\cdot\{\log(n+1)\cdot\log(m+1)\}\Bigg] & if \alpha=\beta, a-b=-1. \end{cases}$$

**Proof:** Putting $\xi_1(s)+\xi_2(l)=\frac{(sl)^{\alpha}}{(\log\frac{1}{s})^{a}\cdot(\log\frac{1}{l})^{a}},\ \eta_1(s)+\eta_2(l)=\frac{(sl)^{\beta}}{(\log\frac{1}{s})^{b}\cdot(\log\frac{1}{l})^{b}}$ and $0\leq\beta<\alpha\leq 1,\ 0<s,l\leq\pi$, in Theorem 2.1, we have

$$\left\|\tilde{t}^{H}_{n,m}(x,y)-\tilde{g}(x,y)\right\|_r^{(\eta)}=O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\cdot\log(m+1)$$

$$+\log(n+1)(m+1)+2\Bigg)\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}(sl)^{\alpha-\beta-2}\cdot\left(\log\frac{1}{s}\right)^{b-a}\cdot\left(\log\frac{1}{l}\right)^{b-a}dlds\right)\Bigg],$$

$$\Longrightarrow \left\|\tilde{t}^H_{n,m}(x,y)-\tilde{g}(x,y)\right\|_r^{(\eta)}$$

$$= \begin{cases} O\Bigg[\Bigg(2\log(n+1)\cdot\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \qquad \Bigg(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}(sl)^{\alpha-\beta-2}\Bigg(\log\frac{1}{s}\Bigg)^{b-a}\cdot\Bigg(\log\frac{1}{l}\Bigg)^{b-a} dlds\Bigg)\Bigg] & if\ \alpha=\beta, a-b>-1, \\ O\Bigg[\Big(\frac{1}{(n+1)(m+1)}\Big)\Bigg(2\log(n+1)\cdot\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \qquad \Bigg(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{(sl)^2}(\log\frac{1}{s})(\log\frac{1}{l})dlds\Bigg)\Bigg] & if\ \alpha=\beta, a-b=-1. \end{cases}$$

$$\therefore\ \left\|\tilde{t}^H_{n,m}(x,y)-\tilde{g}(x,y)\right\|_r^{(\eta)} = \begin{cases} O\Bigg[\Bigg(2\log(n+1)\cdot\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad \{\log(n+1)\log(m+1)\}^{b-a}\Bigg] & if \alpha=\beta, a-b>-1, \\ O\Bigg[\Bigg(2\log(n+1)\cdot\log(m+1)+\log(n+1)(m+1)+2\Bigg) \\ \quad \{\log(n+1)\log(m+1)\}\Bigg] & if\ \alpha=\beta, a-b=-1. \end{cases}$$

□

## 3. Degree of Convergence of a Function of $N$-Dimensional Variable of $N$-Multiple Conjugate Fourier Series ($N \geq 3$)

Let $g(x_1,x_2,\cdots,x_N)$ is periodic integrable function of period $2\pi$ in each variable over $N$-dimensional cube $\mathbb{T}^N$. The $N$-multiple Fourier series of $g(x_1,x_2,\cdots,x_N)$ can be written in the form

$$g(x_1,x_2,\cdots,x_N)\sim \sum\sum\cdots\sum_{(m_1,m_2,\cdots,m_N)\in\mathbb{Z}^{\mathbb{N}}} c_{m_1,m_2,\cdots,m_N}(g)e^{i(m_1x_1+m_2x_2+\cdots+m_Nx_N)},$$

where $c_{m_1,m_2,\cdots,m_N}(g)$ is the Fourier coefficient of $g$ (see [1, p. 300]).
The $N$-multiple conjugate Fourier series of $f(x_1,\cdots,x_N)$ can be written in the form

$$\tilde{g}(x_1,x_2,\cdots,x_N)\sim \sum\sum\cdots\sum_{(m_1,m_2,\cdots,m_N)\in\mathbb{Z}^{\mathbb{N}}} (-i \text{ sign } m_1)(-i \text{ sign } m_2)\cdots(-i \text{ sign } m_N)$$
$$\cdot\, c_{m_1,m_2,\cdots,m_N}(\tilde{g})e^{i(m_1x_1+m_2x_2+\cdots+m_Nx_N)}.$$

The following representation of the symmetric square partial sum of above series is given by

$$\tilde{s}_{m_1,\cdots,m_N}(x_1,\cdots,x_N)$$
$$:=\frac{1}{\pi^N}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\psi_{x_1x_2\cdots x_N}(g;l_1,l_2,\cdots,l_N)\prod_{j=1}^{N}\tilde{D}_{m_j}(l_j)dl_1dl_2\cdots dl_N,$$

where $\tilde{D}_{m_j}(l_j)$ are the conjugate Derichlet kernels for each $j$.
If $H$ is a $N$-dimensional Hausdorff matrix then

$$h_{m_1,q_1;\cdots;m_N,q_N}$$
$$= \begin{cases} \binom{m_1}{q_1}\binom{m_2}{q_2}\cdots\binom{m_N}{q_N}\int_0^1\int_0^1\cdots\int_0^1 l_1^{q_1}(1-l_1)^{m_1-q_1}l_2^{q_2}(1-l_2)^{m_2-q_2}\cdots l_N^{q_N}(1-l_N)^{m_N-q_N} \\ \quad \cdot d\chi(l_1,l_2,\cdots,l_N), \qquad q_1=0,1,...m_1,\ q_2=0,1,\cdots m_2,\cdots,q_N=0,1,...m_N; \\ 0, \qquad\qquad q_1>m_1,q_2>m_2,\cdots,q_N>m_N. \end{cases}$$

Let $\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_N=0}^{\infty} d_{m_1,m_2,\cdots,m_N}$ be a $N$-multiple conjugate Fourier series with $\tilde{s}_{m_1,m_2,\cdots,m_N}=\sum_{q_1=0}^{m_1}\sum_{q_2=0}^{m_2}\cdots\sum_{q_N=0}^{m_N} d_{q_1,q_2,\cdots,q_N}$ as its $(m_1,m_2,\cdots,m_N)^{th}$ partial sums.
The $N$-dimensional Hausdorff means $\tilde{t}^{H}_{m_1,m_2,\cdots,m_N}$ is given by

$$\tilde{t}^{H}_{m_1,m_2,\cdots,m_N}=\sum_{q_1=0}^{m_1}\sum_{q_2=0}^{m_2}\cdots\sum_{q_N=0}^{m_N} h_{m_1,q_1;\cdots;m_N,q_N}\tilde{s}_{q_1,q_2,\cdots,q_N}.$$

If $\tilde{t}^{H}_{m_1,m_2,\cdots,m_N}\to c$ as $m_1,m_2,\cdots,m_N\to\infty$, then the $N$-dimensional conjugate Fourier series $\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_N}^{\infty} d_{m_1,m_2,\cdots,m_N}$ with the sequence of $(m_1,\cdots,m_N)^{th}$ partial sums $(\tilde{s}_{m_1,m_2,\cdots,m_N})$ is said to be summable to some finite value $c$ by the $H$ method.
Now, we give the concept of Hölder classes of functions on $\mathbb{T}^N$. The Hölder class for continuous periodic function $g(x_1,x_2,\cdots,x_N)$ with period $2\pi$ in each variable is defined as

$$\begin{aligned}H_{(\alpha_1,\alpha_2,\cdots,\alpha_N)}&=\{g:|g(x_1,x_2,\cdots,x_N;l_1,l_2,\cdots,l_N)|\\&:=|g(x_1+l_1,\cdots,x_N+l_N)-g(x_1,\cdots,x_N)|\le C_1(|l_1|^{\alpha_1}+|l_2|^{\alpha_2}+\cdots+|l_N|^{\alpha_N})\}\end{aligned}$$

for some $0<\alpha_1,\cdots,\alpha_N\le 1$ and for all $x_1,\cdots,x_N,l_1,,\cdots,l_N$, where $C_1>0$ is a constant may depend on $f$, but not on $x_1,\cdots,x_N,l_1,,\cdots,l_N$. This class of functions is also called Lipschitz class and denoted by $Lip(\alpha_1,\cdots,\alpha_N)$. It can be easily varified that $H_{(\alpha_1,\cdots,\alpha_N)}$ is a Banach space with respect to the norm $\|\cdot\|_{(\alpha_1,\cdots,\alpha_N)}$ defined by

$$\|g\|_{(\alpha_1,\cdots,\alpha_N)}=\|g\|_C+\sup_{x_1\ne l_1,\cdots,x_N\ne l_N}\Delta^{\alpha_1,\cdots,\alpha_N}g(x_1,\cdots,x_N;l_1,\cdots,l_N),$$

where

$$\begin{aligned}&\Delta^{\alpha_1,\cdots,\alpha_N}g(x_1,\cdots,x_N;l_1,\cdots,l_N)\\&\quad=\frac{|g(x_1+l_1,\cdots,x_N+l_N)-g(x_1,\cdots,x_N)|}{|x_1-l_1|^{\alpha_1}+\cdots+|x_N-l_N|^{\alpha_N}}\qquad(x_1\ne l_1,\cdots,x_N\ne l_N).\end{aligned}$$

By convention $\Delta^{0,\cdots,0}g(x_1,\cdots,x_N;l_1,\cdots,l_N)=0$ and

$$\|g\|_C=\sup_{(x_1,\cdots,x_N)\in\mathbb{T}^N}|g(x_1,\cdots,x_N)|.$$

The function space $L^r[[0,2\pi]^N]$ is given by

$$\begin{aligned}&L^r[[0,2\pi]^N]\\&=\left\{g:[[0,2\pi]^N]\to\mathbb{R}^N:\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}|f(x_1,\cdots,x_N)|^r dx_1\cdots dx_N<\infty,r\ge 1\right\}.\end{aligned}\tag{3.1}$$

The norm of (37) is defined by

$$\|g\|_r:=\begin{cases}\left\{\frac{1}{(2\pi)^N}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}|g(x_1,\cdots,x_N)|^r\,dx_1\cdots dx_N\right\}^{\frac{1}{r}}, & r\in[1,\infty);\\ \operatorname{ess\,sup}_{(x_1,\cdots,x_N)\in[0,2\pi]^{\mathbb{N}}}|f(x_1,\cdots,x_N)|, & r=\infty.\end{cases}$$

Let $\xi_1,\cdots,\xi_N:\mathbb{T}^N\to\mathbb{R}^N$ be an arbitrary function. The class of function $H_r^{(\xi_1,\cdots,\xi_N)}$ is defined by

$$H_r^{(\xi_1,\cdots,\xi_N)}=\left\{g\in L^r[[0,2\pi]^N]:\sup_{l_1\ne 0,\cdots,l_N\ne 0}\frac{\|g(x_1+l_1,\cdots,x_N+l_N)-g(x_1,\cdots,x_N)\|_r}{\xi_1(l_1)+\cdots+\xi_N(l_N)}\right\},$$

where $\xi_1,\xi_2,\cdots,\xi_N$ are the moduli of continuity that is $\xi_1,\xi_2,\cdots,\xi_N$ are non-decreasing positive continuous function with the following properties:

$$\lim_{l_1\to 0^+}\xi_1(l_1)=\xi_1(0)=0,\ \lim_{l_2\to 0^+}\xi_2(l_2)=\xi_2(0)=0,\cdots,\ \lim_{l_N\to 0^+}\xi_N(l_N)=\xi_N(0)=0$$

and

$$\xi_1(l_{1,1}+\cdots+l_{1,N})\leq\xi_1(l_{1,1})+\cdots+\xi_1(l_{1,N}),\cdots,\xi_N(l_{N,1}+\cdots+l_{N,N})\leq\xi_N(l_{N,1})+\cdots+\xi_N(l_{N,N}).$$

We define

$$\|g\|_r^{(\xi_1,\cdots,\xi_N)}=\|g\|_r+\sup_{l_1\neq0,\cdots,l_N\neq0}\frac{\|g(x_1+l_1,\cdots,x_N+l_N)-g(x_1,\cdots,x_N)\|_r}{\xi_1(l_1)+\cdots+\xi_N(l_N)}.$$

Clearly, $\|\cdot\|_r^{(\xi_1,\cdots,\xi_N)}$ is a norm on $H_r^{(\xi_1,\cdots,\xi_N)}$.
It can be varified that the completeness of the space $L^r[[0,2\pi]^N]$ implies the completeness of the space $H_r^{(\xi_1,\cdots,\xi_N)}$.
We write

$$\begin{aligned}\Psi(l_1,\cdots,l_N)=\psi(x_1,l_1;\cdots;x_N,l_N)=\frac{1}{(2)^N}\Bigg[&g(x_1+l_1,\cdots,x_N+l_N)\\&-g(x_1+l_1,\cdots,x_N-l_N)-\cdots+g(x_1-l_1,\cdots,x_N-l_N)\Bigg];\end{aligned}$$

$$\Psi(x_1,\cdots,x_N)=\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_N}|\psi(v_1,\cdots,v_N)|dv_1dv_2\cdots dv_N;$$

$$\begin{aligned}&\tilde{K}_{m_1,\cdots,m_N}(l_1,\cdots,l_N)\\&\quad=\frac{1}{(2\pi)^N}\sum_{q_1=0}^{m_1}h_{m_1,q_1}\frac{\cos(q_1+\frac{1}{2})l_1}{\sin(\frac{l_1}{2})}\cdots\sum_{q_N=0}^{m_N}h_{m_N,q_N}\frac{\cos(q_N+\frac{1}{2})l_N}{\sin(\frac{l_N}{2})}.\end{aligned}\tag{3.2}$$

Now, we establish a following theorem:

### 3.1. Main Theorem

**Theorem 3.1** *Let $\tilde{g}(x_1,\cdots,x_N)$ be a function, conjugate to a function $g(x_1,\cdots,x_N)$ (periodic with period $2\pi$ in each variable) Lebesgue integrable on $T^N$. Then, the error estimation of $\tilde{g}(x_1,x_2,\cdots,x_N)$ in the space $(H_r^{(\xi_1,\cdots,\xi_N)};r\geq1)$ using $N$-dimensional Hausdorff matrix $H$ is given by*

$$\begin{aligned}\|\tilde{t}_{m_1m_2\cdots m_N}-\tilde{g}\|_r^{(\eta)}=O\Bigg[&\prod_{i=1}^N\frac{1}{(m_i+1)}\left(2\prod_{i=1}^N\log(m_i+1)+\log\prod_{i=1}^N(m_i+1)+2\right)\\&\cdot\left(\int_{\frac{1}{m_1+1}}^{\pi}\int_{\frac{1}{m_2+1}}^{\pi}\cdots\int_{\frac{1}{m_N+1}}^{\pi}\frac{\xi_1(l_1)+\cdots+\xi_N(l_N)}{\eta_1(l_1)+\cdots+\eta_N(l_N)}\frac{1}{(l_1^N\cdots l_N^N)}dl_1\cdots dl_N\right)\Bigg],\end{aligned}\tag{3.3}$$

*where $(\xi_1+\cdots+\xi_N)$ and $(\eta_1+\cdots+\eta_N)$ denote the moduli of continuity such that $\frac{\xi(l_1,\cdots,l_N)}{\eta(l_1,\cdots,l_N)}$ is non-decreasing and positive.*

### 3.2. Proof of the Main Theorem 3.1

**Proof:** One can extend the Lemmas 2.2 to 2.6 in each variable from double to $N$-multiple. Further, the proof goes along the same lines of the proof of Theorem 2.1. □

### 3.3. Corollaries

Following corollaries are deduced from Theorem 3.1.

**Corollary 3.1** *Following Note 1(i) for $N$-dimensional Cesàro means of order 1, we obtain*

$$\left\|\tilde{t}^{(C,1,\cdots,1)}_{m_1m_2\cdots m_N}-\tilde{g}\right\|^{(\eta)}_r=O\Bigg[\prod_{i=1}^{N}\frac{1}{(m_i+1)}\left(2\prod_{i=1}^{N}\log(m_i+1)+\log\prod_{i=1}^{N}(m_i+1)+2\right)$$
$$\int_{\frac{1}{m_1+1}}^{\pi}\int_{\frac{1}{m_2+1}}^{\pi}\cdots\int_{\frac{1}{m_N+1}}^{\pi}\frac{\xi_1(l_1)+\cdots+\xi_N(l_N)}{\eta_1(l_1)+\cdots+\eta_N(l_N)}\frac{1}{(l_1^N\cdots l_N^N)}dl_1\cdots dl_N\Bigg],$$

*where $(\xi_1+\cdots+\xi_N)$ and $(\eta_1+\cdots+\eta_N)$ denotes the modulus of continuity such that $\frac{\xi(l_1,\cdots,l_N)}{\eta(l_1,\cdots,l_N)}$ is positive and non-decreasing.*

**Corollary 3.2** *Following Note 1(ii) for $N$-dimensional Euler's means of order 1, we obtain*

$$\left\|\tilde{t}^{(E,1,\cdots,1)}_{m_1m_2\cdots m_N}-\tilde{g}\right\|^{(\eta)}_r=O\Bigg[\prod_{i=1}^{N}\frac{1}{(m_i+1)}\left(2\prod_{i=1}^{N}\log(m_i+1)+\log\prod_{i=1}^{N}(m_i+1)+2\right)$$
$$\int_{\frac{1}{m_1+1}}^{\pi}\int_{\frac{1}{m_2+1}}^{\pi}\cdots\int_{\frac{1}{m_N+1}}^{\pi}\frac{\xi_1(l_1)+\cdots+\xi_N(l_N)}{\eta_1(l_1)+\cdots+\eta_N(l_N)}\frac{1}{(l_1^N\cdots l_N^N)}dl_1\cdots dl_N\Bigg],$$

*where $(\xi_1+\cdots+\xi_N)$ and $(\eta_1+\cdots+\eta_N)$ denotes the modulus of continuity such that $\frac{\xi(l_1,\cdots,l_N)}{\eta(l_1,\cdots,l_N)}$ is positive and non-decreasing.*

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