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Mountain Pass Theorem and Applications to Fourth-order PDES with Variable Exponents

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ABSTRACT: The authors establish the existence of multiple weak solutions to a fourth order equation. Their analysis employs the Mountain Pass Theorem combined with variable exponent theory of generalized Lebesgue-Sobolev spaces.

Key Words: Mountain Pass theorem, weak solutions, Palais Smale condition, variable exponent Sobolev space.

Contents

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^{\mathbb{N}}$. In this paper, we deal with existence of multiple weak solutions to the problem

$$\begin{cases} \Delta(\mid x \mid^{p(x)}\mid \Delta u \mid^{p(x)-2} \Delta u) - \lambda \mid u \mid^{q(x)-2} u = \mu f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where p, $q : \bar{\Omega} \to (1, \infty)$ are continuous maps, $\lambda < 0$ and $\mu > 0$ are parameters, and $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Fourth order equations have attracted many authors in different areas of applied mathematics and physics. These kinds of problems have various applications and are widely justified via many physical examples. The Karamoto-Sivashinsky equation (see [32]) as a model of instabilities in a flame front is a good example of this type that has been investigated. There are also examples in non-Newtonian fluids and elastic mechanics, in particular, electro-rheological fluids (smart fluids); see [16,31]. Important references on this can be found in [1,2,3,8,9,10,12,13,14,15,17,18,19,20,24,28,29,30,31].

In [4], Ayoujil considered the problem

$$\begin{cases} \Delta(\mid \Delta u \mid^{p(x)-2} \Delta u) = \lambda V(x) \mid u \mid^{q(x)-2} u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

on the smooth bounded domain $\Omega \subset \mathbb{R}^{\mathbb{N}}$ and obtained some sufficient conditions for the existence and nonexistence of eigenvalues for this problem. Here, $V \in L^{r(x)}(\Omega)$ is a indefinite weight that may change signs in Ω, and p, q, $r : \bar{\Omega} \to (1, \infty)$ are continuous functions.

The case $V \equiv 1$ and $p(x) = q(x)$ was discussed in [5] where Ayoujil and Amrouss used an argument based on Ljusternik-Schnirelmann critical point theory and obtained the existence of infinitely many eigenvalues for the problem (1.2). The situation where $V \equiv 1$ and $p(x) \neq q(x)$ was studied in [6].

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By means of variable exponent theory for generalized Lebesgue-Sobolev spaces and using a variant of the Mountain Pass theorem, Mousaviankhatir *et al.* showed the existence of at least one weak solution to

$$\begin{cases} \Delta(\mid x \mid^{p(x)}\mid \Delta u \mid^{p(x)-2} \Delta u) = \lambda \mid u \mid^{q(x)-2} u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where once again Ω is a smooth bounded domain in $\mathbb{R}^{\mathbb{N}}$, p, $q \in C(\bar{\Omega}, (1, \infty))$, and λ is negative.

Here we use the approach taken in [27] to obtain existence and multiplicity of weak solutions to problem (1.1)

Under appropriate conditions on the nonlinear function f, we show that there exists a $\mu^* > 0$ such that (1.1) has at least two nontrivial weak solutions for each $\mu > \mu^*$.

2. Preliminary results

We begin by recalling some ideas on generalized Lebesgue-Sobolev spaces that will be used in our analysis. The variable exponent Lebesgue space with $p \in C(\bar{\Omega})$ is defined by

$$L^{p(x)}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_\Omega \mid u(x) \mid^{p(x)} dx < \infty\}.$$

This is a separable and reflexive Banach space equipped with what is called the Luxemburg norm

$$\|u\|_{L^{p(x)}} = \inf\left\{\beta > 0 : \int_\Omega \left|\frac{u(x)}{\beta}\right|^{p(x)} dx \leq 1\right\}.$$

We set

$$p^- := \inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \sup_{x\in\Omega} p(x),$$

and for any $x \in \bar{\Omega}$ and positive integer k, we let

$$p_k^*(x) = \begin{cases} \frac{N-p(x)}{N-kp(x)}, & \text{if } kp(x) < N, \\ +\infty, & \text{if } kp(x) \geq N. \end{cases}$$

It is known that if $0 <\mid \Omega \mid< \infty$ and p_1, p_2 are variable exponents with $p_1(x) \leq p_2(x)$ almost everywhere in Ω, then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ whose norm does not exceed $\mid \Omega \mid +1$.

From [11,22], for any positive integer k, the Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), \mid \alpha \mid\leq k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}} u$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index such that $\mid \alpha \mid= \sum_{k=1}^N \alpha_k$. Equipped with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\leq k} \| D^\alpha u \|_{L^{p(x)}(\Omega)},$$

$W^{k,p(x)}(\Omega)$ is a separable, reflexive, and uniformly convex Banach space. If we let $q(x)$ be the conjugate exponent to $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega \mid uv \mid dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \mid u \mid^{p(x)}\mid v \mid^{q(x)}, \ u \in L^{p(x)}(\Omega) \text{ and } v \in L^{q(x)}(\Omega)$$

is known to hold.

Proposition 2.1 ([7, Proposition 2.3], [11, Theorems 1.3 and 1.4]) *For* u, $u_n \in L^{p(x)}(\Omega)$, *set* $\rho_{p(x)}(u) = \int_\Omega \mid u \mid^{p(x)} dx$. *Then:*

(i) $\|u\| \geq 1$ *implies* $\|u\|^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|^{p^+}$;

(ii) $\|u\| \leq 1$ *implies* $\|u\|^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|^{p^-}$.

Also, the following statements are equivalent:

(a) $\lim_{n\to\infty} \|u_n - u\| = 0$.

(b) $\lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

Similar to Proposition 2.1, if $\rho_{p(x)}(u) = \int_\Omega |\, \Delta u \,|^{p(x)}\, dx$, we will have the same results (see [7,23]).

Proposition 2.2 ([11, Theorem 2.2 and 2.3]) *Suppose* $q \in C(\bar{\Omega}, \mathbb{R})$ *is such that* $1 < q^- \leq q^+ < \infty$ *and* $q(x) \leq p_k^*(x)$. *Then the embedding* $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is continuous. If* $q(x) \leq p_k^*(x)$ *is replaced by* $q(x) < p_k^*(x)$, *then the embedding is also compact.*

For our next result we need to recall the Palais-Smale (PS) condition.

Definition 2.1 *A functional* $I \in C^1(X, \mathbb{R})$ *satisfies the Palais-Smale condition if any sequence* $\{u_n\}$ *in* X *for which* $\{I(u_n)\}$ *is bounded and* $\{I'(u_n)\} \to 0$ *as* $n \to \infty$, *has a convergent subsequence.*

Theorem 2.1 ([9, Theorem 2.1]) *Let* X *be a real Banach space,* $I \in C^1(X, \mathbb{R})$ *satisfy the (PS) condition, and* I *be bounded from below on every bounded subset of* X. *Then, the following assertions are equivalent:*

(i) There exist u_0, $u_1 \in X$ *and* $r \in \mathbb{R}$, *with* $0 < r < \|u_1 - u_0\|$, *such that*

$$\inf_{\|u_1 - u_0\| = r} I(u) \geq \max\{I(u_0), I(u_1)\}.$$

(ii) I *admits at least one local minimum which is not strictly global.*

Lemma 2.1 ([23, Lemma 3.3]) *Let* $(X, \|\cdot\|$ *be a real Banach space and* $I \in C^1(X, \mathbb{R})$. *Suppose that* I *satisfies the (PS) condition and there exist* u_0, $u_1 \in X$ *and* $\rho > 0$ *such that*

(i) $u_1 \notin \overline{B_\rho(u_0)}$,

(ii) $\max\{I(u_0), I(u_1)\} < \inf_{u \in \partial B_\rho(u_0)} I(u)$.

Then, I *possesses a critical value which can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)) \geq \inf_{u \in \partial B_\rho(u_0)} I(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

3. Main results

We introduce the notation that $C_c^m(\Omega)$ is the class of all m-times continuously differentiable functions with compact support, and $C_0(\Omega)$ is the class of all continuous functions that tend to zero as $x \to \infty$. It is known that

$$C_c^2(\Omega) \subset \overline{C_0^2(\Omega)} = W_0^{2,p}(\Omega) \hookrightarrow W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$$

and

$$D_0^{2,p(x)}(\Omega) := \overline{C_c^2(\Omega)} \hookrightarrow L^p(\Omega).$$

Then, with the norm

$$\|u\| = \|\, |x|\, |\Delta u|\, \|_{p(x)},$$

the space $(D_0^{2,p(x)}(\Omega), \|\cdot\|)$ is a reflexive Banach space.

Assume that $q \in C(\bar{\Omega}, \mathbb{R})$ with $1 < q^- \leq q < q^+ < \frac{2Np^-}{2N+p^-}$. We say that $u \in D_0^{2,p(x)}(\Omega)$ is a *weak solution* of problem (1.1) if

$$\int_\Omega | x |^{p(x)} | \Delta u |^{p(x)-2} \Delta u \Delta v dx - \lambda \int_\Omega | u |^{q(x)-2} uv dx - \mu \int_\Omega f(x,u) v dx = 0$$

for all $v \in D_0^{2,p(x)}(\Omega)$.

Our main result asserts that under some appropriate conditions, problem (1.1) has at least two nontrivial weak solutions.

To transfer the problem of the existence of weak solutions to problem (1.1) into the question of the existence of critical points of a related energy functional, we associate to the problem (1.1), the functional $\varphi : D_0^{2,p(x)}(\Omega) \to \mathbb{R}$ defined by $\varphi(u) = I_\lambda(u) - \mu\psi(u)$ where $I_\lambda : X = D_0^{2,p(x)}(\Omega) \to \mathbb{R}$ is given by

$$I_\lambda(u) = \int_\Omega \frac{| x |^{p(x)}}{p(x)} | \Delta u |^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} | u |^{q(x)} dx, \quad \psi(u) = \int_\Omega F(x,u) du$$

and

$$F(x,u) = \int_{[0,u]} f(x,s) ds.$$

Using Hölder's inequality and Proposition 2.1, we conclude that I_λ is well defined on $D_0^{2,p(x)}$ since

$$\begin{aligned}
&\int_\Omega \frac{| x |^{p(x)}}{p(x)} | \Delta u |^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} | u |^{q(x)} \\
&\leq \frac{D_1}{p^-} \left(\int_\Omega (| x |^{p(x)} | \Delta u |^{p(x)})^{\frac{q(x)}{p(x)}} dx \right)^{\frac{p(x)}{q(x)}} | 1 |_{(\frac{q(x)}{p(x)})'} \\
&\quad - \frac{\lambda D_2}{q^-} \left(\int_\Omega (| u |^{q(x)})^{\frac{p(x)}{q(x)}} dx \right)^{\frac{q(x)}{p(x)}} | 1 |_{(\frac{p(x)}{q(x)})'} \\
&= \frac{D_1}{p^-} \left(\int_\Omega | x |^{q(x)} | \Delta u |^{q(x)} dx \right)^{\frac{p(x)}{q(x)}} | 1 |_{(\frac{q(x)}{p(x)})'} \\
&\quad - \frac{\lambda D_2}{q^-} \left(\int_\Omega (| u |^{p(x)} dx \right)^{\frac{q(x)}{p(x)}} | 1 |_{(\frac{p(x)}{q(x)})'} \\
&\leq \frac{D_1}{p^-} \left[\left(\int_\Omega | x |^{q(x)} | \Delta u |^{q(x)} dx \right)^{\frac{p(x)}{q^+}} + \left(\int_\Omega | x |^{q(x)} | \Delta u |^{q(x)} dx \right)^{\frac{p(x)}{q^-}} \right] \\
&\quad - \frac{\lambda D_2}{q^-} \left[\left(\int_\Omega | u |^{p(x)} dx \right)^{\frac{q(x)}{p^-}} + \left(\int_\Omega | u |^{p(x)} dx \right)^{\frac{q(x)}{p^+}} \right] \\
&\leq \frac{D_1}{p^-} \left[(\sup_{x\in\Omega} | x | + 1)^{p^+} + (\sup_{x\in\Omega} | x | + 1)^{\frac{p^+ q^+}{q^-}} \right] \\
&\quad \left[\left(\int_\Omega | \Delta u |^{q(x)} dx \right)^{\frac{p(x)}{q^+}} + \int_\Omega | \Delta u |^{q(x)} dx)^{\frac{p(x)}{q^-}} \right] \\
&\quad - \frac{\lambda D_2}{q^-} \left[(\int_\Omega | u |^{p(x)} dx)^{\frac{q(x)}{p^-}} + (\int_\Omega | u |^{p(x)} dx)^{\frac{q(x)}{p^+}} \right] \\
&\leq \frac{D_1}{p^-} \left[(\|u\|^{q^+})^{\frac{p(x)}{q^+}} + (\|u\|^{q^+})^{\frac{p(x)}{q^-}} \right] \\
&\quad - \frac{\lambda D_2}{q^-} \left[(\|u\|^{p^+})^{\frac{q(x)}{p^-}} + (\|u\|^{p^+})^{\frac{q(x)}{p^+}} \right] < \infty.
\end{aligned}$$

where D_1 and D_2 are positive constants.

Also, $\varphi \in C^1(D_0^{2,p(x)}(\Omega), \mathbb{R})$, and for all $u,\ v \in D_0^{2,p(x)}$, we have

$$(\varphi'(u), v) = \int_\Omega \mid x \mid^{p(x)} \mid \Delta u \mid^{p(x)-2} \Delta u \Delta v dx - \lambda \int_\Omega \mid u \mid^{q(x)-2} uv dx - \mu \int_\Omega f(x,u) v dx.$$

It is clear that weak solutions of problem (1.1) correspond to the critical points of φ. More precisely, we have the following theorem.

Theorem 3.1 *Assume that* $X := D_0^{2,p(x)}(\Omega)$, $q \in C(\bar{\Omega}, \mathbb{R})$ *satisfies* $1 < q^- \leq q < q^+ < \frac{2Np^-}{2N+p^-} < p^- < p^+ < \theta$, $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function, and the following properties hold:*

(a) For $q(x) < p(x) < p^*(x)$, *there exist* $c > 0$ *such that*

$$\mid f(x,t) \mid \leq c(1 + \mid t \mid^{q(x)-1}) \quad \textit{for all} \quad x \in \bar{\Omega};$$

(b) $\lim\limits_{|t| \to 0} \dfrac{F(x,t)}{\mid t \mid^{q(x)}} = 0$;

(c) There exists $t^* > 0$ *such that* $0 < \theta F(x,\xi) \leq f(x,\xi)\xi$ *for all* $x \in \bar{\Omega}$ *and* $\mid \xi \mid > t^*$;

(d) $\limsup\limits_{|t| \to \infty} \dfrac{F(x,t)}{\mid t \mid^{q(x)}} \leq 0$;

(e) There exists $e \in X$ *such that*

$$\frac{\int_\Omega \frac{|x|^{p(x)}}{p(x)} \mid \Delta e \mid^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} \mid e(x) \mid^{q(x)} dx}{\mu^*} < \int_\Omega F(x, e(x)) dx.$$

Then, the problem (1.1) *has at least two nontrivial weak solutions for each* $\mu > \mu^*$ *and* $\lambda < 0$, *where*

$$\mu^* = \inf_{e \in D_0^{2,p(x)}, \|e\| > 1} \frac{\int_\Omega \frac{|x|^{p(x)}}{p(x)} \mid \Delta e \mid^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} \mid e(x) \mid^{q(x)} dx}{\int_\Omega \mid e(x) \mid^{q^-} dx}.$$

Before proving our main theorem, we first show that the quantity μ^* is positive.

Lemma 3.1 μ^* *is positive.*

Proof: By the fact that we have a compact embedding of $D_0^{2,p(x)}(\Omega)$ into $L^{q(x)}$ (see [26,27]) and $\lambda < 0$, we see that

$$\begin{aligned} I_\lambda(e) &= \int_\Omega \frac{\mid x \mid^{p(x)}}{p(x)} \mid \Delta e \mid^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} \mid e(x) \mid^{q(x)} dx \\ &\geq \frac{1}{p^+} \|e(x)\|^{q^-} \geq \frac{1}{p^+} c_1^{q^-} \int_\Omega \mid e \mid^{q^-} dx \end{aligned}$$

for all $e \in D_0^{2,p(x)}(\Omega)$ with $\|e\| > 1$ and for some $c_1 > 0$. Therefore,

$$\frac{\int_\Omega \frac{|x|^{p(x)}}{p(x)} \mid \Delta e \mid^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} \mid e(x) \mid^{q(x)} dx}{\int_\Omega \mid e \mid^{q^-} dx} \geq \frac{1}{p^+} c_1^{q^-} > 0,$$

which proves the lemma. $\square$

Proof: [Proof of Theorem 3.1] We first prove that I'_λ has the property (S_+) (see, for example, [7, Proposition 2.5]). To do this, we set

$$U_p = \{x \in \Omega : p(x) \geq 2\} \quad \text{and} \quad V_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

We will make use of the following elementary inequalities:

$$|x-y|^\gamma \leq 2^\gamma (|\, x\,|^{\gamma-2}\, x - |\, y\,|^{\gamma-2}\, y) \cdot (x-y), \quad \gamma \geq 2,$$

$$|x-y|^2 \leq \frac{1}{\gamma-1}(|\, x\,| + |\, y\,|)^{2-\gamma}(|\, x\,|^{\gamma-2}\, x - |\, y\,|^{\gamma-2}\, y) \cdot (x-y), \quad 1 < \gamma < 2,$$

for all x, $y \in \mathbb{R}^{\mathbb{N}}$.

We claim that I'_λ is strictly monotone. Since for $\lambda < 0$ and $p(x) \geq 2$, we have

$$\begin{aligned}
(I'_\lambda(u) - I'_\lambda(v), u-v) &= \left(\int_\Omega |\, x\,|^{p(x)} |\, \Delta u\,|^{p(x)-2}\, \Delta u dx - \int_\Omega |\, x\,|^{p(x)} |\, \Delta v\,|^{p(x)-2}\, \Delta v dx, u-v\right) \\
&\quad + \left(\lambda \int_\Omega |\, v\,|^{q(x)-2}\, v dx - \lambda \int_\Omega |\, u\,|^{q(x)-2}\, u dx, u-v\right) \\
&= \left(\int_\Omega |\, x\,|^{p(x)}\, (|\, \Delta u\,|^{p(x)-2}\, \Delta u dx - |\, \Delta v\,|^{p(x)-2}\, \Delta v) dx\right)(u-v) \\
&\quad - \lambda \left(\int_\Omega |\, u\,|^{q(x)-2}\, u dx - \int_\Omega |\, v\,|^{q(x)-2}\, v dx\right)(u-v) \\
&\geq \int_\Omega \frac{|\, x\,|^{p(x)}}{2^{p(x)}} |\, \Delta u - \Delta v\,|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{2^{p(x)}} |\, u-v\,|^{p(x)}\, dx \\
&\geq \frac{1}{2^{p(x)}} \int_\Omega ||\, x\,||\, \Delta u - \Delta v\,||^{p(x)} = \frac{1}{2^{p(x)}} \|u-v\| > 0
\end{aligned}$$

for all u, $v \in X$ with $u \neq v$. A similar proof holds if $1 < p(x) < 2$.

Now assume that $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty} (I'_\lambda(u_n), u_n - u) \leq 0.$$

To show that $u_n \to u$, first note that from the monotonicity I'_λ, we obtain

$$\limsup_{n\to\infty} (I'_\lambda(u_n) - I'_\lambda(u), u_n - u) = 0. \tag{3.1}$$

On the other hand, a simple calculation shows that if $u \in L^{q(x)}$, then $|\, u\,|^{q(x)-1} \in L^{p(x)}$, and by the compactness of the embedding, we have

$$u_n \to u \text{ in } L^{q(x)}$$

and

$$|\, u_n\,|^{q(x)-2}\, u_n \to |\, u\,|^{q(x)-2}\, u \quad \text{in} \quad L^{p(x)}.$$

Set

$$\xi_n(u) = (|\, u_n\,|^{q(x)-2}\, u_n - |\, u\,|^{q(x)-2}\, u)(u_n - u)$$

and

$$\chi_n(u) = (|\, \Delta u_n\,|^{p(x)-2}\, \Delta u_n - |\, \Delta u\,|^{p(x)-2}\, \Delta u)(\Delta u_n - \Delta u).$$

Then,

$$\int_\Omega \xi_n(u) dx \to 0. \tag{3.2}$$

as $n \to \infty$, and it follows from (3.1) and (3.2) that

$$\limsup_{n\to\infty} \int_\Omega \chi_n(x)dx = 0. \tag{3.3}$$

Again applying elementary inequalities,

$$\int_{U_p} | x |^{p(x)} (| \Delta u_n |^{p(x)-2} \Delta u_n - | \Delta u |^{p(x)-2} \Delta u)(\Delta u_n - \Delta u) \geq \frac{1}{2^{p(x)}} \int_{U_p} || x || \Delta u_n - \Delta u ||^{p(x)} dx.$$

From (3.3), we obtain

$$\int_{U_p} \|u_n - u\|^{p(x)} dx \leq 0.$$

On the other hand,

$$\int_{U_p} (| u_n |^{q(x)-2} u_n - | u |^{q(x)-2} u)(u_n - u) \geq \frac{1}{2^{p(x)}} \int_{U_p} | u_n - u |^{q(x)} dx.$$

From (3.2),

$$\int_{U_p} | u_n - u |^{q(x)} dx \leq 0. \tag{3.4}$$

Therefore,

$$\int_{U_p} \frac{1}{p(x)} | x |^{p(x)} | \Delta u_n - \Delta u |^{p(x)} dx - \lambda \int_{U_p} \frac{1}{q(x)} | u_n - u |^{q(x)} dx \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Hence, by the compactness of the embedding, we can easily obtain

$$\lim_{n\to\infty} \|u_n - u\| \to 0 \text{ on } U_p.$$

Now consider the above process on V_p. Let $\delta_n = | \Delta u_n | + | \Delta u |$; then

$$\int_{V_p} (| \Delta u_n - \Delta u |^{\frac{p(x)}{2}})^2 dx \leq \int_{V_p} \frac{\chi_n^{\frac{p(x)}{2}}}{p(x) - 1} (\delta_n)^{\frac{p(x)}{2}(2-p(x))} dx \leq \frac{1}{p^- - 1} \int_{V_p} \chi_n^{\frac{p(x)}{2}} (\delta_n)^{\frac{p(x)}{2}(2-p(x))} dx.$$

Using Young's inequality for all $d > 0$ with $a = d\chi_n^{\frac{p(x)}{2}}$ and $b = (\delta_n)^{\frac{p(x)}{2}(2-p(x))}$ gives

$$d \int_{V_p} | \Delta u_n - \Delta u |^{p(x)} dx \leq d \int_{V_p} \chi_n^{\frac{p(x)}{2}} (\delta_n)^{\frac{p(x)}{2}(2-p(x))} dx \leq \int_{V_p} \chi_n d^{\frac{2}{p(x)}} dx + \int_{V_p} (\delta_n)^{p(x)} dx. \tag{3.6}$$

Since $\limsup_{n\to\infty} \int_\Omega \chi_n dx = 0$, it can be shown that $0 \leq \int_{V_p} \chi_n dx < 1$. Also, if $\int_{V_p} \chi_n dx = 0$, then since χ_n is positive, we can conclude that $\chi_n = 0$. Therefore,

$$\int_{V_p} | \Delta u_n - \Delta u |^{p(x)} dx \leq 0,$$

since otherwise choosing $d = (\int_{V_p} \chi_n dx)^{\frac{-1}{2}} > 1$ and the fact that $\frac{2}{p(x)} < 2$, we have

$$\int_{V_p} | \Delta u_n - \Delta u |^{p(x)} \, dx \leq \frac{1}{d} \left(\int_{V_p} \chi_n d^2 dx + \int_{V_p} \delta_n^{p(x)} dx \right).$$

Using the value of d in (3.6), we have

$$\begin{aligned} \int_{V_p} | \Delta u_n - \Delta u |^{p(x)} \, dx &\leq \left(\int_{V_p} \chi_n dx \right)^{\frac{1}{2}} \left(\int_{V_p} \chi_n dx \left(\int_{V_p} \chi_n dx \right)^{-1} \right) + \int_{V_p} \delta_n^{p(x)} dx \\ &= \left(\int_{V_p} \chi_n dx \right)^{\frac{1}{2}} \left(1 + \int_{V_p} \delta_n^{p(x)} dx \right). \end{aligned}$$

On the other hand, $\int_{V_p} \delta_n^{p(x)} dx$ is bounded as $n \to \infty$, so

$$\int_{V_p} | \Delta u_n - \Delta u |^{p(x)} \, dx \to 0.$$

Also, it can be proved in a similar way that

$$\int_{V_p} | u_n - u |^{q(x)} \, dx \to 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$\int_{V_p} \frac{1}{p(x)} \mid x \mid^{p(x)} \mid \Delta u_n - \Delta u \mid^{p(x)} \, dx - \lambda \int_{V_p} \frac{1}{q(x)} \mid u_n - u \mid^{q(x)} \, dx \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\lim_{n \to \infty} \|u_n - u\| \to 0 \quad \text{on} \quad V_p.$$

This shows that I'_λ satisfies property (S_+) on Ω.

Next, we will show that φ satisfies the Palais-Smale condition. Suppose, to the contrary, that $\| u_n \| \to \infty$ as $n \to \infty$. In this case, from the definition of the norm and the facts that the embedding of $D_0^{2,p(x)}(\Omega)$ into $L^{q(x)}$ is compact and $I_\lambda \in C^1(X, \mathbb{R})$, we can write

$$\begin{aligned} \varphi(u_n) - \frac{(\varphi'(u_n), u_n)}{\theta} &= \int_\Omega \frac{| x |^{p(x)}}{p(x)} \mid \Delta u_n \mid^{p(x)} \, dx - \lambda \int_\Omega \frac{1}{q(x)} \mid u_n \mid^{q(x)} \, dx - \mu \int_\Omega F(x, u_n) dx \\ &\quad - \frac{1}{\theta} \left((I'(u_n), u_n) - \mu \int_\Omega f(x, u_n) u_n dx \right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p(x)} - \lambda \left(\frac{1}{q^-} - \frac{1}{\theta} \right) \int_\Omega \frac{1}{q(x)} \mid u_n \mid^{q(x)} \, dx \\ &\quad + \mu \int_\Omega \left[\frac{f(x, u_n) u_n}{\theta} - F(x, u_n) \right] dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p(x)} - \lambda \left(\frac{1}{q^-} - \frac{1}{\theta} \right) \left(\mid u_n \mid_{q(x)}^{q-} + \mid u_n \mid_{q(x)}^{q+} \right) \\ &\quad + \mu \int_\Omega \left[\frac{f(x, u_n) u_n}{\theta} - F(x, u_n) \right] dx. \end{aligned}$$

Since $\lambda < 0$, we see that

$$\varphi(u_n) - \frac{(\varphi'(u_n), u_n)}{\theta} \geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|^{p^-} - \lambda k \left(\frac{1}{q^-} - \frac{1}{\theta}\right) (\|u_n\|^{q^-} + \|u_n\|^{q^+})$$
$$+ \mu \int_\Omega \left[\frac{f(x,u_n)u_n}{\theta} - F(x,u_n)\right] dx$$
$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|^{p^-} + \mu \int_\Omega \left[\frac{f(x,u_n)u_n}{\theta} - F(x,u_n)\right] dx.$$

By condition (c) and setting

$$V = \sup\left\{\left|\frac{f(x,t)t}{\theta} - F(x,t)\right| : x \in \bar{\Omega},\ \mid t \mid < t^*\right\},$$

we have

$$\left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|^{p^-} \leq \varphi(u_n) - \frac{1}{\theta}(\varphi'(u_n), u_n) - \mu \int_{|u_n|>t^*} \left[\frac{f(x,u_n)u_n}{\theta} - F(x,u_n)\right] dx$$
$$+ \mu V \operatorname{meas}(\Omega).$$

This can be written as

$$\left(\frac{\theta - p^+}{p^+\theta}\right) \|u_n\|^{p^-} \leq \varphi(u_n) - \frac{1}{\theta}(\varphi'(u_n), u_n) + \mu V \operatorname{meas}(\Omega).$$

Letting $T = \left(\frac{\theta - p^+}{p^+\theta}\right)$ and dividing by $\|u_n\|$ gives

$$T\|u_n\|^{p^- - 1} \leq 0,$$

which implies $\|u_n\| \to 0$. This is a contradiction, and so $\{u_n\}$ is bounded in X, and the (PS) condition is satisfied.

Since X is reflexive, there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$. Then,

$$(\varphi'(u_n), u_n - u) = (I'_\lambda(u_n), u_n - u) + \mu \int_\Omega f(x,u_n)(u_n - u)dx.$$

By using the boundedness of $\{u_n - u\}$ and the (PS) condition, we can write

$$(\varphi'(u_n), u_n - u) \leq \|\varphi'(u_n)\| |u_n - u| \to 0 \quad \text{as} \quad n \to \infty.$$

By condition (a) and Hölder's inequality,

$$\int_\Omega \mid f(x,u_n) \mid (u_n - u)dx \leq \left(\int_\Omega c(1+ \mid u_n \mid^{q(x)-1})^{p(x)} dx\right)^{\frac{1}{p(x)}} \left(\int_\Omega (u_n - u)^{q(x)} dx\right)^{\frac{1}{q(x)}},$$

and as $n \to \infty$, we obtain $\int_\Omega \mid f(x,u_n) \| u_n - u \mid dx \to 0$. Thus, $(I'_\lambda(u_n), u_n - u) \to 0$ as $n \to \infty$. Taking into account that the I'_λ has the property (S_+), we have $u_n \to u$ in X.

Next we use condition (b) to show that 0 is a strict local minimizer of φ, but it is not a global minimizer. Notice that $\varphi(0) = I_\lambda(0) - \mu\psi(0)$. From (b), for $\epsilon < \frac{1}{p^+ c_1 \mu}$, there exists $E > 0$ such that

$$\mid F(x,t) \mid \leq \varepsilon \mid t \mid^{q(x)} \quad \text{for all } \mid t \mid \leq E. \tag{3.7}$$

By the embedding properties, for all $u \in X \setminus \{0\}$ with $\|u\| > 1$,

$$\varphi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda k}{q^-}(\|u\|^{q^-} + \|u\|^{q^+}) - \mu\epsilon \int_\Omega \mid u \mid^{q(x)} dx$$
$$\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda k}{q^-}(\|u\|^{q^-} + \|u\|^{q^+}) - \mu\epsilon c_1 \|u\|^{p^-}$$
$$= \left(\frac{1}{p^+} - \mu\epsilon c_1\right) \|u\|^{p^-} - \frac{\lambda k}{q^-}(\|u\|^{q^-} + \|u\|^{q^+}) > 0.$$

where k is a positive constant. Condition (e) and the fact that $\mu^* < \mu$ imply that

$$\varphi(e) < \int_\Omega \frac{|\,x\,|^{p(x)}}{p(x)} \mid \Delta e \mid^{p(x)} dx - \lambda \int_\Omega \frac{|\,e(x)\,|^{q(x)}}{q(x)} - \mu^* \int_\Omega F(x,e)dx < 0.$$

To show that φ has a global minimizer, we first show that φ is coercive. From the continuity of f and condition (d), there exists $c > 0$ and $l \in L^1(\Omega, \mathbb{R}^+)$ such that

$$|\,F(x,t)\,| \leq \epsilon \mid t \mid^{q(x)} \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R} \quad \text{and} \quad |\,t\,| > c$$

and

$$|\,F(x,t)\,| \leq l(x) \quad \text{for all} \quad (x,t) \in \overline{\Omega} \times \mathbb{R} \quad \text{and} \quad |\,t\,| < c.$$

Now set $\Omega_1 = \{x \in \Omega : |\,u\,| > c\}$ and $\Omega_2 = \{x \in \Omega : |\,u\,| \leq c\}$. Then,

$$\begin{aligned}
\varphi(u) &\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda \int_\Omega \frac{|\,u\,|^{q(x)}}{q(x)} dx - \mu \int_{\Omega_1} F(x,u)dx - \mu \int_{\Omega_2} F(x,u)dx \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda \int_\Omega \frac{|\,u\,|^{q(x)}}{q(x)} dx - \mu\epsilon \int_\Omega |\,u\,|^{q(x)} dx - \mu \int_{\Omega_2} l(x)dx \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda k}{q^-}(\|u\|^{q^-} + \|u\|^{q^+}) - \mu\epsilon k(\|u\|^{q^-} + \|u\|^{q^+}) - \mu\|l\|_{L^1}.
\end{aligned}$$

An immediate computation gives

$$\varphi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - k\left(\frac{\lambda}{q^-} + \frac{1}{p^+c_1}\right)(\|u\|^{q^-} + \|u\|^{q^+}) - \mu\|l\|_{L^1}.$$

since $\epsilon < \frac{1}{p^+c_1\mu}$. By choosing $\lambda < \frac{-q^-}{p^+c_1}$, if $\lim_{n\to\infty}\|u_n\| = \infty$, then $\varphi(u_n) \to \infty$, which means φ is coercive.

If for every $\alpha \in \mathbb{R}$ with $\varphi(e) < \alpha$, we set $Y = \{u \in X : \varphi(u) \leq \alpha\}$, then $Y \neq \emptyset$, and since φ is coercive, Y is bounded.

Next, we show that φ is bounded below on Y. Assume to the contrary that there exists a bounded sequence $\{u_n\}$ in Y such that

$$\lim_{n\to\infty} \varphi(u_n) = -\infty. \tag{3.8}$$

Since X is a reflexive Banach space, there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. Also, by the compactness of the embedding $X \hookrightarrow L^{q(x)}(\Omega)$, we have

$$u_n \to u_0 \quad \text{in} \quad L^{q(x)}(\Omega) \quad \text{and} \quad u_n \to u_0 \quad \text{for a.e.} \quad x \in \Omega.$$

Hence,

$$F(x, u_n(x)) \to F(x,u) \quad \text{for a.e} \quad x \in \Omega.$$

By Fatou's Lemma,

$$\limsup_{n\to\infty} \int_\Omega F(x,u_n(x))dx \leq \int_\Omega F(x,u)dx.$$

On the other hand, in [27] it was proved that I_λ is lower semicontinuous and

$$\begin{aligned}
\liminf_{n\to\infty} \varphi(u_n) &= \liminf_{n\to\infty} I_\lambda(u_n) - \mu \limsup_{n\to\infty} \int_\Omega F(x,u_n(x))dx \\
&\geq I_\lambda(u_n) - \mu \int_\Omega F(x,u_n)dx \\
&= \varphi(u_0) > -\infty,
\end{aligned}$$

which contradicts (3.8). Thus, if $\{u_n\} \subset Y$ and $\lim_{n\to\infty} \varphi(u_n) = v$, then

$$0 > v := \inf_{u\in Y} \varphi(u) = \inf_{u\in X} \varphi(u) > -\infty.$$

Hence, there exists $u_1 \in X$ such that, up to taking a subsequence, $u_n \to u_1$. Therefore,

$$\varphi(u_1) = v < 0, \tag{3.9}$$

and u_1 is a nontrivial solution.

We will show that φ satisfies the hypotheses of Lemma 2.1. By Theorem 2.1 and the fact that 0 is a strict local minimizer of φ, there exists $0 < \rho < \|u_1\|$ such that $r := \inf_{u\in\partial B_\rho(u_0)} \varphi(u) > 0$. Now (3.9) and the fact that $\varphi(0) = 0$ imply that $u_1 \notin \overline{\partial B_\rho(u_0)}$ and $\max\{\varphi(0), \varphi(u_1)\} < \inf_{u\in\partial B_\rho(u_0)} \varphi(u) = r$. By Lemma 2.1, there exists a critical point u_2 of φ such that

$$\varphi(u_2) \geq r > 0. \tag{3.10}$$

Inequalities (3.9) and (3.10) imply $u_1 \neq u_2$ and $u_2 \neq 0$. This completes the proof of the theorem. □

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