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On Riemannian Sequential Warped Product Submersions

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ABSTRACT: As a natural generalization of Riemannian warped product submersions, we introduce the idea of Riemannian sequential warped product submersions in this paper and study some fundamental properties like totally geodesic, totally umbilical and minimality. Further, we work with characterization of geodesic for Riemannian sequential warped product submersions and study conformal and killing vector fields for that submersions.

Key Words: Riemannian submersions, sequential warped product, sequential warped product submersions.

Contents

1. Introduction

J. F. Nash [9] started the study of warped product manifolds and proved that every warped product manifold could be embedded as a Riemannian submanifold in some Euclidean spaces. B. O'Neill and Bishop [2] in 1969 studied warped product manifold as a fruitful generalization of the Riemannian product manifold. Warped product manifold is used to study Riemannian manifolds with negative sectional curvature (pseudo-Riemannian manifold).
Robert-Walker, static, Schwarzschild, and Kruskal space-time can be considered as a warped product. The Warped product technique has been used to construct essential examples in relativity and differential geometry [4]. Also, many exact solutions of the Einstein field equations and modified field equations are warped products.
De, Shenway and Unal [5] introduced the notion of sequential warped product manifolds to study a larger class of exact solution of Einstein equation. They obtained curvature tensor, Ricci curvature and some classes of sequential warped product models. Further, Sahin [14] studied sequential warped product submanifold of Kaehler manifolds. In [11], Perktas and Blaga extended these results to Sasakian manifold.

B. O'Neill [10], in the 1960s, introduced the notion of Riemannian submersion as a tool to study the geometry of a manifold in terms of simpler components, namely, base space and fibers. It is well known that Riemannian submersion has many applications in physics. It is applicable in Yang-Mill theory [3], supergravity, and superstring theories [6,8].

Besse [1] considered warped product Riemannian submersion. Further, I. K. Erken and C. Murathan [7] studied warped product Riemannian submersion and obtained fundamental geometric properties.

We define Riemannian sequential warped product submersion as follows:

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Definition 1.1 Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are two sequential warped product manifolds and $\pi_i : \mathbb{M}_i \longrightarrow \mathbb{N}_i,\ i \in \{1,2,3\}$ are the Riemannian submersion between the manifold $\mathbb{M}_i$ and $\mathbb{N}_i$. Then the map

$$\pi = \pi_1 \times \pi_2 \times \pi_3 : \mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3 \longrightarrow \mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3,$$

given by $\pi(X_1, X_2, X_3) = (\pi_1(X_1), \pi_2(X_2), \pi_3(X_3))$ is a Riemannian submersion, which is called Riemannian sequential warped product submersion.

The main aim of this paper is to study some properties of Riemannian sequential warped product submersions. This work is the generalisation of the work based on Riemannian doubly warped product submerions and Riemannian wapred-twsited product submersions [12,13].

This paper is divided into four sections. In section 2: we define Riemannian submersion and Sequential warped product manifolds. In section 3: we study some properties of Riemannian sequential warped product submersion. In section 4: we work with characterization of geodesic for Riemannian sequential warped product submersions and study conformal and killing vector fields for that submersions. This work is the generalisation of the work [12,13]

2. Preliminaries

In this section, we recall some necessary definitions, results, and notations which are useful for the paper. This section is divided into two subsections.

2.1. Riemannian Submersion

Let $(\mathbb{M}, g_{\mathbb{M}})$ and $(\mathbb{N}, g_{\mathbb{N}})$ be two Riemannian manifolds with $dim\mathbb{M} = m$ and $dim\mathbb{N} = n$, where $m > n$. A smooth map $\pi : (\mathbb{M}, g_{\mathbb{M}}) \longrightarrow (\mathbb{N}, g_{\mathbb{N}})$ is said to be Riemannian submersion if the following axioms are satisfied,

1. π_*(derivative map of π) is onto
2. π_* preserves the length of horizontal vectors, i.e.,

$$g_{\mathbb{N}}(\pi_* X, \pi_* Y) = g_{\mathbb{M}}(X, Y). \tag{2.1}$$

For each $p_2 \in \mathbb{N}$, $\pi^{-1}(p_2)$ is a submanifold of dimension $(m-n)$ called fibers. A vector field on manifold $\mathbb{M}$ is called horizontal if it is always orthogonal to fibers and is called vertical if it is always tangent to fibers. Let $\pi : (\mathbb{M}, g_{\mathbb{M}}) \longrightarrow (\mathbb{N}, g_{\mathbb{N}})$ be a smooth map between Riemannian manifolds. Then $\Gamma(T\mathbb{M})$ has the following decomposition

$$T\mathbb{M} = (ker\pi_*) \oplus (ker\pi_*)^{\perp}.$$

The tangent bundle $T\mathbb{N}$ of $\mathbb{N}$ has the following decomposition.

$$T\mathbb{N} = (rang\pi_*) \oplus (rang\pi_*)^{\perp},$$

where $(rang\pi_*)^{\perp}$ is the orthogonal complement of $(rang\pi_*)$.
B. O'Neill [10] first introduced the fundamental tensors of submersions, and it is given as

$$T(E,F) = T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.2}$$

$$A(E,F) = A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.3}$$

where E and F are vector fields on $\mathbb{M}$; $\mathcal{H}$ and $\mathcal{V}$ are the projection morphism on the distribution $(ker\pi*)^{\perp}$ and $(ker\pi_*)$, respectively. We observe that the tensor fields T and A satisfy

1. $T(U,V) = T(V,U), \qquad U, V \in \Gamma(ker\pi_*)$.

2. $A(X,Y) = -A(Y,X), \quad X,Y \in \Gamma(ker\pi_*)^{\perp}$.

Equation (2.2) and (2.3) gives the following Lemma.

Lemma 2.1 *[10] Let $X, Y \in \Gamma(ker\pi_*)^{\perp}$ and $U, V \in \Gamma(ker\pi_*)$; we have*

$$\nabla_U V = T(U,V) + \hat{\nabla}_U V, \tag{2.4}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + T(U,X), \tag{2.5}$$

$$\nabla_X U = A(X,U) + \mathcal{V}\nabla_X U, \tag{2.6}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A(X,Y), \tag{2.7}$$

where ∇ is the Levi-Civita connection of $(\mathbb{M}, g_{\mathbb{M}})$ and $\hat{\nabla}_U V = \mathcal{V}\nabla_U V$.

It is noted that if the tensor field A (respectively T) vanishes, then the horizontal distribution $\mathcal{H}$ (respectively vertical distribution $\mathcal{V}$ or fibre) is integrable. Also, any fibre of Riemannian submersion π is totally umbilical if and only if

$$T(V,W) = g(V,W)\mathbb{H},$$

where $\mathbb{H}$ is the mean curvature vector field of the fibre and it is given by

$$N = s\mathbb{H}$$

such that

$$N = \sum_{i=1}^{s} T(U_i, U_i), \tag{2.8}$$

and $\{U_1, U_2, \cdots, U_s\}$ denotes the orthonormal basis of vertical distribution and s denotes the dimension of any fibre. It is easy to see that any fibre of Riemannian submersion π is minimal if and only if the horizontal vector field N vanishes.

2.2. Sequential Warped Product Manifolds

Let $(\mathbb{M}_i, g_{\mathbb{M}_i})$ be Riemannian manifolds of dimensions m_i, $i \in \{1,2,3\}$, f_1 and f_2 be two positive differentiable functions on $\mathbb{M}_1$ and $\mathbb{M}_1 \times \mathbb{M}_2$ respectively. Let $\pi_i : \mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3 \longrightarrow \mathbb{M}_i$ be the natural projections from the product manifold $\mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3$ to $\mathbb{M}_i$, $i \in \{1,2,3\}$. Then the sequential warped product manifold $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ is a product manifold $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ equipped with the metric $g_{\mathbb{M}}$ such that

$$\begin{aligned} g_{\mathbb{M}}(X,Y) &= g_{\mathbb{M}_1}(\pi_{1*}(X), \pi_{1*}(Y) + {f_1}^2 g_{\mathbb{M}_2}(\pi_{2*}(X), \pi_{2*}(Y))) \\ &+ {f_2}^2 g_{\mathbb{M}_3}(\pi_{3*}(X), \pi_{3*}(Y)), \end{aligned} \tag{2.9}$$

for any $X, Y \in \Gamma(T\mathbb{M})$, and f_1, f_2 are called the warping function.
For a vector field X on $\mathbb{M}_1$, the lift of X to $(\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ is the vector field $\tilde{X}$ whose value at each (p,q,r) is the lift X_p to (p,q,r). Thus the lift of X is the unique vector field on $(\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ that is π_1-related to X and π_2-related, π_3-related to the zero vector field on $\mathbb{M}_2$ and $\mathbb{M}_3$ respectively.
Let ∇ and ∇^i denote the Levi-Civita connection of $(\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{M}_i$, respectively for $i \in \{1,2,3\}$. Further, we denote the set of lifts of vector fields on $\mathbb{M}_i$ by $\mathfrak{L}(\mathbb{M}_i)$ and use the same notation

for a vector field and its lift. Then, the covariant derivative formulas for a sequential warped product manifold are given [5]

$$\begin{aligned}
\nabla_{X_1} Y_1 &= \nabla^1_{X_1} Y_1, && (2.10)\\
\nabla_{X_1} X_2 &= \nabla_{X_2} X_1 = X_1(\ln f_1) X_2, && (2.11)\\
\nabla_{X_2} Y_2 &= \nabla^2_{X_2} Y_2 - f_1 g_{\mathbb{M}_2}(X_2, Y_2) grad^1 f_1, && (2.12)\\
\nabla_{X_3} X_1 &= \nabla_{X_1} X_3 = X_1(\ln f_2) X_3, && (2.13)\\
\nabla_{X_2} X_3 &= \nabla_{X_3} X_2 = X_2(\ln f_2) X_3, && (2.14)\\
\nabla_{X_3} Y_3 &= \nabla^3_{X_3} Y_3 - f_2 g_{\mathbb{M}_3}(X_3, Y_3) grad f_2, && (2.15)
\end{aligned}$$

for $X_i, Y_i \in \mathfrak{X}(\mathbb{M}_i)$.
Remark: Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ then $grad^1 f_1$ is the gradient of f_1 on $\mathbb{M}_1$ and $grad f_2$is the gardient of f_2 on $(\mathbb{M}_1 \times \mathbb{M}_2)$.

3. Riemannian Sequential Warped Product Submersions

In this section, we define Riemannian sequential warped product submersion and obtain some fruitful results.

Proposition 3.1 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi_i : \mathbb{M}_i \longrightarrow \mathbb{N}_i$, $i \in \{1,2,3\}$ are Riemannian submersion between the manifolds $\mathbb{M}_i$ and $\mathbb{N}_i$. Then the map*

$$\pi = \pi_1 \times \pi_2 \times \pi_3 : \mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3 \longrightarrow \mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3,$$

given by $\pi(X_1, X_2, X_3) = (\pi_1(X_1), \pi_2(X_2), \pi_3(X_3))$ is a Riemannian submersion, which is called Riemannian sequential warped product submersion.

Proof: Let $\pi_i, i = \{1,2,3\}$, are submersions between $\mathbb{M}_i$ and $\mathbb{N}_i$ then the map π is a submersion from $\mathbb{M}$ to $\mathbb{N}$. Since

$$\begin{aligned}
T_{(p_1,p_2,p_3)}(\mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3) &= T_{(p_1,p_2,p_3)}(\mathbb{M}_1 \times \{p_2\} \times \{p_3\})\\
&\oplus T_{(p_1,p_2,p_3)}(\{p_1\} \times \mathbb{M}_2 \times \{p_3\})\\
&\oplus T_{(p_1,p_2,p_3)}(\{p_1\} \times \{p_2\} \times \mathbb{M}_3),
\end{aligned}$$

$ker(\pi_1 \times \pi_2 \times \pi_3)_* = ker(\pi_{1*}) \times ker(\pi_{2*}) \times ker(\pi_{3*})$ and $T_{(p_1,p_2,p_3)}(\mathbb{M}_1 \times \mathbb{M}_2 \times \mathbb{M}_3) = \mathcal{H}_{(p_1,p_2,p_3)} \oplus \mathcal{V}_{(p_1,p_2,p_3)}$. Since

$$\begin{aligned}
T_{(p_1,p_2,p_3)}(\mathbb{M}_1 \times \{p_2\} \times \{p_3\}) &= \Big((\mathcal{H}_1)_{p_1} \times \{p_2\} \times \{p_3\}\Big) \oplus \Big((\mathcal{V}_1)_{p_1} \times \{p_2\} \times \{p_3\}\Big),\\
T_{(p_1,p_2,p_3)}(\{p_1\} \times \mathbb{M}_2 \times \{p_3\}) &= \Big(\{p_1\} \times (\mathcal{H}_2)_{p_2} \times v\{p_3\}\Big) \oplus \Big(\{p_1\} \times (\mathcal{V}_2)_{p_2} \times \{p_3\}\Big)
\end{aligned}$$

and

$$T_{(p_1,p_2,p_3)}(\{p_1\} \times \{p_2\} \times \mathbb{M}_3) = \Big(\{p_1\} \times \{p_2\} \times (\mathcal{H}_3)_{p_3}\Big) \oplus \Big(\{p_1\} \times \{p_2\} \times (\mathcal{V}_3)_{p_3}\Big).$$

Thus, we obtain

$$\mathcal{V}_{(p_1,p_2,p_3)} = \Big((\mathcal{V}_1)_{p_1} \times \{p_2\} \times \{p_3\}\Big) \oplus \Big(\{p_1\} \times (\mathcal{V}_2)_{p_2} \times \{p_3\}\Big) \oplus \Big(\{p_1\} \times \{p_2\} \times (\mathcal{V}_3)_{p_3}\Big)$$

and

$$\mathcal{H}_{(p_1,p_2,p_3)} = \Big((\mathcal{H}_1)_{p_1} \times \{p_2\} \times \{p_3\}\Big) \oplus \Big(\{p_1\} \times (\mathcal{H}_2)_{p_2} \times \{p_3\}\Big) \oplus \Big(\{p_1\} \times \{p_2\} \times (\mathcal{H}_3)_{p_3}\Big)$$

Let $\left(v^{\mathcal{H}}\right) \in (\mathcal{H}_i)_{p_i}$ (resp.$\left(v^{\mathcal{V}}\right) \in (\mathcal{V}_i)_{p_i}$ then $\overline{(v^{\mathcal{H}})}$ (resp. $\left.\overline{(v^{\mathcal{V}})}\right)$ the lift of $v^{\mathcal{H}}$ (resp. $\left.v^{\mathcal{V}}\right)$ to $\mathcal{H}_{(p_1,p_2,p_3)}$ (resp. $\left.\mathcal{V}_{(p_1,p_2,p_3)}\right)$ is $(\bar{v})^{\mathcal{H}}$ (resp. $\left.(\bar{v})^{\mathcal{V}}\right)$. In view of this fact, for a horizontal vector field $X^{\mathcal{H}} \in \Gamma\left(\mathcal{H}_i\right)$ and a vertical vector field $X^{\mathcal{V}} \in \Gamma\left(\mathcal{V}_i\right)$, we are lead to $\overline{(X^{\mathcal{H}})} = (\bar{X})^{\mathcal{H}}$ and $\overline{(X^{\mathcal{V}})} = (\bar{X})^{\mathcal{V}}$, respectively. Next, we show that this submersion is a Riemannian sequential warped product submersion. Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be horizontal vectors on $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3$ respectively. Then we get immediately from (2.9)

$$\begin{aligned}
g_{\mathbb{N}}\left(\pi_*\left(X_1, X_2, X_3\right), \pi_*\left(Y_1, Y_2, Y_3\right)\right) &= g_{\mathbb{N}_1}\left(\pi_{1*}\left(X_1\right), \pi_{1*}\left(Y_1\right)\right) \\
&+ \rho_1^2\left(\pi_1\left(p_1\right)\right) g_{\mathbb{N}_2}\left(\pi_{2*}\left(X_2\right), \pi_{2*}\left(Y_2\right)\right) \\
&+ \rho_2^2\left(\pi_2\left(p_2\right)\right) g_{\mathbb{N}_3}\left(\pi_{3*}\left(X_3\right), \pi_{3*}\left(Y_3\right)\right) \\
&= g_{\mathbb{M}_1}\left(X_1, Y_1\right) + f_1^2\left(p_1\right) g_{\mathbb{M}_2}\left(X_2, Y_2\right) \\
&+ f_2^2\left(p_2\right) g_{\mathbb{M}_3}\left(X_3, Y_3\right) \\
&= g_{\mathbb{M}}\left(\left(X_1, X_2, X_3\right), \left(Y_1, Y_2, Y_3\right)\right).
\end{aligned}$$

So the differential π_* preserves the lenghts of horizontal vectors. This completes the proof of the proposition. □

Lemma 3.1 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$. Then for any $U_i, V_i \in \Gamma(\mathcal{V}_i)$, $i = \{1,2,3\}$ we have*

1. $T(U_1, V_1) = T_1(U_1, V_1)$.
2. $T(U_1, U_2) = 0$.
3. $T(U_2, V_2) = T_2(U_2, V_2) - f_1 g_{\mathbb{M}_2}(U_2, V_2)\mathcal{H}grad^1 f_1$.
4. $T(U_3, U_1) = 0$.
5. $T(U_2, U_3) = 0$.
6. $T(U_3, V_3) = T_3(U_3, V_3) - f_2 g_{\mathbb{M}_3}(U_3, V_3)\mathcal{H}grad f_2$.

Proof: From (2.4) we have

$$\nabla_{U_1} V_1 = \hat{\nabla}_{U_1} V_1 + T(U_1, V_1). \tag{3.1}$$

By equation (2.10) we get

$$\nabla_{U_1} V_1 = \nabla^1_{U_1} V_1. \tag{3.2}$$

Using equation (3.1) in equation (3.2) and comparing the vertical parts we obtain

$$T(U_1, V_1) = T_1(U_1, V_1).$$

Making use of equation (2.11) we obtain

$$\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = U_1(\ln f_1) U_2. \tag{3.3}$$

Combining equation (3.3) with equation (3.1) we get

$$T(U_1, U_2) = 0.$$

From equation (2.2) we have

$$T(U_2, V_2) = \mathcal{H}(\nabla_{U_2} V_2). \tag{3.4}$$

Using equation (2.12) we get

$$\nabla_{U_2} V_2 = \nabla^2_{U_2} V_2 - f_1 g_{\mathbb{M}_2}(U_2, V_2) grad^1 f_1. \tag{3.5}$$

From equation (2.4) we know that

$$\nabla^2_{U_2} V_2 = T_2(U_2, V_2) + \mathcal{V}\nabla^2_{U_2} V_2. \tag{3.6}$$

By using equations (3.5) and (3.6) in equation (3.4) we obtain

$$T(U_2, V_2) = T_2(U_2, V_2) - f_1 g_{\mathbb{M}_2}(U_2, V_2)\mathcal{H} grad^1 f_1.$$

In a similar manner, we prove the rest of the part. □

Lemma 3.2 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$. Then for any $X_i, Y_i \in \Gamma(\mathcal{H}_i)$, $i = \{1, 2, 3\}$ we have*

1. $\mathcal{H}\nabla_{X_1} Y_1 = \mathcal{H}\nabla^1_{X_1} Y_1, \quad A(X_1, Y_1) = A_1(X_1, Y_1).$

2. $\mathcal{H}\nabla_{X_1} X_2 = X_1(\ln f_1)X_2 = \mathcal{H}\nabla_{X_2} X_1, \quad A(X_1, X_2) = 0 = A(X_2, X_1).$

3. $A(X_2, Y_2) = A_2(X_2, Y_2)$ *and* $\mathcal{V} grad^1 f_1 = 0$,
 $\mathcal{H}\nabla_{X_2} Y_2 = \mathcal{H}\nabla^2_{X_2} Y_2 - f_1 g_{\mathbb{M}_2}(X_2, Y_2)\mathcal{H} grad^1 f_1.$

4. $\mathcal{H}\nabla_{X_3} X_1 = X_1(\ln f_2)X_3 = \mathcal{H}\nabla_{X_1} X_3, \quad A(X_3, X_1) = 0 = A(X_1, X_3).$

5. $\mathcal{H}\nabla_{X_2} X_3 = X_2(\ln f_2)X_3 = \mathcal{H}\nabla_{X_3} X_2, \quad A(X_2, X_3) = 0 = A(X_3, X_2).$

6. $A(X_3, Y_3) = A_3(X_3, Y_3)$ *and* $\mathcal{V} grad f_2 = 0$,
 $\mathcal{H}\nabla_{X_3} Y_3 = \mathcal{H}\nabla^3_{X_3} Y_3 - f_2 g_{\mathbb{M}_3}(X_3, Y_3)\mathcal{H} grad f_2.$

Proof: From equation (2.7) we have

$$\nabla_{X_1} Y_1 = \mathcal{H}\nabla_{X_1} Y_1 + A(X_1, Y_1). \tag{3.7}$$

By using equation (2.10) and equation (3.7) we get

$$\mathcal{H}\nabla_{X_1} Y_1 + A(X_1, Y_1) = \mathcal{H}\nabla^1_{X_1} Y_1 + A_1(X_1, Y_1). \tag{3.8}$$

Separating the horizontal and Vertical parts in equation (3.8) we obtain (i).
Again using equation (2.7) we have

$$\nabla_{X_1} X_2 = \mathcal{H}\nabla_{X_1} X_2 + A(X_1, X_2). \tag{3.9}$$
$$\nabla_{X_2} X_1 = \mathcal{H}\nabla_{X_2} X_1 + A(X_2, X_1). \tag{3.10}$$

Combining equations (3.9), (3.10), and (2.11) we obtain (ii).
We know

$$A(X_2, Y_2) = \mathcal{V}\nabla_{X_2} Y_2.$$

Using equation (2.12) in the above expression we get

$$\begin{aligned} A(X_2, Y_2) &= \mathcal{V}\left(\nabla^2_{X_2} Y_2 - f_1 g_{\mathbb{M}_2}(X_2, Y_2) grad^1 f_1\right) \\ &= \mathcal{V}(\nabla^2_{X_2} Y_2) - f_1 g_{\mathbb{M}_2}(X_2, Y_2)\mathcal{V} grad^1 f_1 \\ &= A_2(X_2, Y_2) - f_1 g_{\mathbb{M}_2}(X_2, Y_2)\mathcal{V} grad^1 f_1. \end{aligned} \tag{3.11}$$

and

$$\mathcal{H}\nabla_{X_2} Y_2 = \mathcal{H}\nabla^2_{X_2} Y_2 - f_1 g_{\mathbb{M}_2}(X_2, Y_2)\mathcal{H} grad^1 f_1. \tag{3.12}$$

Since A and A_2 are skew-symmetric tensor fields and $g_{\mathbb{M}_2}$ is symmetric tensor field, by using equations (3.11) and (3.12) we obtain the required result (iii).
In a similar manner, we prove the rest of the part.

□

Lemma 3.3 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$. Then for any $V_i \in \Gamma(\mathcal{V}_i)$, and $X_i \in \Gamma(\mathcal{H}_i)$, $i = \{1, 2, 3\}$ we have*

1. $T(V_1, X_1) = T_1(V_1, X_1), \quad \mathcal{H}\nabla^1_{V_1} X_1 = \mathcal{H}\nabla_{V_1} X_1.$
2. $T(V_1, X_2) = 0 = \mathcal{V}\nabla_{X_2} V_1, \quad \mathcal{H}\nabla_{V_1} X_2 = V_1(\ln f_1)X_2 = A(X_2, V_1).$
3. $T(V_2, X_1) = X_1(\ln f_1)V_2 = \mathcal{V}\nabla_{X_1} V_2, \quad \mathcal{H}\nabla_{V_2} X_1 = 0 = A(X_1, V_2).$
4. $T(V_2, X_2) = T_2(V_2, X_2) - f_1 g_{\mathbb{M}_2}(V_2, X_2)\mathcal{V} grad^1 f_1,$
   $\mathcal{H}\nabla_{V_2} X_2 = \mathcal{H}\nabla^2_{V_2} - f_1 g_{\mathbb{M}_2}(V_2, X_2)\mathcal{H} grad^1 f_1.$
5. $T(V_1, X_3) = 0 = \mathcal{V}\nabla_{X_3} V_1, \quad \mathcal{H}\nabla_{V_1} X_3 = V_1(\ln f_2)X_3 = A(X_3, V_1).$
6. $T(V_3, X_1) = X_1(\ln f_2)V_3 = \mathcal{V}\nabla_{X_1} V_3, \quad \mathcal{H}\nabla_{V_3} X_1 = 0 = A(X_1, V_3).$
7. $T(V_2, X_3) = 0 = \mathcal{V}\nabla_{X_3} V_2, \quad \mathcal{H}\nabla_{V_2} X_3 = V_2(\ln f_2)X_3 = A(X_3, V_2).$
8. $T(V_3, X_2) = X_2(\ln f_2)V_3 = \mathcal{V}\nabla_{X_2} V_3, \quad \mathcal{H}\nabla_{V_3} X_2 = 0 = A(X_2, V_3).$
9. $T(V_3, X_3) = T_3(V_3, X_3) - f_2 g_{\mathbb{M}_3}(V_3, X_3)\mathcal{V} grad f_2,$
   $\mathcal{H}\nabla_{V_3} X_3 = \mathcal{H}\nabla^3_{V_3} X_3 - f_2 g_{\mathbb{M}_3}(V_3, X_3)\mathcal{H} grad f_2.$

Proof: For $V_1 \in \Gamma(\mathcal{V}_1)$ and $X_1 \in \Gamma(\mathcal{H}_1)$, by using equation (2.5) we have

$$\nabla_{V_1} X_1 = \mathcal{H}\nabla_{V_1} X_1 + T(V_1, X_1). \tag{3.13}$$

Making use of equations (2.10) and (2.5) we obtain

$$\begin{aligned} \nabla_{V_1} X_1 &= \nabla^1_{V_1} X \\ &= \mathcal{H}\nabla^1_{V_1} X_1 + T_1(V_1, X_1). \end{aligned} \tag{3.14}$$

Combining equations (3.13) and (3.14) and comparing the vertical and the horizontal parts in the resulting expression we get (i).
From equation (2.11) we have

$$\nabla_{V_1} X_2 = \nabla_{X_2} V_1 = V_1(\ln f_1) X_2. \tag{3.15}$$

Using equations (2.5) and (2.6) we get

$$\nabla_{V_1} X_2 = \mathcal{H}\nabla_{V_1} X_2 + T(V_1, X_2). \tag{3.16}$$
$$\nabla_{X_2} V_1 = A(X_2, V_1) + \mathcal{V}\nabla_{X_2} V_1. \tag{3.17}$$

Combining equations (3.15), (3.16), and (3.17) and comparing the vertical and the horizontal parts in the resulting expression we obtain (ii). On a similar note to prove (ii) we get (iii).
Further, using equation (2.5) and equation (2.12) we obtain (iv).
Similarly, we prove the rest of the parts. □

Theorem 3.1 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$, where $dim\mathbb{M}_i = m_i$ and $dim\mathbb{N}_i = n_i$, $i \in \{1, 2, 3\}$. Then*
(i)π has totally geodesic fibers if and only if π_i has totally geodesic fibers and f_i is constant, $i \in \{1, 2, 3\}$.
(ii)The fundamental metric tensor T satisfies the following inequality

$$\|T\|^2 \geq (m_2 - n_2) f_1^2 \|\mathcal{H}grad^1 f_1\|^2 + (m_3 - n_3) f_2^2 \|\mathcal{H}gradf_2\|^2$$

with the equality hold if and only if π_i has totally geodesic fibers.

Proof: (i) Let $e_k \in \Gamma(\mathcal{V}_1)$ and $k = 1, 2, 3, ..., m_1 - n_1$, $e_c \in \Gamma(\mathcal{V}_2)$ and $c = m_1 - n_1 + 1, ..., m_2 - n_2 + m_1 - n_1$, and $d_1 \in \Gamma(\mathcal{V}_3)$ and $d_1 = m_2 - n_2 + m_1 - n_1 + 1, ..., m_3 - n_3 + m_2 - n_2 + m_1 - n_1$ be orthonormal vectors of vertical spaces of submersion π.
By using Lemma (3.1) we have

$$\begin{aligned}
\|T\|^2 &= \sum_{k,k_1=1}^{m_1-n_1} g_{\mathbb{M}}\left(T\left(e_k, e_{k_1}\right), T\left(e_k, e_{k_1}\right)\right) + \sum_{c,d=m_1-n_1+1}^{m_2-n_2+m_1-n_1} g_{\mathbb{M}}\left(T\left(e_c, e_d\right), T\left(e_c, e_d\right)\right) \\
&+ \sum_{d_1,d_2=m_2-n_2+m_1-n_1+1}^{m_3-n_3+m_2-n_2+m_1-n_1} g_{\mathbb{M}}\left(T\left(e_{d_1}, e_{d_2}\right), T\left(e_{d_1}, e_{d_2}\right)\right) \\
&= \|T_1\|^2 + \|T_2\|^2 + \|T_3\|^2 + (m_2 - n_2) f_1^2 \|\mathcal{H}grad^1 f_1\|^2 + (m_3 - n_3) f_2^2 \|\mathcal{H}gradf_2\|^2.
\end{aligned}$$

(ii)It follows from the above relation. □

Theorem 3.2 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$.Then π has totally umbilical fibers if and only if π_1,π_2 and π_3 have totally geodesic fibers and $\vec{\mathbb{H}}^\pi = \mathcal{H}grad^1 f_1 = \mathcal{H}gradf_2$, where $\vec{\mathbb{H}}^\pi$ denotes the mean curvature of π.*

Proof: From Lemma (3.1) and the fact that π has totally umbilical fiber we have

$$\begin{aligned}
T(U_1, V_1) &= T_1(U_1, V_1) = g_{\mathbb{M}}(U_1, V_1)\vec{\mathbb{H}}^\pi \\
T(U_1, U_2) &= 0 = g_{\mathbb{M}}(U_1, U_2)\vec{\mathbb{H}}^\pi = 0\vec{\mathbb{H}}^\pi \\
T(U_2, V_2) &= T_2(U_2, V_2) - f_1 g_{\mathbb{M}}(U_2, V_2)\mathcal{H}grad^1 f_1 = g_{\mathbb{M}}(U_2, V_2)\vec{\mathbb{H}}^\pi \\
T(U_3, U_1) &= 0 = g_{\mathbb{M}}(U_3, U_1)\vec{\mathbb{H}}^\pi = 0\vec{\mathbb{H}}^\pi \\
T(U_2, U_3) &= 0 = g_{\mathbb{M}}(U_2, U_3)\vec{\mathbb{H}}^\pi = 0\vec{\mathbb{H}}^\pi \\
T(U_3, V_3) &= T_3(U_3, V_3) - f_2 g_{\mathbb{M}}(U_3, V_3)\mathcal{H}gradf_2 = g_{\mathbb{M}}(U_3, V_3)\vec{\mathbb{H}}^\pi,
\end{aligned}$$

for any $U_i, V_i \in \Gamma(\mathcal{V}_i)$, $i \in \{1,2,3\}$. Then the above expression gives the following relation
$\vec{\mathbb{H}}^{\pi} = -\mathcal{H}grad^1 f_1$, $\vec{\mathbb{H}}^{\pi} = -\mathcal{H}gradf_2$ and $T(U_2,V_2)=0$, $T(U_3,V_3)=0$.
Converse follow easily.

□

Theorem 3.3 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$.Then π has minimal fiber if and only if mean curvature of π_1, π_2, and π_3 are given by $\vec{\mathbb{H}}_3 = 0$, $\vec{\mathbb{H}}_2 = \frac{m_3-n_3}{m_2-n_2}(f_2\mathcal{H}grad^2 f_2)$, and $\vec{\mathbb{H}}_1 = \frac{m_3-n_3}{m_1-n_1}(f_2\mathcal{H}grad^1 f_2) + \frac{m_2-n_2}{m_1-n_1}(f_1\mathcal{H}grad^1 f_1)$.*

Proof: We suppose that π has minimal fiber for $\mathbb{M}$.Let $e_k \in \Gamma(\mathcal{V}_1)$ and $k = 1,2,3,...,m_1-n_1$, $e_c \in \Gamma(\mathcal{V}_2)$ and $c = m_1-n_1+1,...,m_2-n_2+m_1-n_1$, and $e_d \in \Gamma(\mathcal{V}_3)$ and $d = m_2-n_2+m_1-n_1+1,...,m_3-n_3+m_2-n_2+m_1-n_1$ be orthonormal vectors of vertical spaces of submersion π.Then using equation (2.8) and Lemma (3.1) we have

$$\vec{\mathbb{H}} = \frac{1}{m_3-n_3+m_2-n_2+m_1-n_1}\begin{pmatrix} \sum_{k=1}^{m_1-n_1} T(e_k,e_k) + \sum_{c=m_1-n_1+1}^{m_2-n_2+m_1-n_1} T(e_c,e_c) \\ + \sum_{d=m_2-n_2+m_1-n_1+1}^{m_3-n_3+m_2-n_2+m_1-n_1} T(e_d,e_d) \end{pmatrix}$$

$$= \frac{1}{m_3-n_3+m_2-n_2+m_1-n_1}\begin{pmatrix} \sum_{k=1}^{m_1-n_1} T_1(e_k,e_k) \\ + \sum_{c=m_1-n_1+1}^{m_2-n_2+m_1-n_1} T_2(e_c,e_c) - f_1 g_{\mathbb{M}}(e_c,e_c)\mathcal{H}grad^1 f_1 \\ + \sum_{d=m_2-n_2+m_1-n_1+1}^{m_3-n_3+m_2-n_2+m_1-n_1} T_3(e_d,e_d) - f_2 g_{\mathbb{M}}(e_d,e_d)\mathcal{H}gradf_2 \end{pmatrix}$$

$$= \frac{1}{m_3-n_3+m_2-n_2+m_1-n_1}\begin{pmatrix} (m_1-n_1)\vec{\mathbb{H}}_1 + (m_2-n_2)\left(\vec{\mathbb{H}}_2 - f_1\mathcal{H}grad^1 f_1\right) \\ + (m_3-n_3)\left(\vec{\mathbb{H}}_3 - f_2\mathcal{H}gradf_2\right) \end{pmatrix}.$$

Since $\vec{\mathbb{H}}_i \in \Gamma(\mathcal{H}_i)$, where $i \in \{1,2,3\}$ and $f_1\mathcal{H}grad^1 f_1 \in \Gamma(\mathcal{H}_1)$.
We know, $f_2\mathcal{H}gradf_2 \in \Gamma(\mathcal{H}_1 \times \mathcal{H}_2)$.
So, we can write $f_2\mathcal{H}gradf_2 = f_2\mathcal{H}grad^1 f_2 + f_2\mathcal{H}grad^2 f_2$, where $f_2\mathcal{H}grad^1 f_2 \in \Gamma(\mathcal{H}_1)$ and $f_2\mathcal{H}grad^2 f_2 \in \Gamma(\mathcal{H}_2)$.
We conclude,
$\vec{\mathbb{H}}_3 = 0$, $\quad \vec{\mathbb{H}}_2 = \frac{m_3-n_3}{m_2-n_2}(f_2\mathcal{H}grad^2 f_2)$, and
$\vec{\mathbb{H}}_1 = \frac{m_3-n_3}{m_1-n_1}(f_2\mathcal{H}grad^1 f_2) + \frac{m_2-n_2}{m_1-n_1}(f_1\mathcal{H}gard^1 f_1)$.
Conversely, it follows quickly from the above relation.

□

4. Conformal Vector Fields

Definition 4.1 A vector field ξ is called Conformal vector field on Riemannian manifold $(\mathbb{M}, g_{\mathbb{M}})$, if

$$\mathcal{L}_{\xi} g_{\mathbb{M}} = \rho g_{\mathbb{M}}, \tag{4.1}$$

where $\mathcal{L}_{\xi}$ is the Lie derivative in the direction of ξ.

In equation (4.1), if $\rho = 0$ then ξ is called killing vector i.e., for any $X, Y \in \mathfrak{X}(\mathbb{M})$ we have

$$g_{\mathbb{M}}(\nabla_X \xi, Y) + g_{\mathbb{M}}(X, \nabla_Y \xi) = 0. \tag{4.2}$$

By the symmetry in equation (4.2), ξ is killing vector field if

$$g_{\mathbb{M}}(\nabla_X \xi, X) = 0. \tag{4.3}$$

Theorem 4.1 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$. A vertical vector field ξ is killing vector if,*

1. ξ_i is killing vector on the horizontal space of $\mathbb{M}_i$, $i \in \{1,2,3\}$.
2. ξ_i is killing vector on the vertical space of $\mathbb{M}_i$, $i \in \{1,2,3\}$.
3. $\xi_1(f_1) = 0$.
4. $(\xi_1 + \xi_2)(f_2) = 0$.

Proof: If ξ is killing vector, then using equation (4.3) we have

$$g_{\mathbb{M}}(\nabla_{\bar{X}}\xi, \bar{X}) = 0,$$

where $\xi \in \Gamma(\mathcal{V})$ and $\bar{X} \in \mathfrak{X}(\mathbb{M})$.
Now

$$\begin{aligned} g_{\mathbb{M}}(\nabla_{\bar{X}}\xi, \bar{X}) &= g_{\mathbb{M}}(\nabla_{X_1+V_1}\xi_1 + \nabla_{X_1+V_1}\xi_2 + \nabla_{X_1+V_1}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(\nabla_{X_2+V_2}\xi_1 + \nabla_{X_2+V_2}\xi_2 + \nabla_{X_2+V_2}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(\nabla_{X_3+V_3}\xi_1 + \nabla_{X_3+V_3}\xi_2 + \nabla_{X_3+V_3}\xi_3, \bar{X}) \end{aligned}$$

Using Lemma (2.1) in above expression we get

$$\begin{aligned} g_{\mathbb{M}}(\nabla_{\bar{X}}\xi, \bar{X}) &= g_{\mathbb{M}}(A(X_1,\xi_1) + \mathcal{V}\nabla_{X_1}\xi_1 + A(X_1,\xi_2) + \mathcal{V}\nabla_{X_1}\xi_2 + A(X_1,\xi_3) + \mathcal{V}\nabla_{X_1}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(T(V_1,\xi_1) + \hat{\nabla}_{V_1}\xi_1 + T(V_1,\xi_2) + \hat{\nabla}_{V_1}\xi_2 + T(V_1,\xi_3) + \hat{\nabla}_{V_1}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(A(X_2,\xi_1) + \mathcal{V}\nabla_{X_2}\xi_1 + A(X_2,\xi_2) + \mathcal{V}\nabla_{X_2}\xi_2 + A(X_2,\xi_3) + \mathcal{V}\nabla_{X_2}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(T(V_2,\xi_1) + \hat{\nabla}_{V_2}\xi_1 + T(V_2,\xi_2) + \hat{\nabla}_{V_2}\xi_2 + T(V_2,\xi_3) + \hat{\nabla}_{V_2}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(A(X_3,\xi_1) + \mathcal{V}\nabla_{X_3}\xi_1 + A(X_3,\xi_2) + \mathcal{V}\nabla_{X_3}\xi_2 + A(X_3,\xi_3) + \mathcal{V}\nabla_{X_3}\xi_3, \bar{X}) \\ &+ g_{\mathbb{M}}(T(V_3,\xi_1) + \hat{\nabla}_{V_3}\xi_1 + T(V_3,\xi_2) + \hat{\nabla}_{V_3}\xi_2 + T(V_3,\xi_3) + \hat{\nabla}_{V_3}\xi_3, \bar{X}). \end{aligned} \tag{4.4}$$

Using Lemma (3.1), (3.2), (3.3) and definition of sequential warped product in equation (4.4) we obtain

$$\begin{aligned} g_{\mathbb{M}}(\nabla_{\bar{X}}\xi, \bar{X}) &= g_{\mathbb{M}_1}(\nabla^1_{X_1}\xi_1, X_1) + g_{\mathbb{M}_1}(\nabla^1_{V_1}\xi_1, V_1) + f_1^2 g_{\mathbb{M}_2}(\nabla^2_{X_2}\xi_2, X_2) + f_1^2 g_{\mathbb{M}_2}(\nabla^2_{V_2}\xi_2, V_2) \\ &+ f_2^2 g_{\mathbb{M}_3}(\nabla^3_{X_3}\xi_3, X_3) + f_2^2 g_{\mathbb{M}_3}(\nabla^3_{V_3}\xi_3, V_3) + f_1\xi_1(f_1) g_{\mathbb{M}_2}(X_2, X_2) \\ &+ f_1\xi_1(f_1) g_{\mathbb{M}_2}(V_2, V_2) + f_2(\xi_1+\xi_2)(f_2) g_{\mathbb{M}_3}(X_3, X_3) \\ &+ f_2(\xi_1+\xi_2)(f_2) g_{\mathbb{M}_3}(V_3, V_3), \end{aligned} \tag{4.5}$$

where $X_i \in \Gamma(\mathcal{H}_i)$ and $V_i \in \Gamma(\mathcal{V}_i)$; $i \in \{1,2,3\}$.
From equation (4.5) we get the required results. □

Proposition 4.1 *If ξ be a vertical vector field on $\mathbb{M}$, then ξ satisfies*

$$\begin{aligned} \mathcal{L}_\xi g_{\mathbb{M}}(\bar{X}, \bar{Y}) &= \mathcal{L}^1_\xi g_{\mathbb{M}_1}(X_1, Y_1) + \mathcal{L}^1_\xi g_{\mathbb{M}_1}(U_1, V_1) + f_1^2 \mathcal{L}^2_\xi g_{\mathbb{M}_2}(X_2, Y_2) + f_1^2 \mathcal{L}^2_\xi g_{\mathbb{M}_2}(U_2, V_2) \\ &+ f_2^2 \mathcal{L}^3_\xi g_{\mathbb{M}_3}(X_3, Y_3) + f_2^2 \mathcal{L}^3_\xi g_{\mathbb{M}_3}(U_3, V_3) + 2f_1\xi_1(f_1) g_{\mathbb{M}_2}(X_2, Y_2) \\ &+ 2f_1\xi_1(f_1) g_{\mathbb{M}_2}(U_2, V_2) + 2f_2(\xi_1+\xi_2)(f_2) g_{\mathbb{M}_3}(X_3, Y_3) \\ &+ 2f_2(\xi_1+\xi_2)(f_2) g_{\mathbb{M}_3}(U_3, V_3), \end{aligned}$$

for $\bar{X}, \bar{Y} \in \mathfrak{X}(\mathbb{M})$, $X_i, Y_i \in \Gamma(\mathcal{H}_i)$ and $U_i, V_i \in \Gamma(\mathcal{V}_i)$; $i \in \{1,2,3\}$.

We will now study the equations of geodesic curves for Riemannian sequential warped product submersion. A curve $\alpha(t)$ can be written as $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, where $\alpha_1(t), \alpha_2(t), \alpha_3(t)$ are the projection of α on $\mathbb{M}_1$,$\mathbb{M}_2$, and $\mathbb{M}_3$ respectively.

Theorem 4.2 *Let $\mathbb{M} = (\mathbb{M}_1 \times_{f_1} \mathbb{M}_2) \times_{f_2} \mathbb{M}_3$ and $\mathbb{N} = (\mathbb{N}_1 \times_{\rho_1} \mathbb{N}_2) \times_{\rho_2} \mathbb{N}_3$ are sequential warped product manifolds and $\pi : \mathbb{M} \to \mathbb{N}$ is Riemannian sequential warped product submersion between the manifolds $\mathbb{M}$ and $\mathbb{N}$. Let $\alpha(t)$ be a smooth curve on $\mathbb{M}$. Then α is geodesic on $\mathbb{M}$ if and only if*

$$\begin{aligned} (i)\nabla^1_{\dot{\alpha}_1}\dot{\alpha}_1 &= f_1\|\dot{\alpha}_2\|_2^2 gard^1 f_1 + f_2\|\dot{\alpha}_3\|_3^2 (gradf_2)^T \ \ on \ \ \mathbb{M}_1. \\ (ii)\nabla^2_{\dot{\alpha}_2}\dot{\alpha}_2 &= f_2\|\dot{\alpha}_3\|_3^2 (gradf_2)^\perp - 2\dot{\alpha}_1(lnf_1)\dot{\alpha}_1 \ \ on \ \ \mathbb{M}_2. \\ (iii)\nabla^3_{\dot{\alpha}_3}\dot{\alpha}_3 &= -2\dot{\alpha}_2(lnf_2)\dot{\alpha}_3 - 2\dot{\alpha}_1(lnf_2)\dot{\alpha}_3 \ \ on \ \ \mathbb{M}_3. \end{aligned}$$

Proof: Let $\alpha_i(t)$ is an integral curve of $\dot{\alpha}_i$ on $\mathbb{M}_i$ and so $\alpha(t)$ is an integral curve of $\dot{\alpha} = \dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3$. Thus

$$\begin{aligned} \nabla_{\dot{\alpha}}\dot{\alpha} &= \nabla_{\dot{\alpha}_1}\dot{\alpha_1} + \nabla_{\dot{\alpha}_1}\dot{\alpha_2} + \nabla_{\dot{\alpha}_1}\dot{\alpha_3} + \nabla_{\dot{\alpha}_2}\dot{\alpha_1} + \nabla_{\dot{\alpha}_2}\dot{\alpha_2} + \nabla_{\dot{\alpha}_2}\dot{\alpha_3} \\ &+ \nabla_{\dot{\alpha}_3}\dot{\alpha_1} + \nabla_{\dot{\alpha}_3}\dot{\alpha_2} + \nabla_{\dot{\alpha}_3}\dot{\alpha_3}. \end{aligned} \tag{4.6}$$

We know $\pi : \mathbb{M} \to \mathbb{N}$ be a Riemannian sequential warped product submersion, let $\alpha_i : \mathbb{I} \to \mathbb{M}_i$ be a curve and $\gamma_i = \pi \circ \alpha_i$ be its projection on $\mathbb{N}_i$. For any $t \in \mathbb{I}$ we have

$$\dot{\alpha_i}(t) = E_i(t) + W_i(t), \tag{4.7}$$

where $E_i(t) \in \mathcal{H}_{i_{\alpha_i(t)}}$ and $W_i(t) \in \mathcal{V}_{i_{\alpha_i(t)}}$ for $i \in \{1, 2, 3\}$.
Using equation (4.7) and lemma (2.1) in equation (4.6) we get

$$\begin{aligned} \nabla_{\dot{\alpha}}\dot{\alpha} &= \mathcal{V}(\nabla_{\dot{\alpha}_1}W_1 + T(W_1,E_1) + A(E_1,E_1)) + \mathcal{H}(\nabla_{E_1}E_1 + 2A(E_1,W_1) + T(W_1,W_1)) \\ &+ \mathcal{V}(\nabla_{\dot{\alpha}_2}W_2 + T(W_2,E_2) + A(E_2,E_2)) + \mathcal{H}(\nabla_{E_2}E_2 + 2A(E_2,W_2) + T(W_2,W_2)) \\ &+ \mathcal{V}(\nabla_{\dot{\alpha}_3}W_3 + T(W_3,E_3) + A(E_3,E_3)) + \mathcal{H}(\nabla_{E_3}E_3 + 2A(E_3,W_3) + T(W_3,W_3)) \\ &+ \mathcal{V}(\nabla_{\dot{\alpha}_1}W_2 + \nabla_{\dot{\alpha}_2}W_1 + T(W_1,E_2) + T(W_2,E_1)) \\ &+ \mathcal{H}(\nabla_{E_2}E_1 + \nabla_{E_1}E_2 + 2A(E_2,W_1) + 2A(E_1,W_2) + 2T(W_1,W_2)) \\ &+ \mathcal{V}(\nabla_{\dot{\alpha}_1}W_3 + \nabla_{\dot{\alpha}_3}W_1 + T(W_1,E_3) + T(W_3,E_1)) \\ &+ \mathcal{H}(\nabla_{E_3}E_1 + \nabla_{E_1}E_3 + 2A(E_3,W_1) + 2A(E_1,W_3) + 2T(W_1,W_3)) \\ &+ \mathcal{V}(\nabla_{\dot{\alpha}_2}W_3 + \nabla_{\dot{\alpha}_3}W_2 + T(W_2,E_3) + T(W_3,E_2)) \\ &+ \mathcal{H}(\nabla_{E_3}E_2 + \nabla_{E_2}E_3 + 2A(E_3,W_2) + 2A(E_2,W_3) + 2T(W_2,W_3)). \end{aligned}$$

Using Lemma (3.1), (3.2) and (3.3) in above expression we get

$$\begin{aligned} \nabla_{\dot{\alpha}}\dot{\alpha} &= \mathcal{V}\left(\nabla_{\dot{\alpha}_1}W_1 + T_1(W_1,E_1) + A_1(E_1,E_1)\right) + \mathcal{H}\left(\nabla_{E_1}E_1 + 2A(E_1,W_1) + T_1(W_1,W_1)\right) \\ &= \mathcal{V}\left(\nabla_{\dot{\alpha}_2}W_2 + T_2(W_2,E_2) - f_1 g_{\mathbb{M}_2}(W_2,E_2) grad^1 f_1 + A_2(E_2,E_2)\right) \\ &+ \mathcal{H}\left(\nabla_{E_2}E_2 + 2A(E_2,W_2) + T(W_2,W_2)\right) \\ &+ \mathcal{V}\left(\nabla_{\dot{\alpha}_3}W_3 + T_3(W_3,E_3) - f_2 g_{\mathbb{M}_3}(W_3,E_3) grad f_2 + A_3(E_3,E_3)\right) \\ &+ \mathcal{H}\left(\nabla_{E_3}E_3 + 2A(E_3,W_3) + T(W_3,W_3)\right) \\ &+ \mathcal{V}\left(\nabla_{\dot{\alpha}_1}W_2 + \nabla_{\dot{\alpha}_2}W_1 + E_1(\ln f_1)W_2\right) + \mathcal{H}\left(\nabla_{E_2}E_1 + \nabla_{E_1}E_2 + 2W_1(\ln f_1)E_2\right) \\ &+ \mathcal{V}\left(\nabla_{\dot{\alpha}_1}W_3 + \nabla_{\dot{\alpha}_3}W_1 + E_1(\ln f_2)W_3\right) + \mathcal{H}\left(\nabla_{E_3}E_1 + \nabla_{E_1}E_3 + 2W_1(\ln f_2)E_3\right) \\ &+ \mathcal{V}\left(\nabla_{\dot{\alpha}_2}W_3 + \nabla_{\dot{\alpha}_3}W_2 + E_2(\ln f_2)W_3\right) + \mathcal{H}\left(\nabla_{E_3}E_2 + \nabla_{E_2}E_3 + 2W_2(\ln f_2)E_3\right). \end{aligned}$$

From the above expression we get the required result. □

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