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Statistical manifolds with the same statistic and Riemann curvature tensor fields

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ABSTRACT: This paper provides a necessary and sufficient condition for an almost contact metric statistical manifold to have the same statistic and Riemann curvature tensor fields, and a condition to determine the uniqueness of such a statistical structures. Additionally, we prove that a contact metric statistical manifold in which the integral curves of the Reeb vector field are geodesics, is a trivial statistical manifold if and only if it has the same statistic and Riemann curvature tensor fields.

Key Words: Almost contact metric statistical manifold, statistical curvature tensor field.

Contents

Introduction

Statistical manifolds are equipped with a torsion-free affine connection ∇ and its dual connection ∇^* in addition to the Levi-Civita connection, introducing a new aspect of geometry. When ∇ coincides with ∇^*, the statistical manifold reduces to a usual Riemannian manifold. A statistical structure is defined as a generalization of a Riemannian metric and its Levi-Civita connection, that means a Riemannian structure is a trivial statistical structure. The concept of statistical structure was initially introduced by S. Amari in 1985 ([3]) while studying statistical distributions. Since then, the geometry of statistical manifolds has been developed in close relation to various families of differential geometry, including affine differential geometry, Hessian geometry, and information geometry. In the past two decades, several families of almost Hermitian and almost contact metric manifolds, such as Kähler manifolds, Sasakian manifolds and Kenmotsu manifolds have been generalized to corresponding statistical manifolds by endowing them with suitable statistical structures ([6], [8], [9]). This generalization has expanded to include all families of almost Hermitian and almost contact metric manifolds ([1], [2]).

Curvature tensor fields play a fundamental role in Riemannian geometry as they characterize the behavior of Riemannian manifolds. The curvature tensor of type (0, 4) constructed by statistical connections does not have enough symmetries, and the problem can be solved by taking the average of the curvature tensor and the curvature tensor for the dual connection, and then it can be comparable with Riemannian curvature. The objective of this article is to characterize statistical structures on almost contact metric statistical manifolds that exhibit the same statistic curvature as the Riemann curvature. Additionally a condition is provided to determine the uniqueness of such a statistical structures.

Section 1 presents the definitions and some properties of almost contact metric statistical manifolds. Section 2 focuses on the definition and some properties of statistic curvature tensor field in almost contact metric statistical manifolds. In section 3, a necessary and sufficient condition is provided for the equality of statistic and Riemann curvature tensor fields in almost contact metric statistical manifolds, along with a condition to determine the uniqueness of such a statistical structure, and finally it is proved that contact metric statistical manifolds with the Reeb vector field as geodesic flow with respect to their

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statistic connection, are trivial if and only if they have the same statistic and Riemann curvature tensor fields.

1. Preliminaries

This section provides a brief overview of statistical manifolds and almost contact metric statistical manifolds, which will be discussed in the next two sections.

Let (M, g) be a Riemannian manifold and $\chi(M)$ denotes the Lie algebra of all smooth vector fields on M. A statistical structure on M is defined as (∇, g), where ∇ is an affine and torsion-free connection, and ∇g is symmetric, meaning $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$, for any X, Y, Z $\in \chi(M)$ ([4]).

The conjugate connection or the dual connection ∇^* of ∇ with respect to g is introduced by the formula,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_Y Z).$$

It can be observed that ∇ coincides with the Levi-Civita connection ∇° if and only if $\nabla = \nabla^*$. In this case, (M, ∇, g) is referred to as a trivial statistical manifold. (∇^*, g) is also a statistical structure on M, and the Levi-Civita connection ∇° is the "mean" of ∇ and ∇^*, i.e., $\nabla^\circ = \frac{1}{2}(\nabla + \nabla^*)$.

In a statistical manifold (M, ∇, ∇^*, g) the (1, 2)-tensor field K is defined as follows:

$$K_X Y := \nabla_X Y - \nabla^\circ_X Y = \nabla^\circ_X Y - \nabla^*_X Y. \tag{1.1}$$

We can also denote $K_X Y$ by $K(X, Y)$.

K is referred to as the difference tensor field of the statistical structure (∇, g), and satisfies the following symmetries:

$$K(X, Y) = K(Y, X), \quad g(K(X, Y), Z) = g(Y, K(X, Z)),$$

for any X, Y, Z $\in \chi(M)$.

It is worth noting that if a Riemannian manifold (M, g) is equipped with a given tensor field K that satisfies the above symmetries, then the pair $(\nabla := \nabla^\circ + K, g)$ forms a statistical structure on M. Thus, the notion of a statistical manifold can be introduced as a Riemannian manifold (M, g) together with a (1, 2)-tensor field K with the mentioned symmetries.

An almost contact metric manifold is an odd-dimensional smooth manifold M^{2n+1} along with a quadruple (ϕ, ξ, η, g) structure consisting of a $(1, 1)$-form ϕ, a vector field ξ, a 1-form η and a metric g satisfying the following properties ([5]),

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta o \phi = 0 \qquad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \forall X, Y \in \chi(M). \end{aligned}$$

Let (∇, g) be a statistical structure, and (ϕ, ξ, η, g) be an almost contact metric structure on M. Then, using equation (1.1), we have,

$$\nabla_X \phi Y - \phi \nabla^*_X Y = (\nabla^\circ_X \phi)Y + K_X \phi Y + \phi K_X Y,$$

the quadruple (∇, g, ϕ, ξ) is called an almost contact metric statistical structure on M if the following equality holds for any $X, Y \in \chi(M)$,

$$K_X \phi Y + \phi K_X Y = 0, \tag{1.2}$$

that is equivalent to

$$\nabla_X \phi Y - \phi \nabla^*_X Y = (\nabla^\circ_X \phi)Y.$$

Let $(M, \nabla, g, \phi, \xi)$ be an almost contact metric statistical manifold, then $(M, \nabla^*, g, \phi, \xi)$ is also an almost contact metric statistical manifold.

Lemma 1.1 *In an almost contact metric statistical manifold* $(M, \nabla, g, \phi, \xi)$*, we have the following two equalities,*

$$(i) K(X,\xi) = \lambda\eta(X)\xi,$$
$$(ii) \eta(K(X,Y)) = \lambda\eta(X)\eta(Y),$$

in which $\lambda = \eta(K(\xi,\xi))$.

Proof: Substituting Y by ξ in (1.2), we have $\phi K_X\xi = 0$, and especially $\phi K_\xi\xi = 0$, that results $K(\xi,\xi) = \lambda\xi$, in which $\lambda = \eta(K(\xi,\xi))$.
Considering once more $\phi K_X\xi = 0$, we have $\phi^2 K_X\xi = 0$, and thus

$$\begin{aligned} K(X,\xi) &= \eta(K(X,\xi))\xi = g(K(X,\xi),\xi)\xi \\ &= g(K(\xi,\xi),X)\xi = g(\lambda\xi,X)\xi = \lambda\eta(X)\xi, \end{aligned}$$

and this proves (i).

The symmetric identity of K with respect to g and the above result, gives

$$\eta(K(X,Y)) = g(K(X,\xi),Y) = \lambda\eta(X)\eta(Y), \tag{1.3}$$

for $X, Y \in \chi(M)$, and then we have (ii). □

The following Lemma is crucial in proving the main theorem of this paper.

Lemma 1.2 *If* $(M, \nabla, g, \phi, \xi)$ *is an almost contact metric statistical manifold, then the following two identities holds for* $X, Y \in \chi(M)$*,*
(i) $[K_\xi, K_X] = 0$*,*
(ii) $[K_X, K_Y]\xi = 0$*.*

Proof: Using $K(X,\xi) = \lambda\eta(X)\xi$ from Lemma 1.1, we have

$$\begin{aligned} [K_\xi, K_X]Z &= K(\xi, K(X,Z)) - K(X, K(\xi,Z)) \\ &= \lambda\eta(K(X,Z))\xi - K(X, \lambda\eta(Z)\xi) \\ &= \lambda g(K(X,Z),\xi)\xi - \lambda\eta(Z)K(X,\xi) \\ &= \lambda g(K(X,\xi),Z)\xi - \lambda\eta(Z)\lambda\eta(X)\xi \\ &= \lambda^2\eta(X)\eta(Z)\xi - \lambda^2\eta(X)\eta(Z)\xi \\ &= 0, \end{aligned}$$

and this proves (i).
The second equality can be easily proved in the same way. □

Notation 1.1 *If* $(M, \nabla, g, \phi, \xi)$ *is an almost contact metric statistical manifold, then according to the above Lemmas, for* $X, Y \in ker\eta$ *we have* $K(X,Y) = K_XY \in ker\eta$*, and since* $K_X : ker\eta \to ker\eta$ *is symmetric relative to g, and skew-symmetric relative to the almost complex structure* $\phi : ker\eta \to ker\eta$*, thus there is an orthonormal basis of* $ker\eta$ *such that the matrix of* K_X *relative to this basis is diagonal with the eigenvalues of the form* $\lambda_1, -\lambda_1, ..., \lambda_n, -\lambda_n$*.*

2. Statistic Curvature Tensor Field

The nature of a manifold mostly depends on its curvature tensor. In this section, we provide the definition and some properties of the statistic curvature tensor field.
The curvature tensor field R for ∇ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

similarly, $R^*(X,Y)$ is defined by replacing ∇ with ∇^* in the above equation.
For a statistical manifold (M,∇,g), the following relations hold for $X,Y,Z,W \in \chi(M)$,

$$g(R(X,Y)Z,W) = -g(R(Y,X)Z,W),$$
$$g(R^*(X,Y)Z,W) = -g(R^*(Y,X)Z,W),$$
$$g(R(X,Y)Z,W) = -g(R^*(X,Y)W,Z).$$

As a result, R(X, Y, Z, W) = g(R(X, Y)Z, W) does not necessarily have the other symmetric properties of the Riemann curvature tensor field.

The statistic curvature tensor field of the statistical structure (∇,∇^*,g) is defined as follows ([7]),

$$S(X,Y)Z = \frac{1}{2}(R(X,Y)Z + R^*(X,Y)Z). \tag{2.1}$$

S satisfies the following symmetries:

$$g(S(X,Y)Z,W) = -g(S(Y,X)Z,W),$$
$$g(S(X,Y)Z,W) = -g(S(X,Y)W,Z),$$
$$S(X,Y)Z + S(Y,Z)X + S(Z,X)Y = 0,$$
$$g(S(X,Y)Z,W) = g(S(Z,W)X,Y).$$

These properties indicates that S is a generalized curvature 4-tensor. Using (1.1) and (1.3), we have the following Proposition,

Proposition 2.1 *([6]) Let $(M,\nabla=\nabla^\circ+K,g)$ be a statistical manifold, and S be the statistical curvature tensor field. Denote the curvature tensor field of ∇° by R°. Then the following formula holds:*

$$S(X,Y)Z = R^\circ(X,Y)Z + [K_X,K_Y]Z, \tag{2.2}$$

in which $[K_X,K_Y]Z := K(X,K(Y,Z)) - K(Y,K(X,Z))$.

3. Statistic and Riemann Curvature Tensor Fields in Almost Contact Metric Statistical Manifolds

In this section, we compare the statistic and Riemann curvature tensor fields of the almost contact metric statistical manifolds and examine the uniqueness of such statistical structures.

Proposition 3.1 *In an almost contact metric statistical manifold (M,∇,g,ϕ,ξ) the following equalities hold,*

$$S(X,Y)\xi = R^\circ(X,Y)\xi, \quad Ric(\xi) = Ric^\circ(\xi).$$

Proof: Using relation (2.2) and Lemma 1.2, the proof is evident. □

The main problems addressed in this paper is to determine the conditions under which the Riemann and statistic curvature tensor fields coincide. The following theorem provides a necessary and sufficient condition for this to happen in almost contact metric statistical manifolds.

Theorem 3.1 *Let (M,∇,g,ϕ,ξ) be an almost contact metric statistical manifold, then $S = R^\circ$, if and only if $K(X,Y) = \lambda\eta(X)\eta(Y)\xi$.*

Proof: In the statistical manifold (M,∇,g), using (2.2) we have,

$$S(X,Y)Z = R^\circ(X,Y)Z + [K_X,K_Y]Z,$$

thus, the necessary and sufficient condition for $S(X,Y) = R^\circ(X,Y)$ is $[K_X, K_Y] = 0$. We prove that, in almost contact metric statistical manifold $(M, \nabla, g, \phi, \xi)$, this condition is equivalent to $K(X,Y) = \lambda\eta(X)\eta(Y)\xi$.

If $K(X,Y) = \lambda\eta(X)\eta(Y)\xi$, then by using the symmetric identity of K and formula (1.3), it can be easily proved that $[K_X, K_Y] = 0$, and thus $S = R^\circ$.

Conversely, if $S = R^\circ$, which means $[K_X, K_Y]Z = 0$ for each $X, Y, Z \in \chi(M)$, then we have

$$K(X, K(Y,Z)) = K(Y, K(X,Z)). \tag{3.1}$$

Substituting Y and Z by ϕY and ϕZ respectively, and using (1.2), we have

$$\begin{aligned} &K(X, K(\phi Y, \phi Z)) = K(\phi Y, K(X, \phi Z)),\\ \Rightarrow\ &K(X, \phi^2 K(Y,Z)) = -K(\phi Y, \phi K(X,Z))\\ \Rightarrow\ &\phi^2 K(X, K(Y,Z) = -\phi^2 K(Y, K(X,Z)). \end{aligned}$$

Comparing the last equality with (3.1), we get $\phi^2 K(X, K(Y,Z) = 0$ and thus for $X, Y, Z \in ker\eta$ we have $K(X, K(Y,Z)) = 0$, in particular $K_X^2 Z = 0$. By the symmetrization of K_X and Notation 1.1, it results that $K_X = 0$ for $X \in ker\eta$ Thus the only non-zero value of K can be $K(\xi,\xi)$, and this proves the assertion. □

Now a more promising approach is to seek those statistical structures (∇, g) that have the same curvature tensor field as the Riemann curvature tensor field in an almost contact metric statistical manifold $(M, \nabla, g, \phi, \xi)$. We show that up to an appropriate condition, such a statistical structure is unique.

Theorem 3.2 *Let $(M, \nabla^1, g, \phi, \xi)$ and $(M, \nabla^2, g, \phi, \xi)$ be almost contact metric statistical manifolds that have the same statistic and Riemann curvature tensor fields (that means $S^1 = R^\circ = S^2$, in which S^1 and S^2 are statistic curvature of ∇^1 and ∇^2 respectively). Then $\nabla^1 = \nabla^2$, if and only if $\nabla^1_\xi \xi = \nabla^2_\xi \xi$.*

Proof: If $\nabla^1 = \nabla^2$ it is evident that $\nabla^1_\xi \xi = \nabla^2_\xi \xi$.

Now, let $\nabla^1_\xi \xi = \nabla^2_\xi \xi$. By Theorem 3.1, $S^i = R^\circ$ if and only if $K^i = \lambda^i \eta \otimes \eta \otimes \xi$, in which K^i is the difference tensor field of the statistical structure (∇^i, g), and $\lambda^i = g(K^i(\xi,\xi),\xi)$, $i = 1, 2$. Then we have,

$$\begin{aligned} K^1(X,Y) - K^2(X,Y) &= (\lambda^1 - \lambda^2)\eta(X)\eta(Y)\xi.\\ &= g(\nabla^1_\xi \xi - \nabla^2_\xi \xi, \xi)\eta(X)\eta(Y)\xi\\ &= 0. \end{aligned}$$

Since by definition $K(X,Y) = \nabla_X Y - \nabla^\circ_X Y$, the above equality proves that $\nabla^1_X Y = \nabla^2_X Y$ for $X, Y \in \chi(M)$, and this proves the assertion.

□

We know that in a contact metric structure, the integral curves of the Reeb vector field are geodesics, that means $\nabla^\circ_\xi \xi = 0$. The following Corollary, states that a contact metric statistical manifold in which the integral curves of the Reeb vector field are geodesics with respect to its statistical connection, is trivial, if and only if it has the same statistic and Riemann curvature tensor fields.

Corollary 3.1 *Let $(M, \nabla, g, \phi, \xi)$ be a contact metric statistical manifold with $\nabla_\xi \xi = 0$. Then $\nabla = \nabla^\circ$ if and only if $S = R^\circ$.*

In other words, there is not any non trivial contact metric statistical structure (∇, g, ϕ, ξ), with the same statistic and Riemann curvature tensor fields and $\nabla_\xi \xi = 0$.

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