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Stabilization results for delayed KdV equation with internal saturation

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ABSTRACT: In this paper, we consider the nonlinear Korteweg-de Vries equation with time-varying delay on the boundary feedback, in the presence of a saturated source term. Under specific assumptions concerning the time-varying delay, we have demonstrated that the studied system is well-posed. Furthermore, we have proven that this system is exponentially stable. Specifically, by introducing an appropriate energy and employing the Lyapunov approach, we ensure that the unique solution of the Korteweg-de Vries equation is exponentially stable. Finally, we present some conclusions.

Key Words: Nonlinear KdV equation, Stability, Saturated source term, Boundary time-varying delay.

Contents

1. Introduction

In real life, all dynamic systems are subject to constraints on input, state and output variables, originating primarily from physical limitations and restrictions within the controller actuators. Dynamical systems with hard limit constraints on the amplitude and changing rate of control inputs are the most common cases, where the hard limit constraint is modeled by a saturation nonlinearity. In many cases, the saturation nonlinearity dominates the performance of the system. It is interesting to note that the design of control loops to effectively manage this non-linearity often requires much more effort than that devoted to the design of the linear components of the system. The study of dynamic systems with saturation nonlinearities is of both practical and theoretical importance. The problem of dynamical systems with saturation nonlinearities that has attracted the most interest is stability. The standard approach used to analyze stability with nonlinear controls is a two-stage process. First, the design is carried out without taking saturation into account. Subsequently, a nonlinear analysis is performed on the closed-loop system after saturation has been added. This approach yields local stabilization results.

For finite dimension, using an appropriate Lyapunov function and a sector condition for the saturation map, several techniques are used for stability analysis (see for instance [34,38,39]). In an infinite-dimensional setting, few works study this topic among them, we can refer to [29] which studies a wave equation with a distributed and saturated feedback law at the boundaries, while [8], where a coupled PDE/ODE system modeling a switched power converter with a transmission line is considered. In addition, there are also papers using nonlinear semigroup theory and focusing on abstract systems [18,32,33].

The Korteweg-de Vries equation, formulated as:

$$u_t + u_x + u_{xxx} + uu_x = 0$$

originated to model long waves in shallow water channels. It serves as a pivotal model for understanding nonlinear wave dynamics and soliton behaviors. Various techniques have been employed to investigate

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the well-posedness of the KdV equation, as detailed in literature such as [17,35,2,30,15]. Furthermore, in several works, stabilisation and controllability properties have been studied without any constraint on control, as reviewed in [5,7,31]. Several works examine this equation, notably with a focus on control theory aspects see for instance [7,5,31,28]. Specific research on boundary and distributed control for the KdV equation is particularly discussed in works such as [19,6], and [28,26].

In the literature, there are several articles that studies the stability of KdV equation with input saturation (see e.g. [21,36,20,3]. In particular, the study by [21] examines the global stabilization of the following nonlinear KdV equation:

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) & \\ +\mathbf{sat}(a(x)u(x,t)) = 0, & t \geq 0, ; x \in (0,L); \\ u(t,0) = u(L,t) = u_x(L,t) = 0, & t \geq 0; \\ u(0,x) = u_0(x), & \end{cases} \tag{1.1}$$

To investigate this problem, the authors worked with two different types of saturation, the saturation function given by

$$\mathbf{sat}(y) = \begin{cases} y, & \text{if } \|y\|_{L^2(0,L)} \leq 1 \\ \dfrac{y}{\|y\|_{L^2(0,L)}}, & \text{if } \|y\|_{L^2(0,L)} \geq 1 \end{cases} \tag{1.2}$$

and the following saturation function

$$\mathbf{sat}(y) = \begin{cases} -u_0, & \text{if } y \leq -u_0 \\ y, & \text{if } -u_0 \leq y \leq u_0 \\ u_0, & \text{if } y \geq u_0 \end{cases} \tag{1.3}$$

where u_0 represents positive constants. Thanks to the Banach fixed-point theorem, well-posedness is proven. To establish asymptotic stability, the authors have considered two cases. When the control acts on the entire saturated domain, the saturation under consideration is the function given by (1.2). In this case, they used a sector condition and Lyapunov theory for infinite-dimensional systems. The two saturation functions given by (1.2) and (1.3) were used when the control acts only on a portion of the saturated domain. For this second case, they prove the asymptotic stability of the closed-loop system using an argument by contradiction.

In this paper, we focus on the nonlinear Korteweg-de Vries equation with a time-varying delay in the boundary feedback, in the presence of a saturated source term:

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = f(x,t), \quad t > 0, \quad x \in (0,L); \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\rho(t)), \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(0,t-\rho(0)) = z_0(t-\rho(0)), \quad 0 < t < \rho(0), \end{cases} \tag{1.4}$$

Here, the variable u signifies the system's state, while $f(x,t) = -\mathbf{sat}(a(x)u(x,t))$ represents the source term. $L > 0$ denotes the length of the spatial domain, the delay denoted by ρ is a function of time t. Additionally, $a(\cdot) \in L^\infty((L,0))$ is a nonnegative function satisfying some conditions, and $\mathbf{sat}(.)$ is given by (1.2).

The presence of delays in certain domains raises important practical problems. The stability problems of partial differential equations with delay have been widely studied in control theory, where many problems have been successfully solved. Such systems can be used to model various processes such as turbojet engines, nuclear reactors and chemical processes. Studies have shown that even a small delay in feedback can make a system unstable, as discussed in several references [10,9]. However, the existence of a delay can also improve system performance, as observed in studies such as [1]. When the delay is time-varying, the analysis of the stability becomes much more complex. Several studies have examined the stability of partial differential equations involving a time-varying delay, see for instance [22,11,25,13].

In recent years, researchers have shown increasing interest in solving stability and robustness problems related to constant delay for the Korteweg-de Vries equation. Notable contributions have been made by researchers such as Baudouin et al. and Parada et al., as mentioned in [4,23], where they studied the Korteweg-de Vries equation with time-delay feedback, establishing the well-posedness and proving exponential stability through the use of the observability inequality. For more details on the KdV equation with time-delay, the readers can find more details in [37,12,16]. Concerning the Korteweg-de Vries equation with time-varying delay, there is a notably singular study conducted by Parada et al. [24]. This study examined the issue of time-varying delay both on the boundary or internal feedback. With specific assumptions concerning time-varying delay, they proved the well-posedness and the stability results is analyzed, using an appropriate Lyapunov functional.

In our best knowledge, there are no work dealing with the stability of a nonlinear Korteweg-de Vries equations with time-varying delay on the boundary feedback in the presence of a saturated source term. The objective of our work is to investigate the question of the local stability of system (1.4) with the saturated source term (1.2). Specifically, we analyze the well-posedness and stability result of this equation. We demonstrate exponential stability using an appropriate Lyapunov functional.

Now, let us present the outline of our work: In the next section, we formulate our problem and recall some properties of the solutions of the linear system associated with (1.4) without a saturated source term and with a saturated source term. We then show that the kdv equation (1.4) is well-posed. Section 3 is devoted to the exponential stabilization of (1.4). Finally, in Section 4, we present some additional conclusions and open questions.

2. Problem formulation and well-posedness

In this paper, we will consider the following KdV equation with a time-varying boundary delay

$$\begin{cases} u_t(x,t)+u_x(x,t)+u_{xxx}(x,t)+uu_x(x,t)=-\mathbf{sat}(a(x)u(x,t)), \quad t>0, \quad x\in(0,L);\\ u(0,t)=u(L,t)=0, \quad t>0;\\ u_x(L,t)=\alpha u_x(0,t)+\beta u_x(0,t-\rho(t)), \quad t>0;\\ u(x,0)=u_0(x), \quad x\in(0,L);\\ u_x(0,t-\rho(0))=z_0(t-\rho(0)), \quad 0<t<\rho(0), \end{cases} \tag{2.1}$$

where the coefficient function $a=a(x)\in L^\infty(0,L)$ satisfies the condition:

$$\begin{cases} a_1\geq a=a(x)\geq a_0>0 \quad \text{on} \quad \omega\subseteq(0,L),\\ \omega \quad \text{is a nonempty open subset of} \quad (0,L), \end{cases} \tag{2.2}$$

Furthermore, the delay function $\rho(\cdot)$ is assumed to belong to $W^{2,\infty}(0,T)$ for all $T>0$ and satisfies the following conditions:

$$0<\rho_0\leq\rho(t)\leq K, \text{ for all } t\geq 0, \tag{2.3}$$

and

$$\dot\rho(t)\leq d\leq 1, \text{ for all } t\geq 0, \tag{2.4}$$

where $d\geq 0$. Additionally, we suppose that the real constants α, β, and d must satisfy the inequality:

$$|\alpha|+|\beta|+d<1. \tag{2.5}$$

Under this assumption and according to [24], the matrix M_1 defined as:

$$M_1=\begin{pmatrix} \alpha^2-1+|\beta| & \alpha\beta\\ \alpha\beta & \beta^2+|\beta|(d-1) \end{pmatrix} \tag{2.6}$$

is definite negative.
Next, we consider the Hilbert space $H = L^2(0,L) \times L^2(0,1)$ equipped with the following usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_H = \int_0^L u u_1 dx + \int_0^1 z z_1 d\mu,$$

and its norm

$$\left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 = \int_0^L u^2 dx + \int_0^1 z^2 d\mu$$

and we introduce a new inner product on H. This inner product depends on time t and is defined as follows:

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_t = \int_0^L u u_1 dx + |\beta|\rho(t) \int_0^1 z z_1 d\mu,$$

with associated norm denoted by $\|\cdot\|_t$.
Using (2.3), the norm $\|\cdot\|_t$ and $\|\cdot\|_H$ are equivalent in H. Indeed, for all $t \geq 0$, and all $\begin{pmatrix} u \\ z \end{pmatrix} \in H$, we have

$$(1+|\beta|\rho_0) \left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 \leq \left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_t^2 \leq (1+|\beta|K) \left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 \tag{2.7}$$

Now, let us recall the mild solution of the following abstract system in an Hilbert space .

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + f(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.8}$$

where $\mathcal{A}$ is an infinitesimal generator of linear C_0-semigroup $(T(t))_{t\geq 0}$ defined on its domain $D(\mathcal{A}) \subseteq H$, where H is an Hilbert space and $f \in L^1_{loc}((0,T), Z)$.

Definition 2.1 *[27, Definition 2.3] Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Let $u_0 \in Z$ and $f \in L^1(0,T,Z)$. Then the function $u \in \mathcal{C}((0,T), Z)$ given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \quad 0 \leq t \leq T, \tag{2.9}$$

is the unique mild solution of the initial value problem (2.8) on $(0,T)$.

Before discussing the well-posedness of (2.1), we first review the well-posedness of the linearized system associated with (2.1), where no source term is present. This treatment of the linearized system has already been addressed by Parada et al. [24].

The linearized KdV equation around 0 of (2.1), is given by

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = 0, \quad t > 0, \quad x \in (0,L); \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0, t-\rho(t)), \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(0, t-\rho(0)) = z_0(t-\rho(0)), \quad 0 < t < \rho(0), \end{cases} \tag{2.10}$$

In this context, following a conventional approach as previously employed in various studies (see, for instance, [22,4]), Parada et al. introduced a new variable $z(\mu,t) = u_x(0, t-\rho(t)\mu)$, where $\mu \in (0,1)$ and $t > 0$, explicitly accounting for the time-delay term. Consequently, $z(\cdot,\cdot)$ satisfies the following system:

$$\begin{cases} \rho(t)z_t(\mu,t) + (1-\dot{\rho}(t)\mu)z_\mu(\mu,t) = 0, & t > 0, \mu \in (0,1); \\ z(0,t) = u_x(0,t), & t > 0; \\ z(\mu,0) = z_0(-\rho(0)\mu), & \mu \in (0,1). \end{cases} \tag{2.11}$$

The systems (2.10) and (2.11) are combined as follows:

$$\begin{cases} u_t(x,t)+u_x(x,t)+u_{xxx}(x,t)=0, & t>0, x\in(0,L);\\ \rho(t)z_t(\mu,t)+(1-\rho_t(t)\mu)z_\mu(\mu,t)=0, & t>0, \mu\in(0,1);\\ u(0,t)=u(L,t)=0, & t>0\\ u_x(L,t)=\alpha u_x(0,t)+\beta u_x(0,t-\rho(t)), & t>0;\\ u(x,0)=u_0(x), & x\in(0,L);\\ u_x(0,t-\rho(0))=z_0(t-\rho(0)), & 0<t<\rho(0).\\ z(0,t)=u_x(0,t), & t>0;\\ z(\mu,0)=z_0(-\rho(0)\mu), & \mu\in(0,1). \end{cases} \tag{2.12}$$

In this case, the state-space system (2.12) can be written for $Y=\begin{pmatrix} u\\ z\end{pmatrix}$ as

$$\begin{cases} Y_t=A(t)Y(t), & t>0,\\ Y(0)=\begin{pmatrix} u_0, & z_0(-\rho(0))\end{pmatrix}^T. \end{cases} \tag{2.13}$$

such that $A(t)$ is the operator defined on its domain:

$$\begin{aligned} D(A(t)) \quad = \quad & \{(u,z)\in H^3(0,L)\times H^1(0,1), u(0)=u(L)=0\\ & z(0)=u_x(0), u_x(L)=\alpha u_x(0,t)+\beta u_x(0,t-\rho(t))\} \end{aligned}$$

by

$$A(t)\begin{pmatrix} u\\ z\end{pmatrix}=\begin{pmatrix} -u_x-u_{xxx}\\ \frac{\dot\rho(t)\mu-1}{\rho(t)}z_\mu\end{pmatrix} \text{ for all } \begin{pmatrix} u\\ z\end{pmatrix}\in D(A(t)). \tag{2.14}$$

Parada et al. [24, Theorem 2.2] establish the well-posedness of (2.13) by using the following theorem

Theorem 2.1 *[14, Theorem 4.1] Assume that*

1. *$D(A(0))$, is a dense subset of H.*
2. *$D(A(t))=D(A(0)) \quad \forall t\geq 0$.*
3. *For all $t\in(0,T)$ $A(t)$ generates a strongly continuous semigroup on H and the family $A=\{A(t): t\in(0,T)\}$ is stable with stability constant C and m independent of t, i.e. the semigroup $(T_t(s))_{s\geq 0}$ generated by $A(t)$ satisfies*

$$\|T_t(s)Y\|_H\leq Ce^{ms}\|Y\|_H, \text{ for all } Y\in H, \text{ and } s\geq 0.$$

4. *$\partial_t A(t)$ belong to $L^\infty_*((0,T),B(D(A(0))))$, the space of equivalent of essentially classes bounded strongly measure functions from $(0,T)$ into the set $B(D(A(0)),H)$ of bounded operator from $D(A(0)$ to H.*

Then the system (2.13) has a unique solution $Y\in\mathcal{C}((0,T),D(A(0)))\cap\mathcal{C}^1((0,T),H)$

Parada et al. in [24] show that if (2.3)-(2.5) are satisfied, the operator $A(t)$ satisfies all the hypotheses of Theorem 2.1 and consequently (2.13) has a unique solution $u\in\mathcal{C}([0,+\infty[,H)$. Moreover if $Y_0\in D(A(0))$, then $Y\in\mathcal{C}([0,+\infty[,D(A(0)))\cap\mathcal{C}^1([0,+\infty[,H)$.

Before moving to the well-posedness of the linear system with saturated source term , it should be noted that the saturation function is Lipschitzian in $L^2(0,L)$, as shown in the following lemma:

Lemma 2.1 *[33, Theorem 5.1] For all $(u,v)\in L^2(0,L)$, we have*

$$\|\mathbf{sat}(u)-\mathbf{sat}(v)\|_{L^2(0,L)}\leq 3\|u-v\|_{L^2(0,L)}$$

In the next, we recall the properties of the nonlinearities, as given by the following lemmas:

Lemma 2.2 *[21, Proposition 3.4] Assume that a satisfies (2.2). If $u \in L^2(0,T;H^1(0,L))$, then $\mathbf{sat}(au) \in L^1(0,T;L^2(0,L))$, and the map*

$$\psi : u \in L^2(0,T;H^1(0,L)) \mapsto \mathbf{sat}(au) \in L^1(0,T;L^2(0,L))$$

is continuous.

Lemma 2.3 *[30, Proposition 4.1] Let $u \in L^2(0,T;H^1(0,L))$. Then, uu_x belong to $L^1(0,T;L^2(0,L))$and the map*

$$\phi : u \in L^2(0,T;H^1(0,L)) \mapsto uu_x \in L^1(0,T;L^2(0,L))$$

is continuous.

Besides, there exists positive constants K_1 , depending on L, such that for every $u, \tilde{u} \in L^2(0,T;H^2(0,L))$, one has

$$\int_0^T \|uu_x - u\tilde{u}_x\|_{L^2(0,L)}dt \leq K_1\|u-\tilde{u}\|_{L^2(0,T;H^1(0,L))}\times\left(\|u\|_{L^2(0,T;H^1(0,L))} + \|\tilde{u}\|_{L^2(0,T;H^1(0,L))}\right)$$

Before moving to the study of the existence and uniqueness of solutions for the nonlinear system (2.1), we reccal that the authors of [24] have previously demonstrated the existence and uniqueness of solution for the following linear system

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = f(x,t), \quad t > 0, \quad x \in (0,L); \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\rho(t)), \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(0,t-\rho(0)) = z_0(t-\rho(0)), \quad 0 < t < \rho(0), \end{cases} \tag{2.15}$$

where $f \in L^1(0,T;L^1(0,L))$ represents the source term.

2.1. Well-posedness: Nonlinear system with saturated source term

In this subsection, our aim is to establish the well-posedness of the system (2.1). Before presenting our main result in this subsection, we need to introduce the following space $\mathcal{B} = \mathcal{C}((0,T),L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ with $T > 0$. We equipped the space $\mathcal{B}$ with the following norm

$$\|u\|_{\mathcal{B}} = \|u\|_{\mathcal{C}((0,T),L^2(0,L))} + \|u\|_{L^2(0,T,H^1(0,L))}.$$

Theorem 2.2 *Let $T > 0, L > 0$ and α, β satisfying (2.5). Assume $a \in L^\infty(0,L)$ such that (2.2) hold. Then, there exists $r > 0$ and $\Delta > 0$ such that for every $(u_0,z_0) \in H$ satisfying $\|(u_0,z_0)\|_H \leq r$, there exists a unique solution $u \in \mathcal{B}$ of system (2.1) satisfying $\|u\|_{\mathcal{B}} \leq \Delta\|(u_0,z_0)\|_H$*

Proof: Let $(u_0,z_0) \in H$ such that $\|(u_0,z_0)\|_H \leq r$ for $r > 0$ chosen small enough later.
Consider $u \in \mathcal{B}$ and the following map:

$$\begin{array}{rcl} \digamma : \mathcal{B} & \to & \mathcal{B} \\ u & \mapsto & \digamma(u) = \tilde{u} \end{array}$$

where $\tilde{u}$ is the solution of the following system

$$\begin{cases} \tilde{u}_t(x,t) + \tilde{u}_x(x,t) + \tilde{u}_{xxx}(x,t) + \tilde{u}\tilde{u}_x(x,t) = -\mathbf{sat}(a(x)\tilde{u}(x,t)), \quad t > 0, \quad x \in (0,L); \\ \tilde{u}(0,t) = \tilde{u}(L,t) = 0, \quad t > 0; \\ \tilde{u}_x(L,t) = \alpha\tilde{u}_x(0,t) + \beta\tilde{u}_x(0,t-\rho(t)), \quad t > 0; \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad x \in (0,L); \\ \tilde{u}_x(0,t-\rho(0)) = z_0(t-\rho(0)), \quad 0 < t < \rho(0), \end{cases} \tag{2.16}$$

Therefore, $u \in B$ is a solution of (2.1) if and only if u is a fixed point of $\digamma$. Let

$$f(x,t) = -u(x,t)u_x(x,t) - \mathbf{sat}(au(x,t)).$$

From Lemma 2.3, if $u \in L^2(0,T,H^1(0,L))$, then $uu_x \in L^1(0,T,L^2(0,L)$ and according to Lemma 2.2, if $u \in L^2(0,T,H^1(0,L))$, hence, $\mathbf{sat}(au(x,t) \in L^1(0,T,L^2(0,L))$. Thus $f(x,t) \in L^1(0,T,L^2(0,L))$. Consequently, from [4, Proposition 2], if (2.5) is satisfied, then there exists $C > 0$ such that

$$\begin{aligned} \|(u,z)\|^2_{\mathcal{C}((0,T),H)} \leq & C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H\right. \\ & \left. + \|uu_x + \mathbf{sat}(au)\|_{L^1(0,T,L^2(0,L))}\right). \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \|u_x\|^2_{L^2(0,T,L^2(0,L))} \leq & C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H\right. \\ & \left. + \|uu_x + \mathbf{sat}(au)\|_{L^1(0,T,L^2(0,L))}\right). \end{aligned} \tag{2.18}$$

Consequently, (2.17), (2.18), Lemma 2.3, and Lemma 2.1, imply that

$$\begin{aligned} \|\digamma(u)\|_B &= \|\tilde{u}\|_B \\ &\leq C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + \int_0^T \|uu_x + \mathbf{sat}(au)\|_{L^2(0,L)}dt\right) \\ &\leq C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + \int_0^T \|uu_x\|_{L^2(0,L)}dt + \int_0^T \|\mathbf{sat}(au)\|_{L^2(0,L)}dt\right) \\ &\leq C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}}dt + 3\int_0^T \|au\|_{L^2(0,L)}dt\right) \\ &\leq C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}}dt + 3a_1\int_0^T \|u\|_{L^2(0,L)}dt\right) \\ &\leq C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}}dt + 3a_1\sqrt{T}\sqrt{L}\|u\|^2_{L^1(0,T,L^2(0,L))}\right) \\ &\leq C\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}} + 3a_1\sqrt{T}\sqrt{L}\|u\|^2_{\mathcal{B}}\right) \\ &\leq CK_1K_2\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + \|u\|^2_{\mathcal{B}}\right) \end{aligned}$$

where $K_2 = 3a_1\sqrt{T}\sqrt{L} > 0$ and $K_1 > 0$ is given from Lemma 2.3. Therefore,

$$\|\digamma(u)\|_{\mathcal{B}} \leq K\left(\|(u_0, z_0(-\rho(0)\cdot))\|_H + \|u\|^2_{\mathcal{B}}\right)$$

where $\Delta = C \times K_1 \times K_2 = C \times K_1 \times 3a_1\sqrt{T}\sqrt{L} > 0$.
Following the previous argument, we have for every u_1 and u_2 which are elements of the space B;

$$\begin{aligned} &\|\digamma(u_1) - \digamma(u_2)\|_{\mathcal{B}} \\ &\leq C\left(\int_0^T \|-u_1u_{1,x} + u_2u_{2,x} - \mathbf{sat}(au_1) + \mathbf{sat}(au_1)\|_{L^2(0,L)}dt\right) \\ &\leq C\left(\int_0^T \|u_1u_{1,x} - u_2u_{2,x}\|_{L^2(0,L)}dt + \int_0^T \|\mathbf{sat}(au_1) - \mathbf{sat}(au_1)\|_{L^2(0,L)}dt\right) \\ &\leq C\left(K_1(\|u_1\|_{\mathcal{B}} + \|u_2\|_{\mathcal{B}})\|u_1 - u_2\|_{\mathcal{B}} + K_2\|u_1 - u_2\|_{L^1(0,T,L^2(0,L))}\right) \\ &\leq C\left(K_1(\|u_1\|_{\mathcal{B}} + \|u_2\|_{\mathcal{B}})\|u_1 - u_2\|_{\mathcal{B}} + K_2\|u_1 - u_2\|_{\mathcal{B}}\right) \\ &\leq \Delta\left((\|u_1\|_{\mathcal{B}} + \|u_2\|_{\mathcal{B}})\|u_1 - u_2\|_{\mathcal{B}} + \|u_1 - u_2\|_{\mathcal{B}}\right) \\ &\leq 2\Delta(\|u_1\|_{\mathcal{B}} + \|u_2\|_{\mathcal{B}})\|u_1 - u_2\|_{\mathcal{B}} \end{aligned}$$

We restricted $\digamma$ to the closed ball $\{u \in \mathcal{B}; \|u\|_{\mathcal{B}} \leq R\}$, where $R > 0$ to be choosen later. Thus,

$$\|\digamma(u)\|_{\mathcal{B}} \leq \Delta(r + R^2),$$

and

$$\|\digamma(u_1) - \digamma(u_2)\|_{\mathcal{B}} \leq 4\Delta R\|u_1 - u_2\|_{\mathcal{B}}$$

Hence, it is enough to take R and r satisfying

$$R < \frac{1}{4\Delta} \text{ and } r < \frac{R}{4\Delta},$$

Therefore, $\|\mathcal{F}(u)\|_{\mathcal{B}} \leq R$ and $\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{\mathcal{B}} < 4KR\|u_1 - u_2\|_{\mathcal{B}}$, with $4\Delta R < 1$. Thus, we can apply the Banach fixed-point theorem and we deduce that the map $\mathcal{F}$ has a unique fixed-point. Consequently, the nonlinear system (2.1) has a unique solution $u \in \mathcal{B}$. □

3. Stability result via Lyapunov approach

The goal of this section is to establish exponential stability results. In order to investigate the stability of our system, let us analyze the characteristics of the following Lyapunov functional

$$V(t) = E(t) + \lambda V_1(t) + \gamma V_2(t), \tag{3.1}$$

where $\lambda, \gamma \geq 0$

$$V_1(t) = \int_0^L x u^2(x,t)dx \tag{3.2}$$

$$V_2(t) = \rho(t) \int_0^1 (1-\mu) u_x^2(0, t - \rho(t)\mu) d\mu. \tag{3.3}$$

$E(\cdot)$ is the energy of equation (2.1) given by

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{|\beta|}{2}\rho(t)\int_0^1 u_x^2(0, t-\rho(t)\mu)d\mu. \tag{3.4}$$

We commence by demonstrating that, for a solution of equation (2.1), the energy is a non-increasing function of time.

Lemma 3.1 *Assume that assumptions (2.3), (2.4) and (2.5) are satisfied. Moreover suppose also that $u \in L^2(0,T,H^1(0,L))$ and $a = a(x) \in L^\infty(0,L)$, satisfying (2.2). Then, for any regular solution of (2.1), the energy (3.4) satisfies the following inequality*

$$\begin{aligned} \frac{d}{dt}E(t) \leq & \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ \leq & 0. \end{aligned} \tag{3.5}$$

Proof: Let u a regular solution of (2.1). By definition $z(\mu,t) = u_x(0, t-\rho(t)\mu)$, hence we rewrite the energy (3.4) as follows

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{|\beta|}{2}\rho(t)\int_0^1 z^2(\mu,t)d\mu.$$

Differentiating $E(\cdot)$, we get

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^L u u_t dx + \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2 d\mu + |\beta|\rho(t)\int_0^1 z z_t d\mu \\ &= -\int_0^L u u_x dx - \int_0^L u u_{xxx} dx - \int_0^L u^2 u_x dx - \int_0^L \mathbf{sat}(au) u dx \\ &+ \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2 d\mu + |\beta|\rho(t)\int_0^1 z z_t d\mu \end{aligned} \tag{3.6}$$

Since, $u \in H_0^1(0,L)$ we have $\int_0^L u^2 u_x dx = 0$, and by some integrations by parts, we obtain

$$-\int_0^L u u_x dx = 0; \quad -\int_0^L u u_{xxx} dx = \frac{1}{2}u_x^2(L,t) - \frac{1}{2}u_x^2(0,t),$$

and

$$|\beta|\rho(t)\int_0^1 zz_t d\mu = |\beta|\rho(t)\int_0^1 \frac{\dot\rho(t)\mu - 1}{\rho(t)} zz_\mu d\mu$$
$$= |\beta|\dot\rho(t)\int_0^1 \mu zz_\mu d\mu - |\beta|\int_0^1 zz_\mu d\mu.$$

Thus

$$\begin{aligned} |\beta|\dot\rho(t)\int_0^1 \mu zz_\mu d\mu &= \frac{|\beta|}{2}\dot\rho(t)\left[\mu z^2(\mu,t)\right]_0^1 - \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2(\mu,t)d\mu \\ &= \frac{|\beta|}{2}\dot\rho(t)z^2(1,t) - \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2(\mu,t)d\mu \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} -|\beta|\int_0^1 zz_\mu d\mu &= -\frac{|\beta|}{2}\left[z^2(\mu,t)\right]_0^1 \\ &= -\frac{|\beta|}{2}\left[z^2(1,t) - z^2(0,t)\right] \\ &= \frac{|\beta|}{2}u_x^2(0,t) - \frac{|\beta|}{2}z^2(1,t). \end{aligned} \tag{3.8}$$

Using (3.6), (3.7) and (3.8), we get

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2}(\alpha u_x(0,t) + \beta z(1,t))^2 - \frac{1}{2}u_x^2(0,t) - \int_0^L \mathbf{sat}(au)udx \\ &+ \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2 d\mu + |\beta|\dot\rho(t)\int_0^1 \mu zz_\mu d\mu - |\beta|\int_0^1 zz_\mu d\mu \\ &= \frac{1}{2}\alpha^2u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) + \frac{1}{2}\beta^2z^2(1,t) - \frac{1}{2}u_x^2(0,t) \\ &- \int_0^L \mathbf{sat}(au)udx + \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2d\mu + \frac{|\beta|}{2}\dot\rho(t)z^2(1,t) \\ &- \frac{|\beta|}{2}\dot\rho(t)\int_0^1 z^2d\mu + \frac{|\beta|}{2}u_x^2(0,t) - \frac{|\beta|}{2}z^2(1,t) \\ &= \frac{1}{2}(\alpha^2 - 1 + |\beta|)u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) \\ &+ \frac{1}{2}\left(\beta^2 + |\beta|(\dot\rho(t) - 1)\right)z^2(1,t) - \int_0^L \mathbf{sat}(au)udx. \end{aligned}$$

Therefore using (2.4), we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right)\begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ &\leq \frac{1}{2}(\alpha^2 - 1 + |\beta|)u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) \\ &+ \frac{1}{2}\left(\beta^2 + |\beta|(\dot\rho(t) - 1)\right)z^2(1,t) - \int_0^L \mathbf{sat}(au)udx \\ &+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right)\begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ &= -\int_0^L \mathbf{sat}(au)udx \\ &\leq 0. \end{aligned}$$

Because $\int_0^L \mathbf{sat}(au)udx \geq 0$, indeed, if $\|au\|_{L^2} \leq 1$, then

$$\mathbf{sat}(au)u = au^2 \geq 0.$$

If $\|au\|_{L^2} \geq 1$,

$$\mathbf{sat}(au)u = \frac{au}{\|au\|_{L^2}}u = \frac{au^2}{\|au\|_{L^2}} \geq 0.$$

where $a = a(x)$ is a nonnegative function. Consequently, using (2.5), we have

$$\begin{aligned}\frac{d}{dt}E(t) \leq & \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ \leq & 0.\end{aligned}$$

□

The proof of the exponential stability of the system (2.1) relies on the following auxiliary lemmas:

Lemma 3.2 *Assume that* $a = a(x) \in L^\infty(0,L)$ *satisfies* (2.2), $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ *and* $u \in L^2(0,T,H^1(0,L))$, *then for any regular solution of* (2.1), *the following equality is satisfied*

$$\begin{aligned}\dot{V}_1(t) =& L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-\rho(t)) + \beta^2 u_x^2(0,t-\rho(t))) \\ &+ \int_0^L u^2 dx - 3\int_0^L u_x^2 dx + \frac{2}{3}\int_0^L u^3 dx - 2\int_0^L x\mathbf{sat}(au)udx \end{aligned} \tag{3.9}$$

Proof: Consider a sufficiently regular solution u of (2.1). Differentiating $V_1(\cdot)$ we have

$$\begin{aligned}\dot{V}_1(t) &= 2\int_0^L xuu_t dx \\ &= -2\int_0^L xuu_x dx - 2\int_0^L xuu_{xxx}dx - 2\int_0^L xu^2u_x dx - 2\int_0^L x\mathbf{sat}(au)udx\end{aligned}$$

using integration by parts and the boundary condition, it follows that

$$-2\int_0^L xuu_x dx = \int_0^L u^2 dx;$$

$$\begin{aligned}-2\int_0^L xuu_{xxx}dx &= Lu^2(L,t) - 3\int_0^L u_x^2 dx \\ &= L(\alpha u_x(0,t) + \beta u_x(0,t-\rho(t)))^2 - 3\int_0^L u_x^2 dx\end{aligned}$$

and

$$-2\int_0^L xu^2u_x dx = \frac{2}{3}\int_0^L u^3 dx;$$

Then,

$$\begin{aligned}\dot{V}_1(t) &= \int_0^L u^2 dx + L(\alpha u_x(0,t) + \beta u_x(0,t-\rho(t)))^2 \\ &- 3\int_0^L u_x^2 dx + \frac{2}{3}\int_0^L u^3 dx - 2\int_0^L x\mathbf{sat}(au)udx \\ &= L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-\rho(t)) + \beta^2 u_x^2(0,t-\rho(t))) \\ &+ \int_0^L u^2 dx - 3\int_0^L u_x^2 dx + \frac{2}{3}\int_0^L u^3 dx - 2\int_0^L x\mathbf{sat}(au)udx\end{aligned}$$

□

Lemma 3.3 *Assume that (2.4) is satisfied. Suppose also* $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ *and* $u \in L^2(0,T,H^1(0,L))$, *then for any regular solution of (2.1), the following inequality is satisfied*

$$\dot{V}_2(t) \leq -(1-d)\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu + u_x^2(0,t). \tag{3.10}$$

Proof: Using the same logic as the previous proof. Consider a regular solution, differentiating $V_2(\cdot)$, we obtain

$$\begin{aligned}
\dot{V}_2(t) =& \dot{\rho}(t)\int_0^1 (1-\mu)u_x^2(0,t-\rho(t)\mu)d\mu \\
&+2\rho(t)\int_0^1 (1-\mu)\partial_t u_x(0,t-\rho(t)\mu)u_x(0,t-\rho(t)\mu)d\mu \\
=& \dot{\rho}(t)\int_0^1 (1-\mu)u_x^2(0,t-\rho(t)\mu)d\mu + 2\int_0^1 \rho(t)\partial_t u_x(0,t-\rho(t)\mu)u_x(0,t-\rho(t)\mu)d\mu \\
&-2\int_0^1 \mu\rho(t)\partial_t u_x(0,t-\rho(t)\mu)u_x(0,t-\rho(t)\mu)d\mu
\end{aligned} \tag{3.11}$$

Using the following equation

$$-\rho(t)\partial_t u_x(0,t-\rho(t)\mu) = (1-\dot{\rho}(t)\mu)\partial_\mu u_x(0,t-\rho(t)\mu),$$

By applying integration by parts and considering boundary conditions, it follows that

$$\begin{aligned}
2\int_0^1 \rho(t)\partial_t u_x(0,t-\rho(t)\mu)u_x(0,t-\rho(t)\mu)d\mu =& u_x^2(0,t) - (1-\dot{\rho}(t))u_x^2(0,t-\rho(t)) \\
&-\dot{\rho}(t)\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
-2\int_0^1 \mu\rho(t)\partial_t u_x(0,t-\rho(t)\mu)u_x(0,t-\rho(t)\mu)d\mu =& (1-\dot{\rho}(t))u_x^2(0,t-\rho(t)) \\
&-\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu \\
&+2\int_0^1 \mu\dot{\rho}(t)u_x^2(0,t-\rho(t)\mu)d\mu
\end{aligned} \tag{3.13}$$

Consequently, (3.11), (3.12), (3.13) and (2.4) imply that

$$\begin{aligned}
\dot{V}_2(t) &= -\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu \\
&+ \dot{\rho}(t)\int_0^1 \mu u_x^2(0,t-\rho(t)\mu)d\mu + u_x^2(0,t) \\
&\leq -\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu + d\int_0^1 \mu u_x^2(0,t-\rho(t)\mu)d\mu + u_x^2(0,t) \\
&= -(1-d)\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu + u_x^2(0,t)
\end{aligned}$$

□

Now, based on the previous lemmas we establish the main result of this section:

Theorem 3.1 *Assume that $a = a(x) \in L^\infty(0, L)$ satisfying (2.2), and $L < \pi\sqrt{3}$. Moreover, suppose that the assumptions (2.3) , (2.4) and (2.5) are satisfied. Then, there exists $r > 0$ sufficiently small, such that for every $(u_0, z_0) \in \mathcal{H}$ satisfying $\|(u_0, z_0)\|_{\mathcal{H}} \le r$, the energy of the system (2.1) denoted by E and defined by (3.4) exponentially decays, that is, there exist two positive constants κ and $\delta > 0$ such that*

$$E(t) \le \kappa e^{-2\delta t} E(0), \quad \forall t > 0. \tag{3.14}$$

Here, λ and γ sufficiently small and the positive constants δ and κ satisfy the following inequality:

$$\delta \le \min\left\{ \frac{(9\pi^2 - 3L^2 - 2L^{\frac{3}{2}} r\pi^2)}{3L^2(1 + 2L\lambda)}\lambda, \frac{\gamma}{h(2\gamma + |\beta|)} \right\} \tag{3.15}$$

and

$$\kappa \le 1 + \max\left\{ L\lambda, \frac{2\gamma}{|\beta|} \right\}.$$

Where λ and γ, satisfying the following inequality

$$\lambda \le \left\{ \begin{array}{l} \dfrac{(1-|\beta|)(1-|\beta|-d) + \alpha^2(d-1) + 2\gamma(|\beta| + d - 1)}{2L(|\beta| - \alpha^2(d-1) - \beta^2 - 2\gamma|\beta|)} \\ \dfrac{1 - \alpha^2 - \beta^2 - |\beta| d + 2\gamma}{2L(\alpha^2 + \beta^2)} \end{array} \right\}. \tag{3.16}$$

and

$$\gamma \le \left\{ \begin{array}{l} \dfrac{1 - \alpha^2 - \beta^2 - |\beta| d}{2} \\ \dfrac{(1-|\beta|)(1-|\beta|-d) + \alpha^2(d-1)}{2(1 - |\beta| - d)} \\ \dfrac{|\beta| - \alpha^2(d-1) - \beta^2}{2\beta} \end{array} \right\}. \tag{3.17}$$

Proof: Before proving this result, it is clear that for any regular solution of (2.1), we have the following inequality

$$E(t) \le V(t) \qquad \text{for all } t \ge 0. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned} \lambda V_1(t) + \gamma V_2(t) &= \lambda \int_0^L x u^2(x,t)dx + \gamma\rho(t)\int_0^1 (1-\mu)u_x^2(0, t - \rho(t)\mu)d\mu \\ &\le \lambda L \int_0^L u^2(x,t)dx + \gamma\frac{\rho(t)}{|\beta|}|\beta| \int_0^1 (1-\mu)u_x^2(0, t-\rho(t)\mu)d\mu \\ &\le (1 + \max\left\{ L\lambda, \frac{2\gamma}{|\beta|} \right\})E(t) \end{aligned}$$

that is,

$$E(t) \le V(t) \le \left(1 + \max\left\{ L\lambda, \frac{2\gamma}{|\beta|}\right\}\right) E(t) \quad \forall t > 0, \tag{3.19}$$

Thanks to inequality (3.19), in order to prove the exponential stability of system (2.1), it is sufficient to show that for all $\delta > 0$,

$$\frac{d}{dt}V(t) + 2\delta V(t) \le 0.$$

Now, let $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ such that $\| \begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \|_0 \le r$, with $r > 0$ chosen later. Using (3.5), (3.9) and (3.10), we get

$$\begin{aligned}\dot{V}(t)\leq&\frac{1}{2}Y^TM_1Y+L\lambda\alpha^2u_x^2(0,t)+2L\lambda\alpha\beta u_x(0,t)u_x(0,t-\rho(t))\\&+\lambda L\beta^2u_x^2(0,t-\rho(t))+\frac{2}{3}\lambda\int_0^Lu^3dx+\lambda\int_0^Lu^2dx-3\lambda\int_0^Lu_x^2dx\\&-\lambda\int_0^Lx\mathbf{sat}(au)udx-\gamma(1-d)\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu+\gamma u_x^2(0,t)\\=&Y^T\left[\frac{1}{2}M_1+M_2\right]Y+\lambda\int_0^Lu^2dx+\frac{2}{3}\lambda\int_0^Lu^3dx-3\lambda\int_0^Lu_x^2dx\\&-\int_0^Lx\mathbf{sat}(au)udx-\gamma(1-d)\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu,\end{aligned}$$

where $Y=\begin{pmatrix}u_x(0,t)\\u_x(0,t-\rho(t))\end{pmatrix}$ and $M_2=\begin{pmatrix}L\lambda\alpha^2+\gamma & L\lambda\alpha\beta\\L\lambda\alpha\beta & L\lambda\beta^2\end{pmatrix}$ and the matrix M_1 is given by (2.6). Since $x\in(0,L)$ and $\mathbf{sat}(au)u\geq0$, we have $\int_0^Lx\mathbf{sat}(au)udx\geq0$. As a result, we can conclude that

$$\begin{aligned}\dot{V}(t)\leq&Y^T\left[\frac{1}{2}M_1+M_2\right]Y+\lambda\int_0^Lu^2dx+\frac{2}{3}\lambda\int_0^Lu^3dx\\&-3\lambda\int_0^Lu_x^2dx-\gamma(1-d)\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu.\end{aligned}\tag{3.20}$$

Now, we calculate $2\delta V(t)$ and using (2.3) we have

$$\begin{aligned}2\delta V(t)=&2\delta E(t)+2\delta\lambda V_1(t)+2\delta\gamma V_2(t)\\=&\delta\int_0^Lu^2dx+\delta|\beta|\rho(t)\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu+2\delta\lambda\int_0^Lxu^2dx\\&+2\delta\gamma\rho(t)\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu-2\delta\gamma\rho(t)\int_0^1\mu u_x^2(0,t-\rho(t)\mu)d\mu\\\leq&\delta\int_0^Lu^2dx+\delta|\beta|K\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu\\&+2\delta\lambda L\int_0^Lu^2dx+2\delta\gamma K\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu\end{aligned}\tag{3.21}$$

According to [24, Theorem 3.2], for λ and γ small enough, the matrix $\frac{1}{2}M_1+M_2$ is definite negative, and from (3.20) and (3.21) we deduce that

$$\begin{aligned}\dot{V}(t)+2\delta V(t)\leq&Y^T\left[\frac{1}{2}M_1+M_2\right]Y+(\lambda+\delta+2L\lambda\delta)\int_0^Lu^2dx+\frac{2}{3}\lambda\int_0^Lu^3dx\\&-3\lambda\int_0^Lu_x^2dx(\delta|\beta|K+2\gamma\delta K-\gamma(1-d))\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu\\\leq&(\lambda+\delta+2L\lambda\delta)\int_0^Lu^2dx+\frac{2}{3}\lambda\int_0^Lu^3dx-3\lambda\int_0^Lu_x^2dx\\&+(\delta|\beta|K+2\gamma\delta K-\gamma(1-d))\int_0^1u_x^2(0,t-\rho(t)\mu)d\mu\end{aligned}\tag{3.22}$$

Additionally, applying Cauchy-Schwarz inequality and using the facts that the energy E defined by (3.4) is non-increasing, together with $H_0^1(0,L)$ into $L^\infty(0,L)$, we obtain

$$\begin{aligned}\int_0^L u^3(x,t)dx &\leq \|u(\cdot,t)\|^2_{L^\infty(0,L)}\int_0^L u(x,t)dx \\ &\leq L\sqrt{L}\|u_x(\cdot,t)\|^2_{L^2(0,L)}\|u(\cdot,t)\|_{L^2(0,L)}\end{aligned}$$

From Lemma 3.1, we deduce that $\|u(\cdot,t)\|_{L^2(0,L)} \leq r$, thus we have

$$\int_0^L u^3(x,t)dx \leq L^{\frac{3}{2}} r\|u_x(\cdot,t)\|^2_{L^2(0,L)}$$

By using the Poincaré inequality, we get

$$\begin{aligned}\dot{V}(t)+2\delta V(t) \leq &\left(\frac{L^2}{\pi^2}(\lambda+\delta+2L\lambda\delta)+\frac{2}{3}L^{\frac{3}{2}}r\lambda-3\lambda\right)\int_0^L u_x^2 dx \\ &+(\delta|\beta|K+2\gamma\delta K-\gamma(1-d))\int_0^1 u_x^2(0,t-\rho(t)\mu)d\mu.\end{aligned} \tag{3.23}$$

By assumption $L<\pi\sqrt{3}$, then from [4], it is possible to choose r small enough to have

$$r<\frac{3(3\pi^2-L^2)}{2L^{\frac{3}{2}}\pi^2}.$$

Consequently, we can choose $\delta>0$ such that (3.15) holds in order to obtain that

$$\frac{L^2}{\pi^2}(\lambda+\delta+2L\lambda\delta)+\frac{2}{3}L^{\frac{3}{2}}r\lambda-3\lambda \leq 0,$$

and

$$\delta|\beta|K+2\gamma\delta K-\gamma(1-d)\leq 0,$$

therefore

$$\dot{V}(t)+2\delta V(t)\leq 0 \quad \forall t\geq 0. \tag{3.24}$$

Hence, , integrating (3.24) over (0, t), and thanks to (3.19), yields that

$$E(t)\leq\left(1+\max\left\{L\lambda,\frac{2\gamma}{|\beta|}\right\}\right)e^{-2\delta t}E(0) \quad \forall t\geq 0 \tag{3.25}$$

Using the density of $D(A(0))$, we conclude the proof by extending the result to any initial condition within H. □

4. Conclusion

In this work, we have established the well-posedness and exponential stability of the nonlinear Korteweg–de Vries equation with a time-varying delay on the boundary feedback in the presence of a saturated source term. Our study illustrates that the incorporation of a time-varying delay, along with a saturated source term, results in a well-posed system under specific conditions. Using a suitable Lyapunov functional, we prove that the system (2.1) is locally exponentially stable with an estimate of the decay rate. However, the length of the spatial domain L must satisfy the condition $L<\pi\sqrt{3}$. This limitation arises from the choice of the multipliers x in the expression of V_1 defined by (3.2). This leads to two natural questions: can we choose another Lyapunov function, distinct from the previous one, to eliminate the restriction on the constraint on L? Furthermore, we can wonder about the possibility of studying the global exponential stability of this equation?

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