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Common neighborhood (signless) Laplacian spectrum and energy of CCC-graph

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ABSTRACT: In this paper, we consider commuting conjugacy class graph (abbreviated as CCC-graph) of a finite group G which is a graph with vertex set $\{x^G : x \in G \setminus Z(G)\}$ (where x^G denotes the conjugacy class containing x) and two distinct vertices x^G and y^G are joined by an edge if there exist some elements $x' \in x^G$ and $y' \in y^G$ such that they commute. We compute common neighborhood (signless) Laplacian spectrum and energy of CCC-graph of finite non-abelian groups whose central quotient is isomorphic to either $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is any prime) or the dihedral group D_{2n} $(n \geq 3)$; and determine whether CCC-graphs of these groups are common neighborhood (signless) Laplacian hyperenergetic/borderenergetic. As a consequence, we characterize certain finite non-abelian groups viz. D_{2n}, T_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that their CCC-graphs are common neighborhood (signless) Laplacian hyperenergetic/borderenergetic. Further, we compare various common neighborhood energies of CCC-graphs of these groups and describe their closeness graphically.

Key Words: Common Neighborhood; Spectrum; Energy; Commuting Conjugacy Class Graph.

Contents

1. Introduction

Characterizing finite groups through various graphs defined on them have been an active area of research over the last 50 years. A number of graphs have been defined on groups [6]. Among those, in our paper, we consider commuting conjugacy class graph (abbreviated as CCC-graph) of a finite non-abelian group G. For any element $x \in G$, we write x^G to denote the conjugacy class of G containing x. The CCC-graph of G, denoted by Γ_G, is defined as a simple undirected graph whose vertex set is the set of conjugacy classes of non-central elements of G and two vertices x^G and y^G are adjacent if there exists some elements $x' \in x^G$ and $y' \in y^G$ such that $x'y' = y'x'$. In 2009, Herzog et al. [27] introduced the concept of CCC-graph of a group. In 2016, Mohammadian et al. [32] have characterized finite groups such that their CCC-graph is triangle-free. Later on Salahshour and Ashrafi [34,35], obtained structures of CCC-graph of several families of finite CA-groups. Salahshour [36] also described Γ_G for the groups whose central quotient is isomorphic to a dihedral group. Characterizations of various classes of finite non-abelian groups through energy, (signless) Laplacian energy, common neighborhood energy (abbreviated as CN-energy) and genus of their CCC-graphs can be found in [4,5,28].

The energy of a graph $\mathcal{G}$ (denoted by $E(\mathcal{G})$) is the sum of absolute values of all the eigenvalues of the adjacency matrix of $\mathcal{G}$. This notion was also used in obtaining π-electron energy of a conjugated carbon molecule in theoretical chemistry. The study of energy of a graph was initiated by Gutman [20] in 1978. After a long time, in 2006 and 2008, Gutman et al. introduced two more graph-energy-like quantity, known as Laplacian energy [25] (denoted by $LE(\mathcal{G})$) and signless Laplacian energy [22] (denoted by $LE^+(\mathcal{G})$), using Laplacian and signless Laplacian eigenvalues of a graph respectively. These energies are

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used to study various properties of graphs (see [3,7,16,17]). Later on, mathematicians have introduced several kinds of graph energies (see [23,24]) and studied graph properties. In 2011, Alwardi et al. [2] have introduced CN-energy of a graph. Let $\mathcal{G}$ be a graph with vertex set $V(\mathcal{G}) = \{v_1, v_2, v_3, \cdots, v_n\}$. Let $C(v_i, v_j)$ be the set of vertices of a graph $\mathcal{G}$ other than v_i and v_j which are adjacent to both v_i and v_j. Then the common neighborhood matrix (CN-matrix) of $\mathcal{G}$, denoted by $CN(\mathcal{G})$, is a matrix of size n whose (i, j)-th entry is given by

$$\mathrm{CN}(\mathcal{G})_{i,j} = \begin{cases} |C(v_i, v_j)|, & \text{if } i \neq j \\ 0, & \text{otherwise.} \end{cases}$$

The CN-energy of $\mathcal{G}$ (denoted by $E_{CN}(\mathcal{G})$) is the sum of absolute values of all the eigenvalues of $CN(\mathcal{G})$. Motivated by the study of (signless) Laplacian energy, Jannat et al. [30] have introduced the notions of common neighborhood Laplacian energy (CNL-energy) and common neighborhood signless Laplacian energy (CNSL-energy) of a graph.

The common neighborhood Laplacian matrix (CNL-matrix) and the common neighborhood signless Laplacian matrix (CNSL-matrix) of $\mathcal{G}$, denoted by $\mathrm{CNL}(\mathcal{G})$ and $\mathrm{CNSL}(\mathcal{G})$, respectively, are given by

$$\mathrm{CNL}(\mathcal{G}) := \mathrm{CNRS}(\mathcal{G}) - \mathrm{CN}(\mathcal{G}) \text{ and } \mathrm{CNSL}(\mathcal{G}) := \mathrm{CNRS}(\mathcal{G}) + \mathrm{CN}(\mathcal{G}),$$

where $\mathrm{CNRS}(\mathcal{G})$ is a matrix of size $|V(\mathcal{G})| = n$ whose (i, j)-th entry is given by

$$\mathrm{CNRS}(\mathcal{G})_{i,j} = \begin{cases} \sum\limits_{k=1}^{n} \mathrm{CN}(\mathcal{G})_{i,k}, & \text{if } i = j \text{ and } i = 1, 2, \ldots, n \\ 0, & \text{if } i \neq j. \end{cases}$$

The common neighborhood Laplacian spectrum of $\mathcal{G}$ (abbreviated as CNL-spectrum and denoted by CNL-spec($\mathcal{G}$)) is the set of eigenvalues of $\mathrm{CNL}(\mathcal{G})$ with multiplicities. We write CNL-spec$(\mathcal{G}) = \{(\alpha_1)^{a_1}, (\alpha_2)^{a_2}, \ldots, (\alpha_k)^{a_k}\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the distinct eigenvalues of $\mathrm{CNL}(\mathcal{G})$ with corresponding multiplicities $a_1, a_2, \ldots, a_k$. Similarly, common neighborhood signless Laplacian spectrum of $\mathcal{G}$ (abbreviated as CNSL-spectrum and denoted by CNSL-spec($\mathcal{G}$)) is the set of eigenvalues of $\mathrm{CNSL}(\mathcal{G})$ with multiplicities. We write CNSL-spec$(\mathcal{G}) = \{(\beta_1)^{b_1}, (\beta_2)^{b_2}, \ldots, (\beta_\ell)^{b_\ell}\}$, where $\beta_1, \beta_2, \ldots, \beta_\ell$ are the distinct eigenvalues of $\mathrm{CNSL}(\mathcal{G})$ with corresponding multiplicities $b_1, b_2, \ldots, b_\ell$. A graph $\mathcal{G}$ is called CNL-integral (CNSL-integral) if CNL-spectrum (CNSL-spectrum) contains only integers. The notions of CNL-integral and CNSL-integral graphs were introduced in [30] motivated by the notions of integral (introduced by Harary and Schwenk [26]), L-integral (introduced by Grone and Merris [19]), Q-integral (introduced by Simic and Stanic [37]) and CN-integral (introduced by Alwardi et al. [2]) graphs. A finite graph is called super integral if it is integral, L-integral and Q-integral (see [4]). Integral graphs have some interests for designing the network topology of perfect state transfer networks (see [1] and the references there in).

The CNL-energy and CNSL-energy of $\mathcal{G}$, denoted by $LE_{CN}(\mathcal{G})$ and $LE^+_{CN}(\mathcal{G})$ respectively, are defined as

$$LE_{CN}(\mathcal{G}) := \sum_{i=1}^{k} a_i \left|\alpha_i - \Delta(\mathcal{G})\right| \tag{1.1}$$

and

$$LE^+_{CN}(\mathcal{G}) := \sum_{i=1}^{\ell} b_i \left|\beta_i - \Delta(\mathcal{G})\right|, \tag{1.2}$$

where $\Delta(\mathcal{G}) = \frac{\mathrm{tr}(\mathrm{CNRS}(\mathcal{G}))}{|V(\mathcal{G})|}$ and $\mathrm{tr}(\mathrm{CNRS}(\mathcal{G}))$ is the trace of $\mathrm{CNRS}(\mathcal{G})$. In [30], various facets of the CNL-spectrum, CNL-energy, CNSL-spectrum and CNSL-energy of graphs were discussed; their connections with other well-known graph energies and Zagreb indices were also established. It was observed that

$$LE_{CN}(K_n) = LE^+_{CN}(K_n) = 2(n-1)(n-2), \tag{1.3}$$

where K_n is the complete graph of order n. A graph $\mathcal{G}$ of order n is called CNL-hyperenergetic or CNSL-hyperenergetic according as $LE_{CN}(\mathcal{G}) > 2(n-1)(n-2)$ or $LE_{CN}^{+}(\mathcal{G}) > 2(n-1)(n-2)$. Further, it is called CNL-borderenergetic (CNSL-borderenergetic) if CNL-energy (CNSL-energy) of $\mathcal{G}$ is equal to $2(n-1)(n-2)$. These classes of graphs were introduced in [29,30] motivated by the notions of various types of hyperenergetic graphs (see [2,14,18,21,39,40,41]).

In Section 2, we compute CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graphs of finite non-abelian groups whose central quotient is isomorphic to either $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is any prime) or the dihedral group D_{2n} ($n \geq 3$). In Section 3, we determine whether CCC-graphs of these groups are CNL-integral, CNSL-integral, CNL-hyperenergetic, CNSL-hyperenergetic, CNL-borderenergetic and CNSL-borderenergetic. As a consequence, we characterize the groups viz. D_{2n}, T_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that their CCC-graphs have above mentioned properties. In Subsection 3.1, we compare various CN-energies of CCC-graphs of the groups G considered in Section 2 and show that $\mathrm{E_{CN}}(\Gamma_G) = LE_{CN}(\Gamma_G) = LE_{CN}^{+}(\Gamma_G)$ or $\mathrm{E_{CN}}(\Gamma_G) < LE_{CN}^{+}(\Gamma_G) < LE_{CN}(\Gamma_G)$. We also characterize the groups viz. D_{2n}, T_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n} such that their CCC-graphs satisfy above mentioned equality/inequality. For the groups satisfying the inequality $\mathrm{E_{CN}}(\Gamma_G) < LE_{CN}^{+}(\Gamma_G) < LE_{CN}(\Gamma_G)$, the closeness of various CN-energies of CCC-graphs of G are depicted graphically in Figures 1 – 8. Finally, we conclude the paper in Section 4 by listing certain problems that arise naturally after our investigation.

2. Computations of spectrum and energies

In this section, we compute CNL-spectrum, CNSL-spectrum and their respective energies of CCC-graphs of various families of non-abelian finite groups. In particular, we consider finite non-abelian groups whose central quotients are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is any prime) or $D_{2n} = \langle x, y : x^n = y^2 = 1, yxy^{-1} = x^{-1}\rangle$ (for $n \geq 3$) in the following subsections. We shall also consider a generalization of dihedral groups, namely $U_{(n,m)} = \langle x, y : x^{2n} = y^m = 1, x^{-1}yx = y^{-1}\rangle$ (for $n \geq 2$, $m \geq 3$) along with the dicyclic group $T_{4n} = \langle x, y : x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$ (for $n \geq 2$), the semidihedral groups $SD_{8n} = \langle x, y : x^{4n} = y^2 = 1, yxy = x^{2n-1}\rangle$ (for $n \geq 2$), the groups $U_{6n} = \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1}\rangle$ (for $n \geq 2$) and $V_{8n} = \langle x, y : x^{2n} = y^4 = 1, yx = x^{-1}y^{-1}, y^{-1}x = x^{-1}y\rangle$ (for $n \geq 2$). The following result is useful in our computation.

Theorem 2.1 *[30] Let $\mathcal{G} = l_1K_{m_1} \cup l_2K_{m_2} \cup l_3K_{m_3}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $i = 1, 2, 3$. Then*

$$\text{CNL-spec}(\mathcal{G}) = \left\{(0)^{l_1+l_2+l_3}, (m_1(m_1-2))^{l_1(m_1-1)}, (m_2(m_2-2))^{l_2(m_2-1)}, (m_3(m_3-2))^{l_3(m_3-1)}\right\}$$

and

$$\text{CNSL-spec}(\mathcal{G}) = \Big\{(2(m_1-1)(m_1-2))^{l_1}, ((m_1-2)^2)^{l_1(m_1-1)}, (2(m_2-1)(m_2-2))^{l_2}, ((m_2-2)^2)^{l_2(m_2-1)}, (2(m_3-1)(m_3-2))^{l_3}, ((m_3-2)^2)^{l_3(m_3-1)}\Big\}.$$

2.1. Groups whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$

This class of groups have been considered by Salahshour and Ashrafi [34, Theorem 3.1] and showed that

$$\Gamma_G = (p+1)K_{\frac{(p-1)|Z(G)|}{p}}. \tag{2.1}$$

CN-energy of CCC-graphs of this class of groups have been studied in [28]. It is worth mentioning that commuting and non-commuting graphs of this class of groups are also studied in [8,9,10,11,12,15,13,33]. In the following theorem, we derive CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graphs of this class of groups.

Theorem 2.2 *Let G be a finite non-abelian group with $|Z(G)| = z \geq 2$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{ (0)^{p+1}, \left(\tfrac{1}{p^2}(pz-z)(pz-z-2p)\right)^{\frac{(p+1)}{p}(pz-z-p)} \right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{ \left(\tfrac{2}{p^2}(pz-z-p)(pz-z-2p)\right)^{p+1}, \left(\tfrac{1}{p^2}(pz-z-2p)^2\right)^{\frac{(p+1)}{p}(pz-z-p)} \right\} \textit{ and}$$

$$LE_{CN}(\Gamma_G) = LE^{+}_{CN}(\Gamma_G) = \begin{cases} \frac{3}{2}, & \textit{for } p=2\ \&\ z=3 \\ \frac{4(p-2)(p+1)}{p^2}, & \textit{for } p \geq 2\ \&\ z=2 \\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \textit{otherwise.} \end{cases}$$

Proof: From (2.1), we have $\Gamma_G = (p+1)K_n$, where $n = \frac{(p-1)z}{p}$. Therefore, by Theorem 2.1, we get

$$\text{CNL-spec}(\Gamma_G) = \{(0)^{p+1}, (\tfrac{1}{p^2}(pz-z)(pz-z-2p))^{\frac{(p+1)}{p}(pz-z-p)}\} \text{ and}$$

$$\text{CNSL-spec}(\Gamma_G) = \{(\tfrac{2}{p^2}(pz-z-p)(pz-z-2p))^{p+1}, (\tfrac{1}{p^2}(pz-z-2p)^2)^{(p+1)\frac{1}{p}(pz-z-p)}\}.$$

Here $|V(\Gamma_G)| = \frac{(p^2-1)z}{p}$ and $\text{tr}(\text{CNRS}(\Gamma_G)) = \frac{(p-1)(p+1)z(p(z-2)-z)(p(z-1)-z)}{p^3}$. Therefore, $\Delta(\Gamma_G) = \frac{(p(z-2)-z)(p(z-1)-z)}{p^2}$.

Now we calculate CNL-energy of Γ_G. We have

$$L_1 := |0 - \Delta(\Gamma_G)| = \left| -\frac{(p(z-2)-z)(p(z-1)-z)}{p^2} \right|.$$

Let $\alpha_1(p,z) = -(p(z-2)-z)(p(z-1)-z)$. Then $\alpha_1(p,z) = -2p^2 - 3pz - z^2 + \frac{1}{2}p^2z(6-z) + \frac{1}{2}pz^2(4-p) < 0$ for $p \geq 4$ and $z \geq 6$. It can be seen that $\alpha_1(2,z) = -z(z-6) - 8 = 1$ or ≤ 0 according as $z = 3$ or $z \neq 3$; $\alpha_1(3,z) = -2(z-3)(2z-3) = 2$ or ≤ 0 according as $z = 2$ or $z \neq 2$; $\alpha_1(p,2) = 2p - 4 \geq 0$; $\alpha_1(p,3) = -2p^2 + 9p - 9 = 1$ or ≤ 0 according as $p = 2$ or $p \neq 2$; $\alpha_1(p,4) = -6p^2 + 20p - 16 \leq 0$ and $\alpha_1(p,5) = -12p^2 + 35p - 25 \leq 0$. Therefore

$$L_1 = \begin{cases} -\frac{(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{for } p=2\ \&\ z=3; p \geq 2\ \&\ z=2 \\ \frac{(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.} \end{cases}$$

Also

$$L_2 := \left| \frac{(pz-z)(pz-2p-z)}{p^2} - \Delta(\Gamma_G) \right| = \left| \frac{pz-2p-z}{p} \right|.$$

Let $\alpha_2(p,z) = pz - 2p - z$. Then $\alpha_2(p,z) = (z-2)p - z \geq z - 4 \geq 0$ for all $z \geq 4$ since $p \geq 2$. It can be seen that $\alpha_2(p,2) = -2 < 0$ and $\alpha_2(p,3) = p - 3 \geq 0$ or < 0 according as $p \geq 3$ or $p = 2$. Therefore

$$L_2 = \begin{cases} -\frac{pz-2p-z}{p}, & \text{for } p=2\ \&\ z=3; p \geq 2\ \&\ z=2 \\ \frac{pz-2p-z}{p}, & \text{otherwise.} \end{cases}$$

Hence, by (1.1), we get

$$\begin{aligned} LE_{CN}(\Gamma_G) &= (p+1) \times L_1 + \frac{p+1}{p}(pz-z-p) \times L_2 \\ &= \begin{cases} \frac{3}{2}, & \text{for } p=2\ \&\ z=3 \\ \frac{4(p-2)(p+1)}{p^2}, & \text{for } p \geq 2\ \&\ z=2 \\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.} \end{cases} \end{aligned}$$

For CNSL-energy of Γ_G we have

$$\begin{aligned} B_1 &:= \left| \frac{2(pz-p-z)(pz-2p-z)}{p^2} - \Delta(\Gamma_G) \right| = \left| \frac{(pz-2p-z)(pz-p-z)}{p^2} \right| \\ &= -L_1 = \begin{cases} -\frac{(pz-2p-z)(pz-p-z)}{p^2}, & \text{for } p=2\ \&\ z=3; p \geq 2\ \&\ z=2 \\ \frac{(pz-2p-z)(pz-p-z)}{p^2}, & \text{otherwise.} \end{cases} \end{aligned}$$

Also

$$B_2 := \left|\frac{(pz-2p-z)^2}{p^2} - \Delta(\Gamma_G)\right| = \left|\frac{-pz+2p+z}{p}\right| = -L_2$$
$$= \begin{cases} \frac{2p+z-pz}{p}, & \text{for } p=2 \,\&\, z=3; p\geq 2 \,\&\, z=2 \\ -\frac{2p+z-pz}{p}, & \text{otherwise.} \end{cases}$$

Hence, by (1.2), we get

$$LE^+_{CN}(\Gamma_G) = (p+1)\times B_1 + \frac{p+1}{p}(pz-z-p)\times B_2$$
$$= \begin{cases} \frac{3}{2}, & \text{for } p=2 \,\&\, z=3 \\ \frac{4(p-2)(p+1)}{p^2}, & \text{for } p\geq 2 \,\&\, z=2 \\ \frac{2(p+1)(p(z-2)-z)(p(z-1)-z)}{p^2}, & \text{otherwise.} \end{cases}$$

Hence the result follows. □

As a corollary of the above theorem we get the following result.

Corollary 2.1 *Let G be a non-abelian group of order p^n with $|Z(G)| = p^{n-2}$. Then CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{(0)^{p+1}, \left(\tfrac{1}{p^2}(p^{n-1}-p^{n-2})(p^{n-1}-p^{n-2}-2p)\right)^{\frac{(p+1)}{p}(p^{n-1}-p^{n-2}-p)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{\left(\tfrac{2}{p^2}(p^{n-1}-p^{n-2}-p)(p^{n-1}-p^{n-2}-2p)\right)^{p+1}, \left(\tfrac{1}{p^2}(p^{n-1}-p^{n-2}-2p)^2\right)^{\frac{(p+1)}{p}(p^{n-1}-p^{n-2}-p)}\right\}$$

and

$$LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = \begin{cases} 0, & \textit{for } p=2 \,\&\, n=3 \\ \frac{2(p+1)\left(p(p^{n-2}-2)-p^{n-2}\right)\left(p(p^{n-2}-1)-p^{n-2}\right)}{p^2}, & \textit{otherwise.} \end{cases}$$

Proof: Here $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.2. □

2.2. Groups whose central quotient is isomorphic to a dihedral group

The CCC-graph of this class of group was first studied by Salahshour [36] in 2020. Salahshour [36, Theorem 1.2] obtained the following structures of Γ_G (where $\frac{G}{Z(G)} \cong D_{2n}$)

$$\Gamma_G = \begin{cases} K_{\frac{(n-1)|Z(G)|}{2}} \cup 2K_{\frac{|Z(G)|}{2}}, & \text{if } 2 \mid n \\ K_{\frac{(n-1)|Z(G)|}{2}} \cup K_{|Z(G)|}, & \text{if } 2 \nmid n. \end{cases} \tag{2.2}$$

The CN-energy of such Γ_G was studied in [28]. Also spectrum, L-spectrum and Q-spectrum of commuting and non-commuting graphs of this class of groups was studied in [8,9,10,11,12,13] along with their respective energies.

Theorem 2.3 *Let G be a finite group such that $|Z(G)| = z$ and $\frac{G}{Z(G)} \cong D_{2n}$ (where $n \geq 3$). Then CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of G are as given below:*

(a) *If n is even then*

(i) $\text{CNL-spec}(\Gamma_G) = \left\{(0)^3, (\frac{1}{4}(nz-z)(nz-z-4))^{\frac{1}{2}(nz-z-2)}, (\frac{1}{4}z(z-4))^{z-2}\right\}$ *and*

$$LE_{CN}(\Gamma_G) = \frac{((n-1)z-2)(n(z+1)((n-2)z-4)+11z-4)}{2(n+1)}.$$

(ii) $\text{CNSL-spec}(\Gamma_G) = \Big\{(\frac{1}{2}(nz-z-2)(nz-z-4))^1, (\frac{1}{4}(nz-z-4)^2)^{\frac{1}{2}(nz-z-2)}, (\frac{1}{2}(z-2)(z-4))^2, (\frac{1}{4}(z-4)^2)^{z-2}\Big\}$

and

$$LE^+_{CN}(\Gamma_G) = \begin{cases} \frac{28}{5}, & \text{for } n=4\,\&\, z=2\\ \frac{3}{5}z^2(4z-6), & \text{for } n=4\,\&\, z\geq 3\\ \frac{(n-2)(n-1)z^2(nz-6)}{2(n+1)}, & \text{otherwise.}\end{cases}$$

(b) *If n is odd then*

(i) $\text{CNL-spec}(\Gamma_G) = \Big\{(0)^2, (\frac{1}{4}(nz-z)(nz-z-4))^{\frac{1}{2}(nz-z-2)}, (z(z-2))^{z-1}\Big\}$

$$\text{and } LE_{CN}(\Gamma_G) = \begin{cases} 0, & \text{for } n=3\,\&\, z=1\\ 4(z-1)(z-2), & \text{for } n=3\,\&\, z\geq 2\\ \frac{((n-1)z-2)\left((n-3)(n+1)z^2+((n-6)n+17)z-4(n+1)\right)}{2(n+1)}, & \text{otherwise.}\end{cases}$$

(ii) $\text{CNSL-spec}(\Gamma_G) = \Big\{(\frac{1}{2}(nz-z-2)(nz-z-4))^1, (\frac{1}{4}(nz-z-4)^2)^{\frac{1}{2}(nz-z-2)}, (2(z-1)(z-2))^1, ((z-2)^2)^{z-1}\Big\}$

$$\text{and } LE^+_{CN}(\Gamma_G) = \begin{cases} 0, & \text{for } n=3\,\&\, z=1;\ n=5\,\&\, z=1\\ 4(z-1)(z-2), & \text{for } n=3\,\&\, z\geq 2\\ \frac{(n-5)(n-3)(n+3)}{2(n+1)}, & \text{for } n\geq 7\,\&\, z=1\\ \frac{(n-3)(n-1)z^2(nz+z-6)}{2(n+1)}, & \text{otherwise.}\end{cases}$$

Proof: From (2.2), we have $\Gamma_G = K_{\frac{(n-1)z}{2}} \cup 2K_{\frac{z}{2}}$ or $K_{\frac{(n-1)z}{2}} \cup K_z$ according as n is even or odd.

(a)(i) If n is even, then by Theorem 2.1

$$\text{CNL-spec}(\Gamma_G) = \left\{(0)^1, \left(\frac{(n-1)z}{2}\left(\frac{(n-1)z}{2}-2\right)\right)^{\frac{(n-1)z}{2}-1}, (0)^2, \left(\frac{z}{2}\left(\frac{z}{2}-2\right)\right)^{2(\frac{z}{2}-1)}\right\}.$$

Here $|V(\Gamma_G)| = \frac{1}{2}(n+1)z$ and $\text{tr}(\text{CNRS}(\Gamma_G)) = \frac{1}{8}z((n((n-3)n+3)+1)z^2 - 6((n-2)n+3)z + 8(n+1))$. So, $\Delta(\Gamma_G) = \frac{(n((n-3)n+3)+1)z^2-6((n-2)n+3)z+8(n+1)}{4(n+1)}$. Note that $z \geq 2$. We have

$$L_1 := |0 - \Delta(\Gamma_G)| = \left|-\frac{(n((n-3)n+3)+1)z^2-6((n-2)n+3)z+8(n+1)}{4(n+1)}\right|.$$

Let $\alpha_1(n,z) = (n((n-3)n+3)+1)z^2 - 6((n-2)n+3)z + 8(n+1)$. Then $\alpha_1(n,z) = 8+8n+6z(2n-3)+z^2+3nz^2+\frac{n^2z}{2}(nz-12)+\frac{n^2z^2}{2}(n-6) > 0$ for $n \geq 12$. Also, $\alpha_1(4,z) = 29z^2-66z+40 \geq 0$, $\alpha_1(6,z) = 127z^2-162z+56 \geq 0$, $\alpha_1(8,z) = 345z^2-306z+72 \geq 0$ and $\alpha_1(10,z) = 731z^2-498z+88 \geq 0$. Therefore

$$L_1 = \frac{(n((n-3)n+3)+1)z^2-6((n-2)n+3)z+8(n+1)}{4(n+1)}.$$

We have

$$L_2 := \left|\frac{1}{4}(nz-z)(nz-z-4) - \Delta(\Gamma_G)\right| = \left|\frac{n(z+1)((n-2)z-4)+11z-4}{2(n+1)}\right|.$$

Let $\alpha_2(n,z) = \{n(z+1)((n-2)z-4)+11z-4\}$. Then $\alpha_2(n,z) > 0$ for $n \geq 6$, since $n-2 \geq 4 \implies z(n-2)-4 \geq 0 \implies n(z+1)(z(n-2)-4) \geq 0$. Also, $\alpha_2(4,z) = 8z^2+3z-20 \geq 0$. Therefore

$$L_2 = \frac{n(z+1)((n-2)z-4)+11z-4}{2(n+1)}.$$

We have

$$L_3 := \left|\frac{1}{4}z(z-4) - \Delta(\Gamma_G)\right| = \left|\frac{14z-8-8n-nz(2(z+8)+n(-6+(n-3)z))}{4(n+1)}\right|.$$

Let $\alpha_3(n,z) = 14z - 8 - 8n - nz(2(z+8) + n(-6 + (n-3)z))$. For $n \geq 10$, $n((n-3)z - 6) > 0$ and $2(z+8) > 0$. So, $\alpha_3(n,z) < 0$ for all $n \geq 10$. Also, $\alpha_3(4,z) = -24z^2 + 46z - 40 \leq 0$, $\alpha_3(6,z) = -120z^2 + 134z - 56 \leq 0$ and $\alpha_3(8,z) = -336z^2 + 270z - 72 \leq 0$. Therefore

$$L_3 = -\frac{14z - 8 - 8n - nz(2(z+8) + n(-6 + (n-3)z))}{4(n+1)}.$$

Hence, by (1.1), we get

$$\begin{aligned} LE_{CN}(\Gamma_G) &= 3 \times L_1 + \frac{1}{2}(nz - z - 2) \times L_2 + (z-2) \times L_3 \\ &= \frac{((n-1)z - 2)(n(z+1)((n-2)z - 4) + 11z - 4)}{2(n+1)}. \end{aligned}$$

(a)(ii) If n is even, then by Theorem 2.1

$$\text{CNSL-spec}(\Gamma_G) = \left\{ \left(2\left(\frac{(n-1)z}{2} - 1\right)\left(\frac{(n-1)z}{2} - 2\right) \right)^1, \left(\left(\frac{(n-1)z}{2} - 2\right)^2 \right)^{\frac{(n-1)z}{2} - 1}, \left(\frac{1}{2}(z-2)(z-4) \right)^2, \left(\frac{1}{4}(z-4)^2 \right)^{z-2} \right\}.$$

Here $|V(\Gamma_G)| = \frac{1}{2}(n+1)z$ and $\text{tr}(\text{CNRS}(\Gamma_G) = \frac{1}{8}z((n((n-3)n+3)+1)z^2 - 6((n-2)n+3)z + 8(n+1))$. So, $\Delta(\Gamma_G) = \frac{(n((n-3)n+3)+1)z^2 - 6((n-2)n+3)z + 8(n+1)}{4(n+1)}$. Note that $z \geq 2$. We have

$$B_1 := \left| \frac{1}{2}(nz - z - 2)(nz - z - 4) - \Delta(\Gamma_G) \right| = \left| \frac{\left(n\left(n^2 + n - 5\right) + 1\right)z^2 - 6n(n+2)z + 8n + 30z + 8}{4(n+1)} \right|.$$

Let $\beta_1(n,z) = (n(n^2+n-5)+1)z^2 - 6n(n+2)z + 8n + 30z + 8$. Then $\beta_1(n,z) = 8 + 8n + 30z + z^2 + nz^2(n-5) + \frac{nz}{2}(n^2z - 24) + \frac{n^2z}{2}(nz - 12)$. Clearly for $n \geq 12$, $\beta_1(n,z) > 0$, as $n^2z - 24 \geq 0$ and $nz - 12 \geq 0$. It can be seen that $\beta_1(4,z) = 61z^2 - 114z + 40 \geq 0$, $\beta_1(6,z) = 223z^2 - 258z + 56 \geq 0$, $\beta_1(8,z) = 537z^2 - 450z + 72 \geq 0$ and $\beta_1(10,z) = 1051z^2 - 690z + 88 \geq 0$. Therefore

$$B_1 = \frac{\left(n\left(n^2 + n - 5\right) + 1\right)z^2 - 6n(n+2)z + 8n + 30z + 8}{4(n+1)}.$$

We have

$$B_2 := \left| \frac{1}{4}(nz - z - 4)^2 - \Delta(\Gamma_G) \right| = \left| \frac{n((n-2)z^2 - (n+6)z + 4) + 13z + 4}{2(n+1)} \right|.$$

Let $\beta_2(n,z) = n((n-2)z^2 - (n+6)z + 4) + 13z + 4$. Then $\beta_2(n,z) = 4 + 4n + 13z + \frac{nz}{3}(nz - 18) + \frac{n^2z}{3}(z-3) + \frac{nz^2}{3}(n-6) > 0$ for $n \geq 6$ and $z \geq 3$. It can be seen that $\beta_2(4,z) = 8z^2 - 27z + 20 = -2$ or ≥ 0 according as $z = 2$ or $z \geq 3$ and $\beta_2(n,2) = 2n(n-8) + 30 = -2$ or ≥ 0 according as $n = 4$ or $n \geq 6$. Therefore

$$B_2 = \begin{cases} \frac{1}{5}, & \text{for } n = 4 \,\&\, z = 2 \\ \frac{n((n-2)z^2 - (n+6)z + 4) + 13z + 4}{2(n+1)}, & \text{otherwise.} \end{cases}$$

We have

$$B_3 := \left| \frac{1}{2}(z-2)(z-4) - \Delta(\Gamma_G) \right| = \left| \frac{-\left(\left(n^3 - 3n^2 + n - 1\right)z^2\right) + 6(n-4)nz + 8n + 6z + 8}{4(n+1)} \right|.$$

Let $\beta_3(n,z) = -((n^3 - 3n^2 + n - 1)z^2) + 6(n-4)nz + 8n + 6z + 8$. Then $\beta_3(n,z) = 8(1 - nz) + 8n(1-z) + 2z(3 - 4n) + z^2(1-n) + \frac{n^2z}{2}(12 - nz) + \frac{n^2z^2}{2}(6-n) < 0$ for $n \geq 12$. It can be seen that $\beta_3(4,z) = -19z^2 + 6z + 40 \leq 0$, $\beta_3(6,z) = -113z^2 + 78z + 56 \leq 0$, $\beta_3(8,z) = -327z^2 + 198z + 72 \leq 0$ and $\beta_3(10,z) = -709z^2 + 366z + 88 \leq 0$. Therefore

$$B_3 = -\frac{-\left(\left(n^3 - 3n^2 + n - 1\right)z^2\right) + 6(n-4)nz + 8n + 6z + 8}{4(n+1)}.$$

We have

$$B_4 := \left|\frac{1}{4}(z-4)^2 - \Delta(\Gamma_G)\right| = \left|\frac{n(8-z(n((n-3)z-6)+2(z+10)))+10z+8}{4(n+1)}\right|.$$

Let $\beta_4(n,z) = n(8-z(n((n-3)z-6)+2(z+10)))+10z+8$. Then $\beta_4(n,z) = 8n-10nz+10z-10nz+8-2nz^2+\frac{n^2z}{2}(12-nz)+\frac{n^2z^2}{2}(6-n) < 0$ for $n \geq 12$. It can be seen that

$\beta_4(4,z) = -24z^2+26z+40 \leq 0$, $\beta_4(6,z) = -120z^2+106z+56 \leq 0$, $\beta_4(8,z) = -336z^2+234z+72 \leq 0$ and $\beta_4(10,z) = -720z^2+410z+88 \leq 0$. Therefore

$$B_4 = -\frac{n(8-z(n((n-3)z-6)+2(z+10)))+10z+8}{4(n+1)}.$$

Hence, by (1.2), we get

$$\begin{aligned} LE^+_{CN}(\Gamma_G) &= 1\times B_1 + \frac{1}{2}(nz-z-2)\times B_2 + 2\times B_3 + (z-2)\times B_4 \\ &= \begin{cases} \frac{28}{5}, & \text{for } n=4 \,\&\, z=2 \\ \frac{3}{5}z^2(4z-6), & \text{for } n=4 \,\&\, z\geq 3 \\ \frac{(n-2)(n-1)z^2(nz-6)}{2(n+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b)(i) If n is odd, then by Theorem 2.1

$$\text{CNL-spec}(\Gamma_G) = \left\{(0)^1, \left(\frac{(n-1)z}{2}\left(\frac{(n-1)z}{2}-2\right)\right)^{\frac{(n-1)z}{2}-1}, (0)^1, (z(z-2))^{z-1}\right\}.$$

Here $|V(\Gamma_G)| = \frac{1}{2}(n+1)z$ and $\text{tr}(\text{CNRS}(\Gamma_G) = \frac{1}{8}z(nz+z-4)(((n-4)n+7)z-2(n+1))$. So, $\Delta(\Gamma_G) = \frac{(nz+z-4)(((n-4)n+7)z-2(n+1))}{4(n+1)}$. We have

$$L_1' := |0-\Delta(\Gamma_G)| = \left|-\frac{(nz+z-4)(((n-4)n+7)z-2(n+1))}{4(n+1)}\right|.$$

Let $\alpha_1'(n,z) = (nz+z-4)(((n-4)n+7)z-2(n+1))$. Then $\alpha_1'(n,z) = (nz+z-4)(7z-2+\frac{nz}{2}(n-8)+\frac{n}{2}(nz-4)) > 0$ for $n \geq 8$, since $z \geq 1$. Again $\alpha_1'(3,z) = 16z^2-48z+32 \geq 0$, $\alpha_1'(5,z) = 72z^2-120z+48 \geq 0$ and $\alpha_1'(7,z) = 224z^2-240z+64 \geq 0$, as $z \geq 1$. Therefore

$$L_1' = \frac{(nz+z-4)(((n-4)n+7)z-2(n+1))}{4(n+1)}.$$

We have

$$L_2' := \left|\frac{1}{4}(nz-z)(nz-z-4) - \Delta(\Gamma_G)\right| = \left|\frac{n^2z^2+n^2z-2nz^2-6nz-4n-3z^2+17z-4}{2(n+1)}\right|.$$

Let $\alpha_2'(n,z) = n^2z^2+n^2z-2nz^2-6nz-4n-3z^2+17z-4$. Then $\alpha_2'(n,z) = 17z-4+\frac{n}{2}(nz-8)+\frac{nz}{2}(n-12)+\frac{z^2}{2}(n^2-6)+\frac{nz^2}{2}(n-4) > 0$ for $n \geq 8$. It can be seen that $\alpha_2'(3,z) = 8z-16 = -8$ or ≥ 0 according as $z=1$ or $z \geq 2$; $\alpha_2'(5,z) = 12z^2+12z-24 \geq 0$ and $\alpha_2'(7,z) = 32z^2+24z-32 \geq 0$, as $z \geq 1$. Therefore

$$L_2' = \begin{cases} 1, & \text{for } n=3 \,\&\, z=1 \\ \frac{n^2z^2+n^2z-2nz^2-6nz-4n-3z^2+17z-4}{2(n+1)}, & \text{otherwise.} \end{cases}$$

We have

$$L_3' := |z(z-2)-\Delta(\Gamma_G)| = \left|\frac{-n^3z^2+3n^2z^2+6n^2z+nz^2-20nz-8n-3z^2+22z-8}{4(n+1)}\right|.$$

Let $\alpha_3'(n,z) = -n^3z^2+3n^2z^2+6n^2z+nz^2-20nz-8n-3z^2+22z-8$. Then $\alpha_3'(n,z) = -8-8n-2z(10n-11)-3z^2-\frac{n^2z}{3}(nz-18)-\frac{nz^2}{2}(n^2-3)-\frac{n^2z^2}{3}(n-9) < 0$ for $n \geq 19$. It can be seen that $\alpha_3'(3,z) = 16z-32 = -16$ or ≥ 0 according as $z=1$ or $z \geq 2$, $\alpha_3'(5,z) = -48z^2+72z-48 \leq 0$, $\alpha_3'(7,z) = -192z^2+176z-64 \leq 0$,

$\alpha'_3(9,z) = -480z^2 + 328z - 80 \leq 0$, $\alpha'_3(11,z) = -960z^2 + 528z - 96 \leq 0$, $\alpha'_3(13,z) = -1680z^2 + 776z - 112 \leq 0$, $\alpha'_3(15,z) = -2688z^2 + 1072z - 128 \leq 0$ and $\alpha'_3(17,z) = -4032z^2 + 1416z - 144 \leq 0$. Therefore

$$L'_3 = \begin{cases} z-2, & \text{for } n=3 \,\&\, z \geq 2 \\ -\frac{-n^3z^2+3n^2z^2+6n^2z+nz^2-20nz-8n-3z^2+22z-8}{4(n+1)}, & \text{otherwise.} \end{cases}$$

Hence, by (1.1), we get

$$\begin{aligned} LE_{CN}(\Gamma_G) &= 2 \times L'_1 + \frac{1}{2}(nz-z-2) \times L'_2 + (z-1) \times L'_3 \\ &= \begin{cases} 0, & \text{for } n=3 \,\&\, z=1 \\ 4(z-1)(z-2), & \text{for } n=3 \,\&\, z \geq 2 \\ \frac{((n-1)z-2)\left((n-3)(n+1)z^2+((n-6)n+17)z-4(n+1)\right)}{2(n+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

(b)(ii) If n is odd, then by Theorem 2.1

$$\text{CNSL-spec}(\Gamma_G) = \left\{ \left(2\left(\frac{(n-1)z}{2}-1\right)\left(\frac{(n-1)z}{2}-2\right)\right)^1, \left(\left(\frac{(n-1)z}{2}-2\right)^2\right)^{\frac{(n-1)z}{2}-1}, (2(z-1)(z-2))^1, \left((z-2)^2\right)^{z-1} \right\}.$$

Here $|V(\Gamma_G)| = \frac{1}{2}(n+1)z$ and $\text{tr}(\text{CNRS}(\Gamma_G) = \frac{1}{8}z(nz+z-4)(((n-4)n+7)z-2(n+1))$. So, $\Delta(\Gamma_G) = \frac{(nz+z-4)(((n-4)n+7)z-2(n+1))}{4(n+1)}$. We have

$$B'_1 := \left|\frac{1}{2}(nz-z-2)(nz-z-4) - \Delta(\Gamma_G)\right| = \left|\frac{n^3z^2+n^2z^2-6n^2z-5nz^2-12nz+8n-5z^2+42z+8}{4(n+1)}\right|.$$

Let $\beta'_1(n,z) = n^3z^2+n^2z^2-6n^2z-5nz^2-12nz+8n-5z^2+42z+8$. Then $\beta'_1(n,z) = 8+8n+42z+z^2(n(n-5)-5)+nz(n(nz-6)-12)$. For $n \geq 9$ we have $nz-6 \geq 3$ which gives $n(nz-6)-12 > 0$ and $n(n-5)-5 > 0$. Thus, $\beta'_1(n,z) > 0$. Again $\beta'_1(3,z) = 16z^2-48z+32 \geq 0$, $\beta'_1(5,z) = 120z^2-168z+48 \geq 0$ and $\beta'_1(7,z) = 352z^2-336z+64 \geq 0$, as $z \geq 1$. Therefore

$$B'_1 = \frac{n^3z^2+n^2z^2-6n^2z-5nz^2-12nz+8n-5z^2+42z+8}{4(n+1)}.$$

We have

$$B'_2 := \left|\frac{1}{4}(nz-z-4)^2 - \Delta(\Gamma_G)\right| = \left|\frac{n^2z^2-n^2z-2nz^2-6nz+4n-3z^2+19z+4}{2(n+1)}\right|.$$

Let $\beta'_2(n,z) = n^2z^2-n^2z-2nz^2-6nz+4n-3z^2+19z+4$. Then $\beta'_2(n,z) = 4+4n+19z+\frac{nz}{4}(nz-24)+\frac{n^2z}{4}(z-4)+\frac{z^2}{4}(n^2-12)+\frac{nz^2}{4}(n-8) > 0$ for $n \geq 9$ and $z \geq 5$. It can be seen that $\beta'_2(3,z) = 16-8z = 8$ or ≤ 0 according as $z=1$ or $z \geq 2$; $\beta'_2(5,z) = 12z^2-36z+24 \geq 0$; $\beta'_2(7,z) = 32z^2-72z+32 = -8$ or ≥ 0 according as $z=1$ or $z \geq 2$; $\beta'_2(n,1) = 20-4n \geq 0$ or < 0 according as $n = 1,3,5$ or $n \geq 7$; $\beta'_2(n,2) = 2n(n-8)+30 \geq 0$ for all n and $\beta'_2(n,3) = 6n^2-32n+34 = -8$ or ≥ 0 according as $n=3$ or $n \neq 3$. Therefore

$$B'_2 = \begin{cases} z-2, & \text{for } n=3 \,\&\, z \geq 2 \\ \frac{n^2z^2-n^2z-2nz^2-6nz+4n-3z^2+19z+4}{2(n+1)}, & \text{otherwise.} \end{cases}$$

We have

$$B'_3 := |2(z-1)(z-2) - \Delta(\Gamma_G)| = \left|\frac{-n^3z^2+3n^2z^2+6n^2z+5nz^2-36nz+8n+z^2+6z+8}{4(n+1)}\right|.$$

Let $\beta'_3(n,z) = -n^3z^2+3n^2z^2+6n^2z+5nz^2-36nz+8n+z^2+6z+8$. Then $\beta'_3(n,z) = -8nz+8-8nz+8n-20nz+6z-\frac{n^2z}{4}(nz-24)-\frac{z^2}{4}(n^3-4)-\frac{nz^2}{4}(n^2-5)-\frac{n^2z^2}{4}(n-12) < 0$ for $n \geq 25$. Again, $\beta'_3(3,z) = 16z(z-3)+32 \geq 0$, $\beta'_3(5,z) = -24z^2-24z+48 \leq 0$, $\beta'_3(7,z) = -160z^2+48z+64 \leq 0$, $\beta'_3(9,z) = -440z^2+168z+80 \leq 0$, $\beta'_3(11,z) = -912z^2+336z+96 \leq 0$, $\beta'_3(13,z) = -1624z^2+552z+112 \leq 0$, $\beta'_3(15,z) = -2624z^2+816z+128 \leq 0$, $\beta'_3(17,z) = -3960z^2+1128z+144 \leq 0$, $\beta'_3(19,z) = -5680z^2+1488z+160 \leq 0$, $\beta'_3(21,z) = -7832z^2+1896z+176 \leq 0$ and $\beta'_3(23,z) = -10464z^2+2352z+192 \leq 0$. Therefore,

$$B'_3 = \begin{cases} z(z-3)+2, & \text{for } n=3 \,\&\, z \geq 1 \\ -\frac{n^3(-z^2)+3n^2z^2+6n^2z+5nz^2-36nz+8n+z^2+6z+8}{4(n+1)}, & \text{for } n \geq 5 \,\&\, z \geq 1. \end{cases}$$

We have

$$B_4' := \left|(z-2)^2 - \Delta(\Gamma_G)\right| = \left|\frac{-n^3z^2+3n^2z^2+6n^2z+nz^2-28nz+8n-3z^2+14z+8}{4(n+1)}\right|.$$

Let $\beta_4'(n,z) = -n^3z^2+3n^2z^2+6n^2z+nz^2-28nz+8n-3z^2+14z+8$. Then $\beta_4'(n,z) = -8n(z-1) - 14z(n-1) + (8-6nz-3z^2) - \frac{n^2z}{3}(nz-18) - \frac{nz^2}{3}(n^2-3) - \frac{n^2z^2}{3}(n-9) < 0$ for $n \geq 19$. It can be seen that $\beta_4'(3,z) = 32-16z = 16$ or ≤ 0 according as $z=1$ or $z \geq 2$; $\beta_4'(5,z) = 24z(1-2z)+48 = 24$ or ≤ 0 according as $z=1$ or $z \geq 2$; $\beta_4'(7,z) = -192z^2+112z+64 \leq 0$; $\beta_4'(9,z) = -480z^2+248z+80 \leq 0$; $\beta_4'(11,z) = -960z^2+432z+96 \leq 0$; $\beta_4'(13,z) = -1680z^2+664z+112 \leq 0$; $\beta_4'(15,z) = -2688z^2+944z+128 \leq 0$ and $\beta_4'(17,z) = -4032z^2+1272z+144 \leq 0$, as $z \geq 1$. Therefore,

$$B_4' = \begin{cases} \frac{-n^3z^2+3n^2z^2+6n^2z+nz^2-28nz+8n-3z^2+14z+8}{4(n+1)}, & \text{for } n=3,5 \,\&\, z=1 \\ -\frac{-n^3z^2+3n^2z^2+6n^2z+nz^2-28nz+8n-3z^2+14z+8}{4(n+1)}, & \text{otherwise.} \end{cases}$$

Hence, by (1.2), we get

$$\begin{aligned} LE^+_{CN}(\Gamma_G) &= 1 \times B_1' + \frac{1}{2}(nz-z-2) \times B_2' + 1 \times B_3' + (z-1) \times B_4' \\ &= \begin{cases} 0, & \text{for } n=3 \,\&\, z=1;\ n=5 \,\&\, z=1 \\ 4(z-1)(z-2), & \text{for } n=3 \,\&\, z \geq 2 \\ \frac{(n-5)(n-3)(n+3)}{2(n+1)}, & \text{for } n \geq 7 \,\&\, z=1 \\ \frac{(n-3)(n-1)z^2(nz+z-6)}{2(n+1)}, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the result follows. □

As a corollary of the above Theorem 2.3, we get the following results.

Corollary 2.2 *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of the dihedral group D_{2n} (where $n \geq 3$) are as given below:*

(a) *If n is odd then*

(i) CNL-spec$(\Gamma_{D_{2n}}) = \{(0)^2, (\frac{1}{4}(n-1)(n-5))^{\frac{1}{2}(n-3)}\}$ *and* $LE_{CN}(\Gamma_{D_{2n}}) = \frac{(n-5)(n-3)(n-1)}{n+1}$.

(ii) CNSL-spec$(\Gamma_{D_{2n}}) = \{(0)^1, (\frac{1}{2}(n-3)(n-5))^1, (\frac{1}{4}(n-5)^2)^{\frac{1}{2}(n-3)}\}$ *and* $LE^+_{CN}(\Gamma_{D_{2n}}) = \frac{(n-5)(n-3)(n+3)}{2(n+1)}$.

(b) *If n is even then*

(i) CNL-spec$(\Gamma_{D_{2n}}) = \{(0)^3, (\frac{1}{4}(n-2)(n-6))^{\frac{1}{2}(n-4)}\}$ *and* $LE_{CN}(\Gamma_{D_{2n}}) = \frac{3(n-6)(n-4)(n-2)}{2(n+2)}$.

(ii) CNSL-spec$(\Gamma_{D_{2n}}) = \{(0)^2, (\frac{1}{2}(n-4)(n-6))^1, (\frac{1}{4}(n-6)^2)^{\frac{1}{4}(n-4)}\}$ *and*

$$LE^+_{CN}(\Gamma_{D_{2n}}) = \begin{cases} \frac{28}{5}, & \text{for } n=8 \\ \frac{(n-6)(n-4)(n-2)}{n+2}, & \text{for } n \neq 8. \end{cases}$$

Proof: We know that $\frac{D_{2n}}{Z(D_{2n})} \cong D_{2\times\frac{n}{2}}$ or D_{2n} according as n is even or odd. Therefore, by Theorem 2.3, we get the required result. □

Corollary 2.3 *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of the dicyclic group T_{4n} (where $n \geq 2$) are as given below:*

(a) CNL-spec$(\Gamma_{T_{4n}}) = \{(0)^3, ((n-1)(n-3))^{n-2}\}$ *and* $LE_{CN}(\Gamma_{T_{4n}}) = \frac{6(n-3)(n-2)(n-1)}{n+1}$.

(b) CNSL-spec$(\Gamma_{T_{4n}}) = \{(0)^2, (2(n-2)(n-3))^1, ((n-3)^2)^{n-2}\}$ *and*

$$LE^+_{CN}(\Gamma_{T_{4n}}) = \begin{cases} \frac{28}{5}, & \text{for } n=4 \\ \frac{4(n-3)(n-2)(n-1)}{n+1}, & \text{for } n \neq 4. \end{cases}$$

Proof: We know that $\frac{T_{4n}}{Z(T_{4n})} \cong D_{2n}$. Therefore, by Theorem 2.2 (for the case $n = 2$) and Theorem 2.3, we get the required result. □

Corollary 2.4 *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of the group* $U_{6n} = \langle x, y : \ x^{2n} = y^3 = 1, \ x^{-1}yx = y^{-1}\rangle$ *(where $n \geq 2$) are as given below:*

(a) $\text{CNL-spec}(\Gamma_{U_{6n}}) = \{(0)^2, (n(n-2))^{2(n-1)}\}$ *and* $LE_{CN}(\Gamma_{U_{6n}}) = 4(n-2)(n-1)$.

(b) $\text{CNSL-spec}(\Gamma_{U_{6n}}) = \{(2(n-1)(n-2))^2, ((n-2)^2)^{2(n-1)}\}$ *and* $LE^+_{CN}(\Gamma_{U_{6n}}) = 4(n-2)(n-1)$.

Proof: We know that $\frac{U_{6n}}{Z(U_{6n})} = D_{2\times 3}$. Therefore, by Theorem 2.3, we get the required result. □

Corollary 2.5 *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of the group* $U_{(n,m)}$ *(where $m > 2$ and $n \geq 2$) are as given below:*

(a) *If m is odd then*

(i) $\text{CNL-spec}\,(\Gamma_{U_{(n,m)}}) = \left\{(0)^2, (n(n-2))^{n-1}, \left(\frac{1}{4}(nm-n)(nm-n-4)\right)^{\frac{1}{2}(nm-n-2)}\right\}$ *and*

$$LE_{CN}(\Gamma_{U_{(n,m)}}) = \begin{cases} 4(n-1)(n-2), & \text{for } m = 3 \,\&\, n \geq 2 \\ \frac{((m-1)n-2)\left((m-3)(m+1)n^2+((m-6)m+17)n-4(m+1)\right)}{2(m+1)}, & \text{otherwise.} \end{cases}$$

(ii) $\text{CNSL-spec}\,(\Gamma_{U_{(n,m)}}) =$
$\left\{(2(n-1)(n-2))^1, ((n-2)^2)^{n-1}, \left(\frac{1}{2}(nm-n-2)(nm-n-4)\right)^1, \left(\frac{1}{4}(nm-n-4)^2\right)^{\frac{1}{2}(nm-n-2)}\right\}$ *and*

$$LE^+_{CN}(\Gamma_{U_{(n,m)}}) = \begin{cases} 4(n-1)(n-2), & \text{for } m = 3 \,\&\, n \geq 2 \\ \frac{(m-3)(m-1)n^2(mn+n-6)}{2(m+1)}, & \text{otherwise.} \end{cases}$$

(b) *If m and $\frac{m}{2}$ are even then*

(i) $\text{CNL-spec}\,(\Gamma_{U_{(n,m)}}) = \left\{(0)^3, \left(\frac{1}{4}(m-2)n((m-2)n-4)\right)^{\frac{1}{2}(mn-2n-2)}, ((n-2)n)^{2(n-1)}\right\}$ *and*

$$LE_{CN}(\Gamma_{U_{(n,m)}}) = \begin{cases} 6(n-1)(n-2), & \text{for } m = 4 \ \& \ n \geq 2 \\ \frac{((m-2)n-2)\left(m^2n(2n+1)-4m\left(2n^2+3n+1\right)+44n-8\right)}{2(m+2)}, & \text{otherwise.} \end{cases}$$

(ii) $\text{CNSL-spec}\,(\Gamma_{U_{(n,m)}}) = \Big\{(\frac{1}{2}(mn-2n-4)(mn-2n-2))^1, \left(\frac{1}{4}(mn-2n-4)^2\right)^{\frac{1}{2}(mn-2n-2)},$
$(2(n-2)(n-1))^2, ((n-2)^2)^{2(n-1)}\Big\}$ *and*

$$LE^+_{CN}(\Gamma_{U_{(n,m)}}) = \begin{cases} 6(n-1)(n-2), & \text{for } m = 4 \ \& \ n \geq 2 \\ \frac{24}{5}n^2(4n-3), & \text{for } m = 8 \ \& \ n \geq 2 \\ \frac{(m-4)(m-2)n^2(mn-6)}{m+2}, & \text{otherwise.} \end{cases}$$

(c) *If m is even and $\frac{m}{2}$ is odd then*

(i) $\text{CNL-spec}\,(\Gamma_{U_{(n,m)}}) = \left\{(0)^2, \left(\frac{1}{4}(mn-2n-4)(mn-2n)\right)^{\frac{1}{2}(mn-2n-2)}, (4(n-1)n)^{2n-1}\right\}$ *and*

$$LE_{CN}(\Gamma_{U_{(n,m)}}) = \begin{cases} 8(n-1)(2n-1), & \text{for } m = 6 \,\&\, n \geq 2 \\ \frac{((m-2)n-2)\left(m^2n(2n+1)-4m\left(2n^2+3n+1\right)-24n^2+68n-8\right)}{2(m+2)}, & \text{otherwise.} \end{cases}$$

(ii) $\text{CNSL-spec}\,(\Gamma_{U_{(n,m)}}) = \Big\{\left(\frac{1}{2}(mn-2n-4)(mn-2n-2)\right)^1, \left(\frac{1}{4}(mn-2n-4)^2\right)^{\frac{1}{2}(mn-2n-2)}$
$(4(n-1)(2n-1))^1, (4(n-1)^2)^{2n-1}\Big\}$ *and*

$$LE^+_{CN}(\Gamma_{U_{(n,m)}}) = \begin{cases} 8(n-1)(2n-1), & \text{for } m = 6 \ \& \ n \geq 2 \\ \frac{(m-6)(m-2)n^2((m+2)n-6)}{m+2}, & \text{otherwise.} \end{cases}$$

Proof: We know that $\frac{U_{(n,m)}}{Z(U_{(n,m)})}$ is isomorphic to $D_{2\times\frac{m}{2}}$ or D_{2m} according as m is even or odd. Therefore, by Theorem 2.3, we get the required result. □

Corollary 2.6 *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of the group SD_{8n} (where $n \geq 2$) are as given below:*

(a) *If n is even then*
(i) $\text{CNL-spec}(\Gamma_{SD_{8n}}) = \left\{(0)^3, ((2n-1)(2n-3))^{2n-2}\right\}$ *and* $LE_{CN}(\Gamma_{SD_{8n}}) = \frac{12(n-1)(4(n-2)n+3)}{2n+1}$.
(ii) $\text{CNSL-spec}(\Gamma_{SD_{8n}}) = \left\{(0)^2, (2(2n-2)(2n-3))^1, ((2n-3)^2)^{2n-2}\right\}$ *and*

$$LE^+_{CN}(\Gamma_{SD_{8n}}) = \begin{cases} \frac{28}{5}, & \text{for } n = 2 \\ \frac{8(n-1)(2n-3)(2n-1)}{2n+1}, & \text{for } n \geq 4. \end{cases}$$

(b) *If n is odd then*
(i) $\text{CNL-spec}(\Gamma_{SD_{8n}}) = \left\{(0)^2, (8)^3, ((2n-2)(2n-4))^{2n-3}\right\}$ *and*

$$LE_{CN}(\Gamma_{SD_{8n}}) = \begin{cases} 24, & \text{for } n = 3 \\ \frac{4(2n-3)(5(n-3)n+4)}{n+1}, & \text{for } n \geq 5. \end{cases}$$

(ii) $\text{CNSL-spec}(\Gamma_{SD_{8n}}) = \left\{(2(2n-3)(2n-4))^1, ((2n-4)^2)^{2n-3}, 12^1, 4^3\right\}$ *and*

$$LE^+_{CN}(\Gamma_{SD_{8n}}) = \begin{cases} 24, & \text{for } n = 3 \\ \frac{16(n-3)(n-1)(2n-1)}{n+1}, & \text{for } n \geq 5. \end{cases}$$

Proof: We know that $\frac{SD_{8n}}{Z(SD_{8n})}$ is isomorphic to $D_{2\times 2n}$ or D_{2n} according as n is even or odd. Therefore, by Theorem 2.3, we get the required result. □

Corollary 2.7 *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of CCC-graph of the group V_{8n} (where $n \geq 2$) are as given below:*

(a) *If n is even then*

(i) $\text{CNL-spec}(\Gamma_{V_{8n}}) = \left\{(0)^5, ((2n-2)(2n-4))^{2n-3}\right\}$ *and* $LE_{CN}(\Gamma_{V_{8n}}) = \frac{20(n-2)(n-1)(2n-3)}{n+1}$.

(ii) $\text{CNSL-spec}(\Gamma_{V_{8n}}) = \left\{(0)^4, (2(2n-3)(2n-4))^1, ((2n-4)^2)^{2n-3}\right\}$ *and*

$$LE^+_{CN}(\Gamma_{V_{8n}}) = \frac{16(n-2)(n-1)(2n-3)}{n+1}.$$

(b) *If n is odd then*

(i) $\text{CNL-spec}(\Gamma_{V_{8n}}) = \left\{(0)^3, ((2n-1)(2n-3))^{2n-2}\right\}$ *and* $LE_{CN}(\Gamma_{V_{8n}}) = \frac{12(n-1)(4(n-2)n+3)}{2n+1}$.

(ii) $\text{CNSL-spec}(\Gamma_{V_{8n}}) = \left\{(0)^2, (2(2n-2)(2n-3))^1, ((2n-3)^2)^{2n-2}\right\}$ *and*

$$LE^+_{CN}(\Gamma_{V_{8n}}) = \frac{8(n-1)(2n-3)(2n-1)}{2n+1}.$$

Proof: (a) If n is even then, by [35, Proposition 2.4], we have $\Gamma_{V_{8n}} = K_{2n-2} \cup 2K_2$.
(i) By Theorem 2.1, we get

$$\text{CNL-spec}(\Gamma_{V_{8n}}) = \{(0)^5, ((2n-2)(2n-4))^{2n-3}\}.$$

Here $|V(\Gamma_{V_{8n}})| = 2(n+1)$ and $\text{tr}(\text{CNRS}(\Gamma_{V_{8n}})) = 4(n-2)(n-1)(2n-3)$. So, $\Delta(\Gamma_{V_{8n}}) = \frac{2(n-2)(n-1)(2n-3)}{n+1}$. We have

$$L_1 := |0 - \Delta(\Gamma_{V_{8n}})| = \left|-\frac{2(n-2)(n-1)(2n-3)}{n+1}\right| = \frac{2(n-2)(n-1)(2n-3)}{n+1},$$

since $-2(n-2)(n-1)(2n-3) < 0$, as $n \geq 2$, so $2n-3 > 0$, $n-2 \geq 0$ and $n-1 > 0$. Also

$$L_2 := |(2n-2)(2n-4) - \Delta(\Gamma_{V_{8n}})| = \left|\frac{10(n-2)(n-1)}{n+1}\right| = \frac{10(n-2)(n-1)}{n+1}, \text{ as } n \geq 2.$$

Therefore, by (1.1), we get

$$LE_{CN}(\Gamma_{V_{8n}}) = 5 \times L_1 + (2n-3) \times L_2 = \frac{20(n-2)(n-1)(2n-3)}{n+1}.$$

(ii) By Theorem 2.1, we get

$$\text{CNSL-spec}(\Gamma_{V_{8n}}) = \left\{(0)^4, (2(2n-3)(2n-4))^1, ((2n-4)^2)^{2n-3}\right\}.$$

We have

$$B_1 := |0 - \Delta(\Gamma_{V_{8n}})| = L_1,$$

$$B_2 := |2(2n-3)(2n-4) - \Delta(\Gamma_{V_{8n}})| = \left|\frac{2(n-2)(n+3)(2n-3)}{n+1}\right| = \frac{2(n-2)(n+3)(2n-3)}{n+1}, \quad \text{as } n \geq 2,$$

and

$$B_3 := \left|(2n-4)^2 - \Delta(\Gamma_{V_{8n}})\right| = \left|\frac{2(n-2)(3n-7)}{n+1}\right| = \frac{2(n-2)(3n-7)}{n+1},$$

as $n \geq 2$, so $2(n-2)(3n-7) \geq 0$. Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_{V_{8n}}) = 4 \times B_1 + 1 \times B_2 + (2n-3) \times B_3 = \frac{16(n-2)(n-1)(2n-3)}{n+1}.$$

(b) If n is odd then, [35, Proposition 2.4], we have $\Gamma_{V_{8n}} = K_{2n-1} \cup 2K_1 = \Gamma_{D_{2\times 4n}}$. Hence, the result follows from Corollary 2.2. □

3. Some consequences

In this section, we discuss some consequences of the results obtained in Section 2. Looking at the CNL-spectrum and CNSL-spectrum of CCC-graphs of the groups considered in Section 2, we get the following result.

Theorem 3.1 *Let G be a finite non-abelian group with center $Z(G)$. Then the CCC-graph of G is (CNSL) CNL-integral if*

(a) $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

(b) $\frac{G}{Z(G)} \cong D_{2n}$.

(c) *G is isomorphic to D_{2n}, T_{4n}, U_{6n}, $U_{(n,m)}$, SD_{8n} and V_{8n}.*

Now we shall determine whether CCC-graphs of these groups are (CNSL) CNL-hyperenergetic.

Theorem 3.2 *Let G be a finite non-abelian group and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then the CCC-graph of G is not (CNSL) CNL-hyperenergetic.*

Proof: Let $|Z(G)| = z$. Then $z \geq 2$ and $|V(\Gamma_G)| = \frac{(p^2-1)z}{p}$. By Theorem 2.2 and (1.3)

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) = \begin{cases} \frac{4\left((p-2)p\left(2p^2+p-2\right)+4\right)}{p^2}, & \text{for } p \geq 2 \,\&\, z = 2 \\ 16, & \text{for } p = 2 \,\&\, z = 3 \\ \frac{2p^3z^2-2p^2z^2-4p^2-2pz^2+2z^2}{p}, & \text{otherwise.} \end{cases}$$

Let $f_1(p) = 4\left((p-2)p\left(2p^2+p-2\right)+4\right)$ and $f_2(p,z) = 2p^3z^2 - 2p^2z^2 - 4p^2 - 2pz^2 + 2z^2$, where $z \geq 3$. Then $f_1(p) > 0$. Also, $f_2(p,z) = \frac{2}{3}(p-3)p^2z^2 + \frac{2}{3}p^2\left(pz^2-6\right) + \frac{2}{3}\left(p^2-3\right)pz^2 + 2z^2 > 0$ for $p \geq 3$. For $p = 2$ we have $f_2(p,z) = 6z^2 - 16 > 0$. Hence, $LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) > 0$. By Theorem 2.2, we also have $LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$. Therefore, $LE^+_{CN}(\Gamma_G) - LE^+_{CN}(K_{|V(\Gamma_G)|}) > 0$. Hence the result follows. □

An immediate corollary of the above theorem is given below.

Corollary 3.1 *Let G be a non-abelian group of order p^n and center $|Z(G)| = p^{n-2}$. Then CCC-graph of G is not (CNSL) CNL-hyperenergetic.*

Theorem 3.3 *Let G be a finite non-abelian group and $\frac{G}{Z(G)} \cong D_{2n}$ (where $n \geq 3$). Then the CCC-graph of G is*

(a) *CNL-borderenergetic if $n = 3, 11$ & $z = 1$.*

(b) *CNL-hyperenergetic except for $n = 4, 6$ & $z = 2$; $n = 4$ & $z = 3$; $n = 4$ & $z = 4$; $n = 3$ & $z \geq 2$; $n = 5, 7, 9$ & $z = 1$ and $n = 5$ & $z = 2, 3$.*

(c) *CNSL-borderenergetic for $n = 3$ & $z = 1$.*

(d) *CNSL-hyperenergetic except for $n = 4$ & $z = 2, 3, 4, 5$; $n = 6, 8$ & $z = 2$; $n = 6$ & $z = 3$; $n = 5$ & $z = 1$; $n = 3$ & $z \geq 2$; $n \geq 7$ (n is odd) & $z = 1$; $n = 5, 7, 9$ & $z = 2$ and $n = 5$ & $z = 3, 4$.*

Proof: We have

$$|V(\Gamma_G)| = \begin{cases} \frac{1}{2}(n+1)z, & \text{for } n \text{ is even} \\ \frac{1}{2}(n+1)z, & \text{for } n \text{ is odd.} \end{cases}$$

Case 1: n is even

In this case $z \geq 2$. By Theorem 2.3 and (1.3), we have

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) = \frac{-n^3z^3 + 3n^2z^3 + 12n^2z^2 - 2nz^3 - 18nz^2 - 24nz + 12z^2 + 12z}{2(n+1)}.$$

Let $f_1(n,z) = -n^3z^3 + 3n^2z^3 + 12n^2z^2 - 2nz^3 - 18nz^2 - 24nz + 12z^2 + 12z$. Then $f_1(n,z) = \frac{1}{2}n^2z^3(6-n) + \frac{1}{2}n^2z^2(24-nz) - 2nz^3 + 6z^2(2-3n) + 12z(1-2n) < 0$ for $n \geq 6$ and $z \geq 4$. We have $f_1(n,2) = 8n^2(9-n) + 72 - 136n < 0$ for $n \geq 10$. Also $f_1(4,2) = 168$, $f_1(6,2) = 120$ and $f_1(8,2) = -504$. Therefore, $f_1(n,2) > 0$ or < 0 according as $n = 4, 6$ or $n \geq 8$. We have $f_1(n,3) = 27n^2(7-n) + 144 - 288n < 0$ for $n \geq 8$.

Also $f_1(4,3) = 288$ and $f_1(6,3) = -612$. Therefore, $f_1(4,3) > 0$ or < 0 according as $n = 4$ or $n \geq 6$.

Now we need to check for $n = 4$ and $z \geq 4$. We have $f_1(4,z) = 12z^2(11-2z) - 84z < 0$ for $z \geq 6$. Also $f_1(4,4) = 240$ and $f_1(4,5) = -120$. Therefore, $f_1(4,z) > 0$ or < 0 according as $z = 4$ or $z \geq 5$. Hence, $LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) > 0$ for $n = 4, 6$ & $z = 2$ and $n = 4$ & $z = 3, 4$. Otherwise, $LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) < 0$.

By Theorem 2.3 and (1.3), we also have

$$LE^+_{CN}(K_{|V(\Gamma_G)|}) - LE^+_{CN}(\Gamma_G) = \begin{cases} \frac{92}{5}, & \text{for } n = 4 \,\&\, z = 2 \\ \frac{1}{10}z\left(-24z^2 + 161z - 150\right) + 4, & \text{for } n = 4 \,\&\, z \geq 3 \\ \frac{-n^3z^3+n^3z^2+3n^2z^3+9n^2z^2-6n^2z-2nz^3-15nz^2-12nz+8n+13z^2-6z+8}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Let $f_2(z) = \frac{1}{10}z\left(-24z^2 + 161z - 150\right) + 4$ and $f_3(n, z) = -n^3z^3 + n^3z^2 + 3n^2z^3 + 9n^2z^2 - 6n^2z - 2nz^3 - 15nz^2 - 12nz + 8n + 13z^2 - 6z + 8$. Then $f_2(z) = \frac{1}{10}z\left(z\left(161 - 24z\right) - 150\right) + 4 > 0$ or < 0 according as $z = 3, 4, 5$ or $z \geq 6$. Also, $f_3(n, z) = \frac{1}{3}n^3(3-z)z^2 + \frac{1}{3}(9-n)n^2z^3 + \frac{1}{3}n^2z^2(27-nz) - 6n^2z + z^2(13-2nz) + \left(8n - 15nz^2\right) + (8-12nz) - 6z < 0$ for $n \geq 10$ and $z \geq 3$.

We have $f_3(n, 2) = 4n^2(12 - n) + 48 - 92n < 0$ for $n \geq 12$. Also, $f_3(6, 2) = 360$, $f_3(8, 2) = 336$ and $f_3(10, 2) = -72$. Therefore, $f_3(n, 2) > 0$ or < 0 according as $n = 6, 8$ or $n \geq 10$.

We have $f_3(6, z) = z^2(463 - 120z) - 294z + 56 < 0$ for $z \geq 4$. Also, $f_3(6, 3) = 101$. Therefore, $f_3(6, z) > 0$ or < 0 according as $z = 3$ or $z \geq 4$. We have $f_3(8, z) = z^2(981 - 336z) - 486z + 72 < 0$ for $z \geq 3$. Therefore, $f_3(n, z) > 0$ if $n = 6, 8 \,\&\, z = 2$ and $n = 6 \,\&\, z = 3$. Otherwise, $f_3(n, z) < 0$. Hence, $LE^+_{CN}(K_{|V(\Gamma_G)|}) - LE^+_{CN}(\Gamma_G) > 0$ if $n = 4$ & $z = 2, 3, 4, 5$; $n = 6, 8$ & $z = 2$ and $n = 6$ & $z = 3$. Otherwise, $LE^+_{CN}(K_{|V(\Gamma_G)|}) - LE^+_{CN}(\Gamma_G) < 0$.

Case 2: n is odd

By Theorem 2.3 and (1.3), we have

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) = \begin{cases} 0, & \text{for } n = 3 \,\&\, z = 1 \\ 4z^2 - 4, & \text{for } n = 3 \,\&\, z \geq 2 \\ \frac{24z-24nz+12z^2-24nz^2+12n^2z^2-3z^3+nz^3+3n^2z^3-n^3z^3}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Clearly $4z^2 - 4 > 0$ for $z \geq 2$. Let $f_4(n, z) = 24z - 24nz + 12z^2 - 24nz^2 + 12n^2z^2 - 3z^3 + nz^3 + 3n^2z^3 - n^3z^3$. Then $f_4(n, z) = \frac{1}{3}n^2z^3(9-n) + \frac{1}{3}nz^3(3-n^2) + \frac{1}{3}n^2z^2(36-nz) - 3z^3 + 12z^2(1-2n) + 24z(1-n) < 0$ for $n \geq 9$ and $z \geq 4$. We have $f_4(n, 1) = n^2(15 - n) + 33 - 47n < 0$ for $n \geq 15$. Also, $f_4(5, 1) = 48$, $f_4(7, 1) = 96$, $f_4(9, 1) = 96$, $f_4(11, 1) = 0$ and $f_4(13, 1) = -240$. Therefore

$$f_4(n, 1) \begin{cases} = 0, & \text{for } n = 11 \\ > 0, & \text{for } n = 5, 7, 9 \\ < 0, & \text{for } n \geq 13. \end{cases}$$

We have $f_4(n, 2) = 8n^2(9-n) + 72 - 136n < 0$ for $n \geq 9$, $f_4(5, 2) = 192$ and $f_4(7, 2) = -96$. Therefore, $f_4(n, 2) > 0$ or < 0 according as $n = 5$ or $n \geq 7$.

We have $f_4(n, 3) = 27n^2(7 - n) + 99 - 261n < 0$ for $n \geq 7$ and $f_4(5, 3) = 144$. Therefore, $f_4(n, 3) > 0$ or < 0 according as $n = 5$ or $n \geq 7$.

Again, we have $f_4(5, z) = 48z^2(4 - z) - 96z < 0$ and $f_4(7, z) = 48z^2(9 - 4z) - 144z < 0$ for $z \geq 4$. Therefore

$$f_4(n, z) \begin{cases} = 0, & \text{for } n = 11 \,\&\, z = 1 \\ > 0, & \text{for } n = 5, 7, 9 \,\&\, z = 1; n = 5 \,\&\, z = 3 \\ < 0, & \text{otherwise.} \end{cases}$$

Hence

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) \begin{cases} = 0, & \text{for } n = 3, 11 \,\&\, z = 1 \\ > 0, & \text{for } n = 3 \,\&\, z \geq 2; \\ & n = 5, 7, 9 \,\&\, z = 1;\ n = 5 \;\&\; z = 2, 3 \\ < 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.3 and (1.3), we also have

$$LE^+_{CN}(K_{|V(\Gamma_G)|}) - LE^+_{CN}(\Gamma_G) = \begin{cases} 0, & \text{for } n = 3 \,\&\, z = 1 \\ 4, & \text{for } n = 5 \,\&\, z = 1 \\ 4(z^2 - 1), & \text{for } n = 3 \,\&\, z \geq 2 \\ \frac{n^2+4n-21}{n+1}, & \text{for } n \geq 7 \,\&\, z = 1 \\ \frac{-n^3z^3+n^3z^2+3n^2z^3+9n^2z^2-6n^2z+nz^3-21nz^2-12nz+8n-3z^3+19z^2-6z+8}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Clearly $4(z^2-1)>0$ for $z\geq 2$ and $n^2+4n-21>0$ for $n\geq 7$. Let $f_5(n,z)=-n^3z^3+n^3z^2+3n^2z^3+9n^2z^2-6n^2z+nz^3-21nz^2-12nz+8n-3z^3+19z^2-6z+8$. Then $f_5(n,z)=\frac{1}{4}n^3z^2(4-z)+\frac{1}{4}n^2z^3(12-n)+\frac{1}{4}n^2z^2(36-nz)+\frac{1}{4}nz^3(4-n^2)-21nz^2+(8n-12nz)-6n^2z+z^2(19-3z)+(8-6z)<0$ for $n\geq 13$ and $z\geq 7$. We have $f_5(n,2)=4n^2(12-n)+48-92n<0$ for $n\geq 13$. Again $f_5(5,2)=288$, $f_5(7,2)=384$, $f_5(9,2)=192$ and $f_5(11,2)=-480$. Therefore, $f_5(n,2)>0$ or <0 according as $n=5,7,9$ or $n\geq 11$.

We have $f_5(n,3)=18n^2(8-n)+80-190n<0$ for $n\geq 9$. Again, $f_5(5,3)=480$ and $f_5(7,3)=-368$. Therefore, $f_5(n,3)>0$ or <0 according as $n=5$ or $n\geq 7$. We have $f_5(n,4)=24n^2(13-2n)+96-312n<0$ for $n\geq 7$. Also, $f_5(5,4)=336$. Therefore, $f_5(n,4)>0$ or <0 according as $n=5$ or $n\geq 7$. We have $f_5(n,5)=10n^2(57-10n)+78-452n<0$ for $n\geq 7$. Also, $f_5(5,5)=-432$. Therefore, $f_5(n,5)<0$. We have $f_5(n,6)=36n^2(26-5n)+8-604n<0$ for $n\geq 7$. Also, $f_5(5,6)=-2112$. Therefore, $f_5(n,6)<0$.

Now we shall check for $n=5,7,9,11$ and $z\geq 7$. We have $f_5(5,z)=24z^2(11-2z)+48-216z<0$, $f_5(7,z)=16z^2(41-12z)+64-384z<0$, $f_5(9,z)=8z^2(161-60z)+80-600z<0$ and $f_5(11,z)=96z^2(23-10z)+96-864z<0$, as $z\geq 7$. Thus, $f_5(n,z)>0$ if $n=5,7,9$ & $z=2$ and $n=5$ & $z=3,4$. Otherwise, $f_5(n,z)<0$.

Hence

$$LE^+_{CN}(K_{|V(\Gamma_G)|})-LE^+_{CN}(\Gamma_G)=\begin{cases}=0, & \text{for } n=3\,\&\,z=1\\ >0, & \text{for } n=5\ \&\ z=1;\ n=3\ \&\ z\geq 2;\\ & n\geq 7\ \&\ z=1;\ n=5,7,9\ \&\ z=2;\ n=5\ \&\ z=3,4\\ <0, & \text{otherwise.}\end{cases}$$

Hence the result follows. □

As a corollary of the above theorem we get the following results.

Corollary 3.2 *The CCC-graph of U_{6n} $(n\geq 2)$ is not (CNSL) CNL-hyperenergetic.*

Corollary 3.3 *Let $G=D_{2n},T_{4n},U_{6n}$, SD_{8n} or $U_{(n,m)}$. Then*

(a) *Γ_G is CNL-borderenergetic if and only if $G=D_6$ and D_{22}.*

(b) *Γ_G is CNL-hyperenergetic if and only if $G=D_{2n}$ for $n\geq 13$; T_{4n} for $n\geq 7$; SD_{8n} for $n\geq 4$ and $U_{(n,m)}$ except for $m=3\,\&\,n\geq 2$, $m=5\,\&\,n=2,3$, $m=4\,\&\,n\geq 2$, $m=8\,\&\,n=2$ and $m=6\,\&\,n\geq 2$.*

(c) *Γ_G is CNSL-borderenergetic if and only if $G=D_6$.*

(d) *Γ_G is CNSL-hyperenergetic if and only if $G=D_{2n}$ for n is even and $n\geq 20$; T_{4n} for $n\geq 10$; SD_{8n} for $n\geq 6$ and $U_{(n,m)}$ except for $m=3\,\&\,n\geq 2$, $m=5,7,9\,\&\,n=2$, $m=5\,\&\,n=3,4$, $m=4\,\&\,n\geq 2$, $m=8\,\&\,n=2$, $m=6\,\&\,n\geq 2$ and $m=10\,\&\,n=2$.*

Corollary 3.4 *The CCC-graph of V_{8n} $(n\geq 2)$ is CNL-hyperenergetic for $n\geq 6$ and CNSL- hyperenergetic for $n\geq 4$.*

Proof: Case 1: n is even

We have $|V(\Gamma_{V_{8n}})|=(2n+2)$. By Corollary 2.7 and (1.3), we get

$$LE_{CN}(K_{|V(\Gamma_{V_{8n}})|})-LE_{CN}(\Gamma_{V_{8n}})=\frac{120-32(n-4)(n-2)n}{n+1}\begin{cases}>0, & \text{for } 2\leq n\leq 4\\ <0, & \text{for } n\geq 6.\end{cases}$$

$$LE^+_{CN}(K_{|V(\Gamma_{V_{8n}})|})-LE^+_{CN}(\Gamma_{V_{8n}})=-\frac{6(n-1)(n(5n-19)+16)}{n+1}\begin{cases}>0, & \text{for } n=2\\ <0, & \text{for } n\geq 4.\end{cases}$$

Thus, $\Gamma_{V_{8n}}$ is not CNL-hyperenergetic if $n=2,4$ and $\Gamma_{V_{8n}}$ is CNL-hyperenergetic if $n\geq 6$. Also, it is not CNSL-hyperenergetic if $n=2$ and $\Gamma_{V_{8n}}$ is CNSL-hyperenergetic if $n\geq 4$.

Case 2: n is odd

We have $\Gamma_{V_{8n}}=K_{2n-1}\cup 2K_1=\Gamma_{D_{2\times 4n}}$. Then, by Corollary 3.3, we have that $\Gamma_{V_{8n}}$ is CNL-hyperenergetic if $n\geq 4$ and CNSL-hyperenergetic if $n\geq 5$. Hence, the result follows. □

3.1. Comparing various CN-energies

In this subsection, we compare various CN-energies of CCC-graphs of the groups considered in Section 2.

Theorem 3.4 *Let G be a finite group such that $|Z(G)| = z \geq 2$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. If $p = 2 \,\&\, z = 3$ or $p \geq 3 \,\&\, z = 2$ then $E_{CN}(\Gamma_G) < LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$. For all other cases, $E_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$.*

Proof: In view of Theorem 2.2, it is sufficient to compare $E_{CN}(\Gamma_G)$ and $LE_{CN}(\Gamma_G)$. By Theorem 2.2 and [28, Theorem 2.9], we have

$$LE_{CN}(\Gamma_G) - E_{CN}(\Gamma_G) = \begin{cases} 3, & \text{for } p = 2 \,\&\, z = 3 \\ \frac{8(p-2)(p+1)}{p^2}, & \text{for } p \geq 2 \,\&\, z = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $8(p-2)(p+1) = 0$ or > 0 according as $p = 2$ or $p > 2$. Hence, the result follows. □

As a corollary to Theorem 3.4 we have the following result.

Corollary 3.5 *Let G be a non-abelian p-group of order p^n and $|Z(G)| = p^{n-2}$, where p is a prime and $n \geq 3$. Then $E_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$.*

Theorem 3.5 *Let G be a finite group and $\frac{G}{Z(G)} \cong D_{2n}$ $(n \geq 3)$. If $n = 3 \,\&\, z \geq 1$ or $n = 5 \,\&\, z = 1$ (where $|Z(G)| = z$) then $E_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$. For all other cases, $E_{CN}(\Gamma_G) < LE^+_{CN}(\Gamma_G) < LE_{CN}(\Gamma_G)$.*

Proof: Case 1: n is even

In this case $z \geq 2$. By Theorem 2.3 and [28, Theorem 2.14], we have

$$LE^+_{CN}(\Gamma_G) - E_{CN}(\Gamma_G) = \begin{cases} \frac{8}{5}, & \text{for } n = 4 \,\&\, z = 2 \\ \frac{z^2(24z-91)+150z-120}{10}, & \text{for } n = 4 \,\&\, z \geq 3 \\ \frac{(n-2)(n-1)nz^3-(n(n(n+5)-17)+15)z^2+6(n+1)^2z-24(n+1)}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Let $f_1(z) = z^2(24z - 91) + 150z - 120$ and $f_2(n, z) = (n-2)(n-1)nz^3 - (n(n(n+5) - 17) + 15)z^2 + 6(n+1)^2 z - 24(n+1)$. Then $f_1(z) > 0$ for $z \geq 4$. Also, $f_1(3) = \frac{159}{10}$. Therefore, $f_1(z) > 0$ for $z \geq 3$. It can be seen that $f_2(n, z) = \frac{1}{3}n^3(z-3)z^2 + \frac{1}{3}(n-9)n^2z^3 + \frac{1}{3}n^2z^2(nz-15) + 6\left(n^2z-4\right) + 2nz^3 + (17n-15)z^2 + 12n(z-2) + 6z > 0$ for $n \geq 10$ and $z \geq 3$. Also, $f_2(n, 2) = 4(n-2)(n-3)^2 > 0$ for $n \geq 6$. We have $f_2(6, z) = z^2(120z - 309) + 294z - 168 > 0$ and $f_2(8, z) = z^2(336z - 711) + 486z - 216 > 0$ for $z \geq 3$. Therefore, $f_2(n, z) > 0$ for $n \geq 6$ and $z \geq 2$. Hence

$$LE^+_{CN}(\Gamma_G) - E_{CN}(\Gamma_G) > 0.$$

Again

$$LE^+_{CN}(\Gamma_G) - LE_{CN}(\Gamma_G) = \begin{cases} -\frac{8}{5}, & \text{for } n = 4 \,\&\, z = 2 \\ \frac{-z(29z-66)-40}{10}, & \text{for } n = 4 \,\&\, z \geq 3 \\ -\frac{(n((n-3)n+3)+1)z^2-6((n-2)n+3)z+8(n+1)}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Clearly, $-z(29z-66)-40 < 0$ for $z \geq 3$. Let $f_3(n, z) = -((n((n-3)n+3)+1)z^2-6((n-2)n+3)z+8(n+1))$. Then $f_3(n, z) = -\frac{1}{2}(n-6)n^2z^2 - \frac{1}{2}n^2z(nz-12) - 3nz^2 - 6(2n-3)z - 8n - z^2 - 8 < 0$ for $n \geq 6$ and $z \geq 2$. Therefore

$$LE^+_{CN}(\Gamma_G) - LE_{CN}(\Gamma_G) < 0.$$

Hence, $E_{CN}(\Gamma_G) < LE^+_{CN}(\Gamma_G) < LE_{CN}(\Gamma_G)$.
Case 2: n is odd

By Theorem 2.3 and [28, Theorem 2.14], we have

$$LE^+_{CN}(\Gamma_G) - E_{CN}(\Gamma_G) = \begin{cases} 0, & \text{for } n = 3, 5 \,\&\, z = 1 \\ 0, & \text{for } n = 3 \,\&\, z \geq 2 \\ \frac{(n-5)(n-3)}{n+1}, & \text{for } n \geq 7 \,\&\, z = 1 \\ \frac{z\left((n-3)(n-1)(n+1)z^2 - (n(n(n+5)-21)+23)z + 6(n+1)^2\right) - 16(n+1)}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Clearly $(n-5)(n-3) > 0$ for $n \geq 7$. Let $f_4(n,z) = z\left((n-3)(n-1)(n+1)z^2 - (n(n(n+5)-21)+23)z + 6(n+1)^2\right) - 16(n+1)$, where $n \geq 5$ and $z \geq 2$. Then $f_4(n,z) = \frac{1}{4}n^3(z-4)z^2 + \frac{1}{4}(n-12)n^2z^3 + \frac{1}{4}\left(n^2-4\right)nz^3 + \frac{1}{4}n^2z^2(nz-20) + 2\left(3n^2z-8\right) + (21n-23)z^2 + 4n(3z-4) + 3z^3 + 6z > 0$ for $n \geq 13$ and $z \geq 4$. We have $f_4(n,2) = 4(n-2)(n-3)^2 > 0$ as $n \geq 5$; $f_4(n,3) = 18n^2(n-6) + 182n - 124 > 0$ for $n \geq 7$; and $f_4(5,3) = 336$. Therefore, $f_4(n,3) > 0$. Further, for $z \geq 4$ we have $f_4(5,z) = 24z^2(2z-7) + 216z - 96 > 0$; $f_4(7,z) = 16z^2(12z-29) + 384z - 128 > 0$; $f_4(9,z) = 8z^2(60z-121) + 600z - 160 > 0$ and $f_4(11,z) = 192z^2(5z-9) + 864z - 192 > 0$. Therefore, $f_4(n,z) > 0$. Thus

$$LE^+_{CN}(\Gamma_G) - E_{CN}(\Gamma_G) \begin{cases} = 0, & \text{for } n = 3 \,\&\, z \geq 1;\ n = 5 \,\&\, z = 1 \\ > 0, & \text{otherwise.} \end{cases}$$

Again

$$LE^+_{CN}(\Gamma_G) - LE_{CN}(\Gamma_G) = \begin{cases} 0, & \text{for } n = 3 \,\&\, z = 1 \\ 0, & \text{for } n = 3 \,\&\, z \geq 2 \\ 0, & \text{for } n = 5 \,\&\, z = 1 \\ -\frac{(n-5)^2(n-3)}{2(n+1)}, & \text{for } n \geq 7 \,\&\, z = 1 \\ -\frac{(nz+z-4)(((n-4)n+7)z-2(n+1))}{2(n+1)}, & \text{otherwise.} \end{cases}$$

Clearly, $-(n-5)^2(n-3) < 0$ for $n \geq 7$. Let $f_5(n,z) = -(nz+z-4)(((n-4)n+7)z-2(n+1))$. Then $f_5(n,z) = -\frac{1}{2}(n-6)n^2z^2 - \frac{1}{2}n^2z(nz-12) - 3nz^2 - 6(2n-5)z - 8n - 7z^2 - 8 < 0$ for $n \geq 7$ and $z \geq 2$. For $z \geq 2$ we have $f_5(5,z) = -24(z-1)(3z-2) < 0$. Therefore, $f_5(n,z) < 0$. Thus

$$LE^+_{CN}(\Gamma_G) - LE_{CN}(\Gamma_G) \begin{cases} = 0, & \text{for } n = 3 \,\&\, z \geq 1;\ n = 5 \,\&\, z = 1 \\ < 0, & \text{otherwise.} \end{cases}$$

Hence, $E_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G)$, if $n = 3 \,\&\, z \geq 1$ or $n = 5 \,\&\, z = 1$. For all other cases, $E_{CN}(\Gamma_G) < LE^+_{CN}(\Gamma_G) < LE_{CN}(\Gamma_G)$. This completes the proof. □

We conclude this section with the following corollary.

Corollary 3.6 *Let $G = D_{2n}, T_{4n}, U_{6n}, SD_{8n}, V_{8n}$ or $U_{(n,m)}$. Then*

(a) $E_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G)$ *if and only if* $G = D_6, D_8, D_{10}, D_{12}, T_8, T_{12}, SD_{28}, V_{16}, U_{6n}$ *for* $n \geq 2$ *and* $U_{(n,m)}$ *for* $m = 3, 4, 6$ *and* $n \geq 2$.

(b) $E_{CN}(\Gamma_G) < LE^+_{CN}(\Gamma_G) < LE_{CN}(\Gamma_G)$ *if and only G is not among the groups listed in* (a).

In Figures 1–8, closeness of various CN-energies of CCC-graphs of $D_{2n}, T_{4n}, SD_{8n}, V_{8n}$ and $U_{(n,m)}$ are depicted.

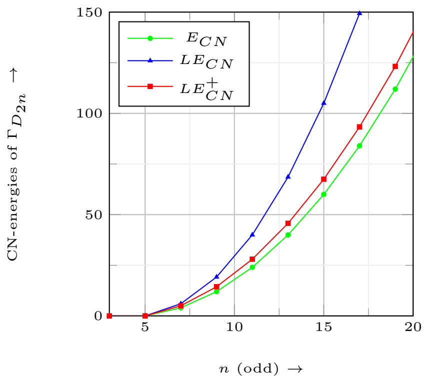

Figure 1: CN-energies of $\Gamma_{D_{2n}}$, n is odd

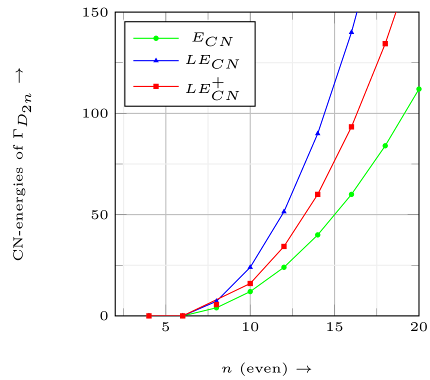

Figure 2: CN-energies of $\Gamma_{D_{2n}}$, n is even

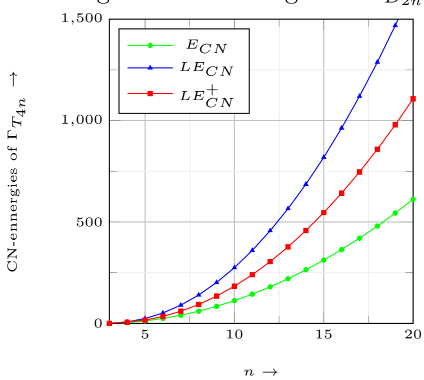

Figure 3: CN-energies of $\Gamma_{T_{4n}}$

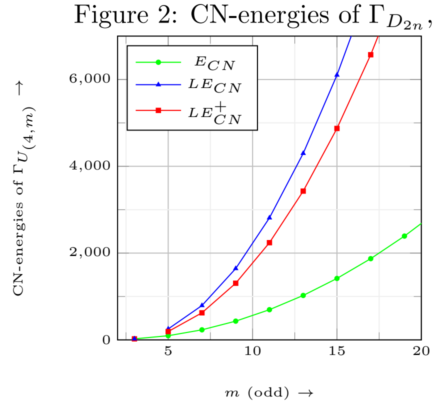

Figure 4: CN-energies of $\Gamma_{U_{(4,m)}}$, m is odd

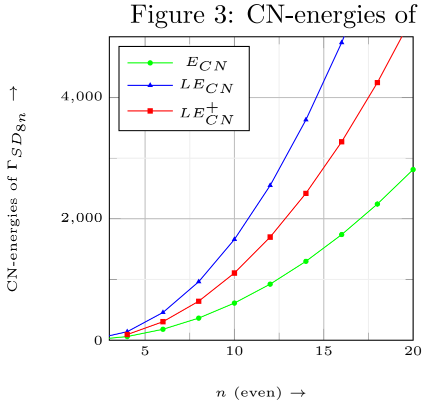

Figure 5: CN-energies of $\Gamma_{SD_{8n}}$, n is even

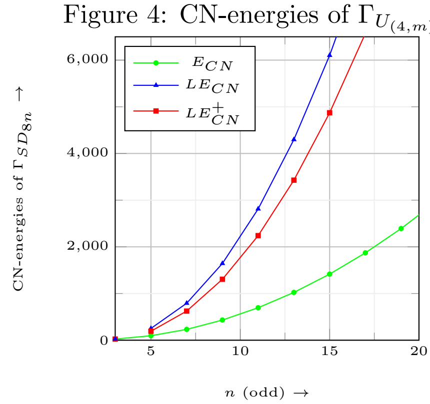

Figure 6: CN-energies of $\Gamma_{SD_{8n}}$, n is odd

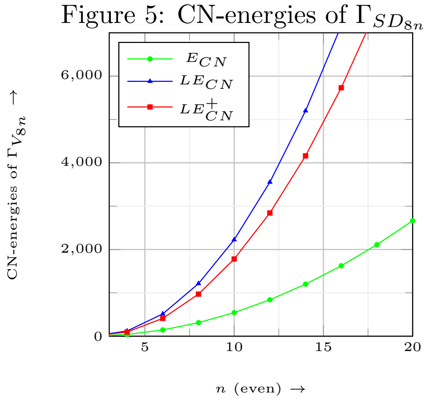

Figure 7: CN-energies of $\Gamma_{V_{8n}}$, n is even

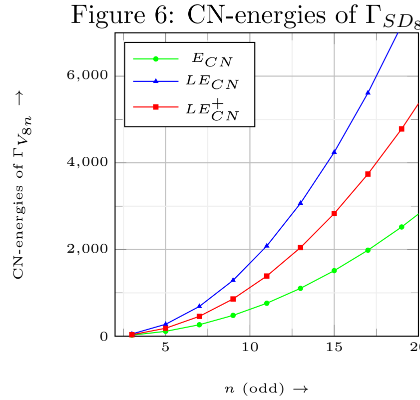

Figure 8: CN-energies of $\Gamma_{V_{8n}}$, n is odd

4. Conclusion

In this paper, we compute common neighborhood (signless) Laplacian spectrum and energy of CCC-graphs of certain finite non-abelian groups. We show that CCC-graphs of all the groups considered in this paper are CNL-integral and CNSL-integral. The common neighborhood spectrum and energy of CCC-graphs of theses groups are already computed in [28]. Analogous to the notion of super integral graph, we call a finite graph *super CN-integral* if it is CN-integral, CNL-integral and CNSL-integral. Thus, CCC-graphs of the groups considered in this paper are super CN-integral. It may be interesting to consider the following problem.

Problem 1 *Characterize all finite non-abelian groups G such that Γ_G is super CN-integral.*

The existence of finite non-abelian groups G such that Γ_G is CN-hyperenergetic is not clear (see [28]). However, there are finite non-abelian groups G such that Γ_G is CN-borderenergetic (See [28, Theorem 3.6]), CNL-hyperenergetic/CNL-borderenergetic and CNSL-hyperenergetic/CSNL-borderenergetic (See Corollary 3.3). Thus the following problem is worth considering.

Problem 2 *Characterize all finite non-abelian groups G such that Γ_G is CN-borderenergetic/ CNL-hyperenergetic/ CNL-borderenergetic/ CNSL-hyperenergetic/ CNL-borderenergetic.*

We have found several classes of finite non-abelian groups G such that $E_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$ in Subsection 3.1. Thus, we pose the following problem.

Problem 3 *Characterize all finite non-abelian groups G such that*

$$E_{CN}(\Gamma_G) = LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G).$$

In Subsection 3.1, we have also found several classes of finite non-abelian groups G such that $E_{CN}(\Gamma_G) < LE^+_{CN}(\Gamma_G) < LE_{CN}(\Gamma_G)$. In [4, Theorem 4.6], it was observed that there are several classes of finite non-abelian groups G such that $E(\Gamma_G) < LE^+(\Gamma_G) < LE(\Gamma_G)$. It follows that there exist finite non-abelian groups such that $E(\Gamma_G)$, $LE^+(\Gamma_G)$, $LE(\Gamma_G)$ and $E_{CN}(\Gamma_G)$, $LE^+_{CN}(\Gamma_G)$, $LE_{CN}(\Gamma_G)$ behave similarly. Thus, the following problem arises naturally.

Problem 4 *Determine all the finite non-abelian groups G such that $E(\Gamma_G)$, $LE^+(\Gamma_G)$, $LE(\Gamma_G)$ and $E_{CN}(\Gamma_G), LE^+_{CN}(\Gamma_G), LE_{CN}(\Gamma_G)$ behave similarly.*

It is worth noting that problem similar to Problem 4 can also be asked for any finite graph.

In [22], Gutman et al. conjectured that $E(\mathcal{G}) \leq LE(\mathcal{G})$ for any finite graph $\mathcal{G}$ but soon after the announcement, this conjecture was refuted [31,38]. For the groups G, we consider in this paper, we have

$$E_{CN}(\Gamma_G) \leq LE_{CN}(\Gamma_G). \tag{4.1}$$

In view of this it is too early to conjecture that the inequality (4.1) holds for CCC-graphs for any finite non-abelian group. However, one may consider the following problem.

Problem 5 *Determine all the finite non-abelian groups such that the inequality (4.1) does not hold. In general, determine all the finite graphs $\mathcal{G}$ such that the inequality $E_{CN}(\mathcal{G}) \leq LE_{CN}(\mathcal{G})$ does not hold.*

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References

1. Ahmadi, O., Alon, N., Blake, I. F. and Shaparlinski, I. E., *Graphs with integral spectrum*, Linear Algebra and its Application 430(1), 547-552, (2009).
2. Alwardi, A., Soner, N. D. and Gutman, I., *On the common-neighborhood energy of a graph*, Bulletin. Classe des Sciences Mathématiques et Naturelles. Sciences Mathématiques 36, 49-59, (2011).
3. Baghipur, M., Ghorbani, M., Ganie, H. A. and Shang, Y., *On the second-largest Reciprocal Distance Signless Laplacian Eigenvalue*, Mathematics 9, 512, (2021).
4. Bhowal, P. and Nath, R. K., *Spectral aspects of commuting conjugacy class graph of finite groups*, Algebraic Structures and Their Applications 8(2), 67-118, (2021).
5. Bhowal, P. and Nath, R. K. *Genus of commuting conjugacy class graph of certain finite groups*, Algebraic Structures and Their Applications 9(1), 93-108, (2022).
6. Cameron, P. J. *Graphs defined on groups*, International Journal of Group Theory 11(2), 53-107, (2022).
7. Das, K. C. and Mojallal, S. A., *On Laplacian energy of graphs*, Discrete Mathematics 325, 52-64, (2014).
8. Dutta, P., Bagchi, B. and Nath, R. K., *Various energies of commuting graphs of finite non-abelian groups*, Khayyam Journal of Mathematics 6(1), 27–45, (2020).
9. Dutta, P., Dutta, J. and Nath, R. K., *Laplacian spectrum of non-commuting graphs of finite groups*, Indian journal of pure and applied mathematics 49(2), 205-216, (2018).
10. Dutta, J. and Nath, R. K., *Finite groups whose commuting graphs are integral*, Matematički Vesnik 69(3), 226–230, (2017).
11. Dutta, J. and Nath, R. K., *Laplacian and signless Laplacian spectrum of commuting graphs of finite groups*, Khayyam Journal of Mathematics 4(1), 77–87, (2018).
12. Dutta, P. and Nath, R. K., *On Laplacian energy of non-commuting graphs of finite groups*, Journal of Linear and Topological Algebra 7(2), 121–132, (2018).
13. Fasfous, W. N. T. and Nath, R. K., Inequalities involving energy and Laplacian energy of non-commuting graphs of finite groups, *Indian Journal of Pure and Applied Mathematics*, 56(2), 791-812, (2025)
14. Fasfous, W. N. T., Nath, R. K. and Sharafdini, R., *Various spectra and energies of commuting graphs of finite rings*, Hacettepe Journal of Mathematics and Statistics 49(6), 1915-1925, (2020).
15. Fasfous, W. N. T., Sharafdini, R. and Nath, R. K., *Common neighborhood spectrum of commuting graphs of finite groups*, Algebra and Discrete Mathematics 32(1), 33-48, (2021).
16. Ganie, H. A. and Pirzada, S. *On the bounds for signless Laplacian energy of a graph*, Discrete Applied Mathematics 228(10), 3–13, (2017).
17. Ganie, H. A. and Shang, Y., *On the spectral radius and energy of signless Laplacian matrix of digraphs*, Heliyon 8(3), (2022).
18. Gong, S., Li, X., Xu, G., Gutman, I. and Furtula, B., *Borderenergetic Graphs*, MATCH Communications in Mathematical and in Computer Chemistry 74, 321-332, (2015).
19. Grone, R. and Merris, R. *The Laplacian spectrum of a graph II*, SIAM Journal of Discrete Mathematics 7(2), 221-299, (1994).
20. Gutman, I., *The energy of a graph*, Berichte der Mathematisch-Statistischen Sektion im Forschungszentrum Graz 103, 1-22, (1978).
21. Gutman, I., *Hyperenergetic molecular graphs*, Journal of the Serbian Chemical Society 64, 199-205, (1999).
22. Gutman, I., Abreu, N. M. M., Vinagre, C. T. M., Bonifacioa, A. S. and Radenkovic, S., *Relation between energy and Laplacian energy*, MATCH Communications in Mathematical and in Computer Chemistry 59, 343-354, (2008).
23. Gutman, I. and Furtula, B., *Survey of Graph Energies*, Mathematics Interdisciplinary Research 2, 85-129, (2017).
24. Gutman, I. and Furtula, B., *Graph energies and their applications*, Bulletin. Classe des Sciences Mathématiques et Naturelles. Sciences Mathématiques 44, 29-45, (2019).
25. Gutman, I. and Zhou, B., *Laplacian energy of a graph*, Linear Algebra and its Applications 414, 29-37, (2006).
26. Harary, F. and Schwenk, A. J., *Which graphs have integral spectra?*, Graphs and Combinatorics, Lect. Notes Maths. 406, 45-51, (1974).
27. Herzog, M., Longobardi, M. and Maj, M., *On a commuting graph on conjugacy classes of groups*, Communications in Algebra 37(10), 3369-3387, (2009).
28. Jannat, F. E. and Nath, R. K., *Common neighbourhood spectrum and energy of commuting conjugacy class graph*, Journal of Algebraic Systems 12(2), 301-326, (2025).

29. Jannat, F. E. and Nath, R. K., *Common neighbourhood Laplacian and signless Laplacian spectra and energies of commuting graph*, Palestine Journal of Mathematics (accepted for publication).

30. Jannat, F. E., Nath, R. K. and Das, K. C., *Common neighborhood energies and their relations with Zagreb index*, (submitted for publication) https://arxiv.org/abs/2402.15416.

31. Liu, J. and Liu, B., *On the relation between energy and Laplacian energy*, MATCH Communications in Mathematical and in Computer Chemistry 61, 403-406, (2009).

32. Mohammadian, A., Erfanian, A., Farrokhi, D. G. M. and Wilkens, B., *Triangle-free commuting conjugacy class graphs*, Journal of Group Theory 19, 1049-1061, (2016).

33. Nath, R. K., Fasfous, W. N. T., Das, K. C. and Shang, Y., *Common neighbourhood energy of commuting graphs of finite groups*, Symmetry 13(9), 1651-1662, (2021).

34. Salahshour, M. A. and Ashrafi, A. R., *Commuting conjugacy class graphs of finite groups*, Algebraic Structures and Their Applications 7(2), 135-145, (2020).

35. Salahshour, M. A. and Ashrafi, A. R., *Commuting conjugacy class graph of finite CA-groups*, Khayyam Journal of Mathematics 6(1), 108-118, (2020).

36. Salahshour, M. A., *Commuting conjugacy class graph of G when* $\frac{G}{Z(G)} \cong D_{2n}$, Mathematics Interdisciplinary Research 1, 379-385, (2020).

37. Simic, S. K. and Stanic, Z., *Q-integral graphs with edge-degrees at most five*, Discrete Mathematics 308, 4625-4634, (2008).

38. Stevanovic, D., Stankovic, I. and Milosevic, M., *On the relation between energy and Laplacian energy of graphs*, MATCH Communications in Mathematical and in Computer Chemistry 61, 395-401, (2009).

39. Tao, Q. and Hou, Y., *Q-borderenergetic graphs*, AKCE International Journal of Graphs and Combinatorics 17(1), 38-44, (2020).

40. Tura, F., *L-borderenergetic graphs*, MATCH Communications in Mathematical and in Computer Chemistry 77, 37-44, (2017).

41. Walikar, H. B., Ramane, H. S. and Hampiholi, P. R., *On the energy of a graph*, Graph connections, Eds. R. Balakrishnan, H. M. Mulder and A. Vijayakumar, Allied publishers, New Delhi, 120-123, (1999).

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