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Enriched Type Contractions in Convex Generalized Super Metric Spaces

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ABSTRACT: In this paper, we present the concept of a generalized super metric space. We then define a generalized enriched contraction within the framework of a convex generalized super metric space and propose several fixed point theorems for this new type of contraction. Additionally, we extend the Dass and Gupta rational contraction in the context of our newly introduced space. To illustrate our findings, we include examples. Finally, we apply our results to solve a fractional differential equation.

Key Words: Enriched contraction, super metric space, generalized contraction.

Contents

1. Introduction and preliminaries

Fréchet pioneered metric space theory in 1906 [12], laying its foundational concepts. Subsequently, numerous scholars have extended the concept of a metric space (MS) by relaxing certain conditions and adjusting the metric function, as evidenced by various works (refer to [4,9,20,14] for examples). In 1922, Banach demonstrated a significant result in fixed point theory, now known as the Banach contraction principle, proving that a contraction self-mapping on a complete MS possesses a unique fixed point. Due to its broad relevance, this principle has been further developed and generalized in various contexts (refer to [13,21,24,25] and references therein). In 1993, Czerwik introduced the concept of a b-MS as a generalization of MS.

Definition 1.1 *Let $A \neq \phi$ and $d : A \times A \to [0, +\infty)$ be a mapping. If for all $a, b, c \in A$*

1. *$0 \leq d(a, b)$ and $d(a, b) = 0$ if and only if $a = b$;*
2. *$d(a, b) = d(b, a)$;*
3. *there exists $s \geq 1$ such that*

$$d(a, b) \leq s(d(a, c) + d(b, c)).$$

Then, (A, d) is said to be a b-metric space.

To enlarge the notion of b-MS, Kamran et al. [16] proposed the concept of an extended b-MS in 2017.

Definition 1.2 *Let $A \neq \phi$ and $d : A \times A \to [0, +\infty)$ be a mapping. If for $a, b, c \in A$*

1. *$0 \leq d(a, b)$ and $d(a, b) = 0$ if and only if $a = b$;*
2. *$d(a, b) = d(b, a)$;*
3. *there exists a function $\phi : A \times A \to [1, +\infty)$ such that*

$$d(a, b) \leq \phi(a, b)(d(a, c) + d(b, c)).$$

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Then, (A, d) is said to be an extended b-metric space.

In 2019, Berinde expanded the scope of contractive mappings in Banach spaces by introducing enrichment, defining an enriched contraction or (b, θ)-enriched contraction for a self-mapping T on X. Here, the terms "enriched contraction" or "(b, θ)-enriched contraction" imply the existence of two constants, $b \in [0, +\infty)$ and $\theta \in [0, b+1)$, such that for all $x, y \in X$, the following inequality holds:

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\|.\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\|.$$

Recent years have seen an increasing interest in expanding the theory of enriched type contractions [22,23]. In 2022, Karapınar and Khojasteh [17] introduced a new generalization of metric space as follows.

Definition 1.3 *[17] Let $A \neq \phi$ and $d : A \times A \to [0, +\infty)$ be a mapping. If for $a, b, c \in A$*

1. *$0 \leq d(a, b)$ and $d(a, b) = 0$ if and only if $a = b$;*
2. *$d(a, b) = d(b, a)$;*
3. *for distinct sequences $\{a_n\}, \{b_n\} \in A$ such that $\lim_{n\to+\infty} d(a_n, b_n) = 0$, there exists $s \geq 1$ such that*

$$\lim_{n\to+\infty} \sup d(a_n, c) \leq s \lim_{n\to+\infty} \sup d(b_n, c),$$

for all $c \in A$.

Then, (A, d) is said to be a super metric space (SMS).

Karapinar and Fulga [18] investigated contractions represented in a rational form within the framework of SMS. They introduced definitions for convergence and Cauchy sequences in a SMS a follows.

Definition 1.4 *[17] Let $\{a_n\}$ be a sequence in a SMS (A, d). Then*

1. *the sequence $\{a_n\}$ is Cauchy if and only if $\lim_{n\to+\infty} \sup\{d(a_n, a_m), m > n\} = 0$.*
2. *the sequence $\{a_n\}$ converges to $a \in A$ if and only if $\lim_{n\to+\infty} d(a_n, a) = 0$.*

We say (A, d) is a complete SMS if and only if every Cauchy sequence converges in A.

In 1970, Takahashi [27] introduced the convexity structure on normed linear spaces, which has been very useful in the context of fixed point theory. Now, we give some basic preliminaries regarding convex metric spaces.

Definition 1.5 *[27] Let (A, d) be a MS. A continuous function $W : A \times A \times [0, 1] \to A$ is said to be a convex structure on A, if for all $a, b \in A$ and $\alpha \in [0, 1]$, the following inequality holds.*

$$d(u, W(a, b; \alpha)) \leq \alpha d(u, a) + (1-\alpha) d(u, b), \text{ for any } u \in A. \tag{1.1}$$

Any MS (A, d) along with a convex structure W is called a Takahashi CMS or simply a CMS and is usually denoted by (A, d, W).

The following lemmas present some basic properties of a CMS.

Lemma 1.1 *[2] Let (A, d, W) be a CMS. For each $a, b \in A$ and $\alpha, \alpha_1, \alpha_2 \in [0, 1]$, the following are true.*

1. *$W(a, a; \alpha) = a$; $W(a, b; 0) = b$ and $W(a, b; 1) = a$.*
2. *$|\alpha_1 - \alpha_2| d(a, b) \leq d(W(a, b; \alpha_1), W(a, b; \alpha_2))$.*

Lemma 1.2 *[2] Let (A, d, W) be a CMS and $T : A \to A$ be a self mapping. The mapping $T_\alpha : A \to A$ is defined by*

$$T_\alpha a = W(a, Ta; \alpha), \ a \in A. \tag{1.2}$$

Then, for any $\alpha \in [0, 1)$, we have $Fix(T) = Fix(T_\alpha)$.

Definition 1.6 *[18] A self-map T on a CMS (A, d, W) is said to be (α, θ)-enriched contraction if there exist $\alpha, \theta \in [0, 1)$, such that*

$$d(W(a, Ta, \alpha), W(b, Tb, \alpha)) \leq \theta \ d(a, b),$$

for all $a, b \in A$.

Motivated by the above results, in this paper, we first generalize the notion of a SMS. Then, we define a generalized (α, θ)-enriched type contraction in the setting of convex generalized super metric space and propose some fixed point results for these introduced contractions. An example is also provided to support our proposed results. In the end an application to fractional differential equations is provided.

2. Main results

We initiate by generalizing the notion of a SMS.

Definition 2.1 *Let $A \neq \phi$ and $d : A \times A \to [0, +\infty)$ be a mapping. If for $a, b, c \in A$*

1. *$0 \leq d(a, b)$ and $d(a, b) = 0$ if and only if $a = b$;*
2. *$d(a, b) = d(b, a)$;*
3. *for distinct sequences $\{a_n\}, \{b_n\} \in A$ such that $\lim_{n\to+\infty} d(a_n, b_n) = 0$, there exists a function $\phi : A \to [1, +\infty)$ so that*

$$\lim_{n\to+\infty} \sup d(a_n, c) \leq \phi(c) \lim_{n\to+\infty} \sup d(b_n, c)$$

for all $c \in A$.

Then, (A, d, ϕ) is said to be a generalized super metric space (GSMS).

Example 2.1 *Let $A = \mathbb{R}$ and define $d : A \times A \to [0, +\infty)$ such that*

$$d(a, b) = (a - b)^2 \text{ for } a, b \in \mathbb{R} - \{1\},$$
$$d(1, b) = d(b, 1) = (1 - b^3)^2 \text{ for } b \in \mathbb{R},$$

and $\phi : \mathbb{R} \to [1, +\infty)$ is defined as $\phi(a, b) = 1 + |a|$.
Clearly, d satisfies condition 1 and 2 of Definition 2.1. Now for condition 3, let $b \neq 1$, and $\{a_n\}, \{b_n\}$ be two sequences in A such that $\lim_{n\to+\infty} \sup d(a_n, b_n) = 0$. This implies that $\lim_{n\to+\infty} a_n = v = \lim_{n\to+\infty} b_n$, for some $v \in \mathbb{R}$. Then, we have

$$\begin{aligned} \lim_{n\to+\infty} \sup d(a_n, c) &= (v - c)^2 \\ &\leq \phi(c).(v - c)^2 \\ &\leq \phi(c) \lim_{n\to+\infty} \sup d(b_n, c) \end{aligned}$$

for all $c \in A$. For $b = 1$, taking both the sequences $\{a_n\}$ and $\{b_n\}$ to be same, we obtain the desired reult. Thus, (A, d, ϕ) is a GSMS.

Remark 2.1 *Clearly, if for all $c \in A$, $\phi(c) = s \geq 1$, then (A, d, ϕ) is a SMS, which leads us to conclude that every SMS is an GSMS.*

Lemma 2.1 *Let $\{a_n\}$ be a sequence in complete GSMS (A, d, ϕ). If $\lim_{n\to+\infty} d(a_n, a_{n+1}) = 0$ and $\lim_{n\to+\infty} \phi(a_n)$ exists. Then the sequence $\{a_n\}$ is Cauchy and converges in A.*

Proof: Let $\{a_n\}$ be a sequence in A such that $\lim_{n\to+\infty} d(a_n, a_{n+1}) = 0$.
Two cases are considered:
Case (i) Suppose the sequence $\{a_n\}$ is infinite. Using mathematical induction we show that

$$\lim_{p\to+\infty} d(a_p, a_{p+k}) = 0 \text{ for all } k > 0.$$

WLOG, we assume that $a_p \neq a_{p+k}$. For $k = 1$, we have $\lim_{p\to+\infty} d(a_p, a_{p+1}) = 0$. Letting $k = 2$, we have

$$\lim_{p\to+\infty} \sup d(a_p, a_{p+2}) \leq \lim_{p\to+\infty} \phi(a_{p+2}) \lim_{p\to+\infty} \sup d(a_{p+1}, a_{p+2}) = 0.$$

This implies that $\lim_{p\to+\infty} d(a_p, a_{p+2}) = 0$. Now, we assume that $\lim_{p\to+\infty} d(a_p, a_{p+k}) = 0$, for $k > 0$. Therefore, we have

$$\lim_{p\to+\infty} \sup d(a_p, a_{p+k+1}) \leq \lim_{p\to+\infty} \phi(a_{p+k+1}) d(a_p, a_{p+k}) = 0.$$

Consequently, we get

$$\lim_{p\to+\infty} \{\sup d(a_p, a_m); m > p\} = 0.$$

Case (ii) Suppose the sequence is finite. We need to find the pair (k, p), where $k, p \in \mathbb{N} \cup \{0\}$ and $k > p$ such that $a_p = a_k$. Picking (k_0, p_0), in such a way that $N = k_0 - p_0$ is minimum. We claim that for all $q \in \mathbb{N}$

$$a_{k_0+Nq} = a_{k_0}. \tag{2.1}$$

Indeed, for $q = 0$, $a_{k_0} = a_{p_0}$ and for $q = 1$, we have

$$a_{k_0+N} = a_{p_0} = a_{k_0}.$$

Let us assume that the result is true for all $q \in \mathbb{N}$. Also, we have

$$a_{k_0+N(q+1)} = a_{k_0+N} = a_{p_0} = a_{k_0}.$$

This proves our claim. Moreover, for $q \in \mathbb{N}$ and $k \in \{0, 1, 2, ..., N-1\}$, we obtain

$$a_{k_0+k+Nq} = a_{k_0+k}.$$

If $N = 1$, we get $k = 0$. Also, $a_{k_0+q} = a_{k_0}$ for all $q \geq 0$ implies $a_n = w$ for all $n \geq k_0$. So w is a limit point of the sequence $\{a_n\}$.
If $N \geq 2$, then we have $a_{k_0+j} \neq a_{k_0+l}$ for all $0 \leq j < l \leq N-1$. Also, N was assumed to be the least number so that (2.1) holds. In particular, taking $l = j + 1$, we have

$$2\epsilon = \min_{0\leq j\leq N-1} d(a_{k_0+j}, a_{k_0+j+1}) > 0.$$

By hypothesis, for $r_0 \in \mathbb{N}$ so that $r_0 \geq k_0$, we have

$$d(a_{r_0}, a_{r_0+1}) < \epsilon.$$

If $r_0 - k_0 = +j_0 \mod N$, then $j_0 \in \{0, 1, 2, .., N-1\}$, there exists $q \geq 1$ such that $r_0 = k_0 + j_0 + Nq$, also

$$a_{r_0} = a_{k_0+j_0+Nq} = a_{k_0+j_0}.$$

Where $k_0 + j_0 \in \{k_0, k_0+1, .., k_0+N-1\}$. Thus, similar as above we have

$$2\epsilon = \min_{0 \le j_0 \le N-1} d(a_{k_0+j_0}, a_{k_0+j_0+1}) \le d(a_{k_0+j_0}, a_{k_0+j_0+1}) < \epsilon.$$

Which leads to a contradiction.
Thus, the sequence $\{a_n\}$ is Cauchy. Since the GSMS (A, d, ϕ) is complete, there exists $a^* \in A$ such that $\lim_{n\to+\infty} d(a_n, a^*) = 0$. □

Now, we define a generalized enriched contraction in the setting of convex GSMS.

Definition 2.2 *Let T be a self-map on a convex GSMS (A, d, ϕ, W). Then T is said to be generalized enriched contration if for $\alpha \in [0, 1)$, there exists $\beta \in [0, 1)$ such that*

$$d(W(a, Ta, \alpha), W(b, Tb, \beta)) \le \theta\ d(a, b), \tag{2.2}$$

for all $a \ne b \in A$ and $\theta \in [0, 1)$.

If $\alpha = \beta$ and considering $a = b$ also, a generalized enriched contration turns out to be an enriched contraction.

Theorem 2.1 *Let T be a generalized enriched contraction on a complete convex GSMS (A, d, ϕ, W). For $a_0 \in A$, define the sequence $\{a_n\}$ as*

$$a_{2n+2} = W(a_{2n+1}, Ta_{2n+1}, \alpha),\ a_{2n+1} = W(a_{2n}, Ta_{2n}, \beta). \tag{2.3}$$

If $\lim_{n\to+\infty} \phi(a_n)$ exists, then the sequence $\{a_n\}$ is convergent and converges to the unique fixed point of T.

Proof: If $a_{2n} = a_{2n+1}$ for some $n \ge 1$, then from equation (2), we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(T_\alpha a_{2n}, T_\beta a_{2n+1}) \\ &\le \theta\ d(a_{2n}, a_{2n+1}) \\ &= 0. \end{aligned}$$

From this we get $a_{2n} = a_{2n+1} = a_{2n+2}$, which further implies $a_{2n} = T_\alpha a_{2n} = T_\beta a_{2n}$. Thus, a_{2n} is a common fixed point of T_α and T_β, and hence a fixed point of T.
Now, suppose $a_{2n} \ne a_{2n+1}$, so we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(T_\alpha a_{2n}, T_\beta a_{2n+1}) \\ &\le \theta\ d(a_{2n}, a_{2n+1}) \\ &\le \theta^2\ d(a_{2n-1}, a_{2n}) \\ &\le . \\ &. \\ &. \\ &\le \theta^{2n} d(a_0, a_1). \end{aligned}$$

Thus, we get $\lim_{n\to+\infty} d(a_{2n+1}, a_{2n+2}) = 0$. From Lemma 2.1, $\{a_n\}$ is a Cauchy sequence and hence converges in A, say to a^*. Again from equation (4), we have

$$\begin{aligned} d(T_\alpha a^*, a_{2n+1}) &= d(T_\alpha a^*, T_\beta a_{2n}) \\ &\le \theta\ d(a^*, a_{2n}). \end{aligned}$$

Letting $n \to +\infty$, we get $\lim_{n\to+\infty} d(T_\alpha a^*, a_{2n+1}) = 0$. Taking $b_n = T_\alpha a^*$, for all $n \in \mathbb{N}$, then from definition of GSMS, we have

$$\lim_{n\to+\infty} \sup d(a^*, T_\alpha a^*) \le \phi(a^*) \lim_{n\to+\infty} \sup d(a^*, a_n).$$

This implies $a^* = T_\alpha a^*$, i.e., a^* is a fixed point of T_α and hence of T.
Uniqueness: Let $a, b \in Fix(T)$, then $a = T_\alpha a$ and $b = T_\beta b$. From equation (2), we have

$$d(a,b) \leq \theta\ d(a,b),$$

which holds only if $d(a,b) = 0$, i.e., $a = b$. □

Example 2.2 *Let $A = [0,1]$ and define a mapping $d : A \times A \to [0, +\infty)$ by*

$$\begin{aligned} &d(a,b) = ab \text{ if } a,b \in (0,1) \\ &d(a,b) = 0 \text{ if and only if } a = b. \\ &d(a,0) = d(0,a) = a \text{ for all } a \in (0,1) \\ &d(a,1) = d(1,a) = 1 - \frac{a}{2} \text{ for all } a \in (0,1). \end{aligned}$$

Also, $\phi : A \to [1, +\infty)$ is defined as $\phi(a) = 1 + a$. Clearly, (A, d, ϕ) is a GSMS.
Now, we define a convex structure in the space. For $\alpha \in [0,1]$, take $W(a,b,\alpha) = (1-\alpha)a + \alpha b$. Let $u \in A$ be arbitrary.
Case(i) If $a, b \in (0,1)$, then clearly $W(a,b,\alpha) \in (0,1)$. For $u \in (0,1)$

$$\begin{aligned} d(u, W(a,b,\alpha)) &= u.((1-\alpha)a + \alpha b) \\ &= (1-\alpha)u.a + \alpha.b \\ &= (1-\alpha)d(u,a) + \alpha d(u,b). \end{aligned}$$

Case(ii) If $a = 0, b \in (0,1)$, then $W(a,b,\alpha) = \alpha b \in [0,1)$. Suppose $W(a,b,\alpha) = 0$, then for $u \in (0,1)$

$$\begin{aligned} d(u, W(a,b,\alpha)) &= u \\ &\leq (1-0)d(u,0) + 0.d(u,b). \end{aligned}$$

Suppose $W(a,b,\alpha) = \alpha.b \in (0,1)$, then for $u \in (0,1)$

$$\begin{aligned} d(u, W(a,b,\alpha)) &= u.\alpha.b \\ &\leq (1-\alpha)d(u,a) + \alpha d(u,b). \end{aligned}$$

Case(iii) If $b = 0, a \in (0,1)$, then $W(a,b,\alpha) = (1-\alpha)a \in [0,1)$. Suppose $W(a,b,\alpha) = 0$, then for $u \in (0,1)$

$$\begin{aligned} d(u, W(a,b,\alpha)) &= u \\ &\leq (1-1)d(u,0) + 1.d(u,0). \end{aligned}$$

Case(iv) If $a = 1, b \in (0,1)$, then $W(a,b,\alpha) = (1-\alpha) + \alpha b \in (0,1]$. Suppose $W(a,b,\alpha) = 1$, then for $u \in (0,1)$

$$\begin{aligned} d(u, W(a,b,\alpha)) &= 1 - \frac{u}{2} \\ &\leq (1-0)d(u,1) + 0.d(u,b). \end{aligned}$$

Suppose $W(a,b,\alpha) = (1-\alpha) + \alpha b \in (0,1)$, then for $u \in (0,1)$

$$\begin{aligned} d(u, W(a,b,\alpha)) &= ((1-\alpha) + \alpha b).u \\ &\leq (1-\alpha)d(u,1) + \alpha.d(u,b). \end{aligned}$$

Case(v) If $b = 1, a \in (0,1)$, then $W(a,b,\alpha) = (1-\alpha)a + \alpha \in (0,1]$. Suppose $W(a,b,\alpha) = 1$, then for $u \in (0,1)$

$$\begin{aligned} d(u, (a,b,\alpha)) &= 1 - \frac{u}{2} \\ &\leq (1-0)d(u,1) + 0.d(u,b). \end{aligned}$$

Suppose $W(a,b,\alpha)=(1-\alpha)a+\alpha\in(0,1)$, *then for* $u\in(0,1)$

$$\begin{aligned} d(u,W(a,b,\alpha)) &= (1-\alpha)a.u+\alpha.u \\ &\leq (1-0)d(u,a)+0.d(u,1). \end{aligned}$$

Similarly, we can prove for $u=0$ *and* $u=1$. *Thus,* (A,d,ϕ,W) *is a convex GSMS. Now, define* $T:A\to A$ *as* $Ta=\frac{a}{2}$. *In particular, taking* $\alpha=\frac{1}{2}$ *and* $\beta=\frac{1}{4}$, *we obtain* $T_\alpha a=W(a,Ta;\alpha)=\frac{3a}{4}$ *and* $T_\beta a=W(a,Ta;\beta)=\frac{5a}{8}$.

Case(i) If $a,b\in(0,1)$, *then* $d(a,b)=ab$ *and* $d(T_\alpha a,T_\beta b)=\frac{15a}{32}$, *so we have*

$$d(T_\alpha a,T_\beta b)\leq\theta d(a,b),$$

for $\frac{15}{32}\leq\theta\leq1$.
Case(ii) If $a=0,b\in(0,1)$, *then* $d(a,b)=b$ *and* $d(T_\alpha a,T_\beta b)=d(0,\frac{5b}{8})=\frac{5b}{8}$, *so we have*

$$d(T_\alpha a,T_\beta b)\leq\theta d(a,b),$$

for $\frac{5}{8}\leq\theta\leq1$.
Case(iii) If $b=0,a\in(0,1)$, *then* $d(a,b)=a$ *and* $d(T_\alpha a,T_\beta b)=d(\frac{3a}{4},0)=\frac{3a}{4}$, *so we have*

$$d(T_\alpha a,T_\beta b)\leq\theta d(a,b),$$

for $\frac{3}{4}\leq\theta\leq1$.
Case(iv) If $a=1,b\in(0,1)$, *then* $d(a,b)=1-\frac{b}{2}\geq\frac{b}{2}$ *and* $d(T_\alpha a,T_\beta b)=d(\frac{3}{4},\frac{5b}{8})=\frac{15b}{32}$, *so we have*

$$d(T_\alpha a,T_\beta b)\leq\theta d(a,b),$$

for $\frac{30}{32}\leq\theta\leq1$.
Case(v) If $b=1,a\in(0,1)$, *then* $d(a,b)=1-\frac{a}{2}\geq\frac{a}{2}$ *and* $d(T_\alpha a,T_\beta b)=d(\frac{3a}{4},\frac{5}{8})=\frac{15a}{32}$, *so we have*

$$d(T_\alpha a,T_\beta b)\leq\theta d(a,b),$$

for $\frac{30}{32}\leq\theta\leq1$.
We see that for $\max\{\frac{15}{32},\frac{5}{8},\frac{3}{4},\frac{30}{32}\}=\frac{30}{32}\leq\theta<1$, T *satisfies all the postulates of Theorem 2.1. Hence* T *has a unique fixed point, which is* 0.

Corollary 2.1 *Let* T *be a* (α,θ)*-enriched contraction on complete convex GSMS* (A,d,ϕ,W). *If for any sequence* $\{a_n\}$ *defined as*

$$a_{n+1}=W(a_n,Ta_n,\alpha),$$

for $a_0\in A$, $\lim\limits_{n\to+\infty}\phi(a_n)$ *exists. Then the sequence* $\{a_n\}$ *converges to the unique fixed point of* T.

Corollary 2.2 *Let* T *be a self-map on a complete GSMS* (A,d,ϕ). *Suppose* $\theta\in[0,1)$, *such that*

$$d(Tx,Ty)\leq\theta\ d(x,y), \tag{2.4}$$

for all $x,y\in A$. *Then* T *has a unique fixed point in* A.

Proof: Using contraction condition (4), T is continuous and $T_\alpha=T=T_\beta$. Following the proof of Theorem 2.1, we obtain that T has a unique fixed point.

□

Definition 2.3 *Let (A, d, ϕ) be a GSMS. A self-map T on A is said to be an interpolative Dass and Gupta rational contraction (IDGRC) if there exist $\alpha \in [0,1)$ and $\theta_1, \theta_2 \in [0,1)$ for which $\theta_1 + \theta_2 < 1$ such that*

$$d(Ta, Tb) \leq \theta \ (d(a,b))^{\theta_1} \left(\frac{[1 + d(a, Ta)d(b, Tb)]}{1 + d(a,b)} \right)^{\theta_2},$$

for all $a, b \in A$.

Definition 2.4 *Let (A, d, ϕ) be a convex GSMS. A self-map T on A is said to be (α,θ)-enriched IDGRC if there exists $\alpha, \theta \in [0,1)$ and $\theta_1, \theta_2 \in [0,1)$ for which $\theta_1 + \theta_2 < 1$ such that*

$$d(W(a, Ta, \alpha), W(b, Tb, \alpha)) \leq \theta \ (d(a,b))^{\theta_1} \left(\frac{[1 + d(a, W(a, Ta, \alpha))d(b, W(b, Tb, \alpha))]}{1 + d(a,b)} \right)^{\theta_2},$$

for all $a, b \in A$.

Definition 2.5 *Let (A, d, ϕ) be a convex GSMS. A self-map T on A is said to be generalized (α,β,θ)-enriched IDGRC if for $\alpha, \theta \in [0,1)$, there exists $\beta \in [0,1)$ and $\theta_1, \theta_2 \in [0,1)$ for which $\theta_1 + \theta_2 < 1$ such that*

$$d(W(a, Ta, \alpha), W(b, Tb, \beta)) \leq \theta \ (d(a,b))^{\theta_1} \left(\frac{[1 + d(a, W(a, Ta, \alpha))d(b, W(b, Tb, \beta))]}{1 + d(a,b)} \right)^{\theta_2}, \tag{2.5}$$

for all $a \neq b \in A \setminus FixT$.

If $\alpha = \beta$ and considering $a = b$, then a generalized enriched IDGRC turns out to be enriched IDGRC .

Theorem 2.2 *Let T be a generalized enriched IDGRC on a complete convex GSMS (A, d, ϕ, W). For $a_0 \in A$, define the sequence $\{a_n\}$ as*

$$a_{2n+2} = W(a_{2n+1}, Ta_{2n+1}, \alpha), \ a_{2n+1} = W(a_{2n}, Ta_{2n}, \beta).$$

Consider $\lim_{n \to +\infty} \phi(a_n)$ exists, then the sequence $\{a_n\}$ is convergent and converges to a unique fixed point of T.

Proof: If $a_{2n} = a_{2n+1}$ for some $n \geq 1$, then from (2.5), we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(T_\alpha a_{2n}, T_\beta a_{2n+1}) \\ &\leq \theta \ d(a_{2n}, a_{2n+1})^{\theta_1} \left(\frac{[1 + d(a_{2n}, a_{2n+1})].d(a_{2n+1}, a_{2n+2})}{1 + d(a_{2n}, a_{2n+1})} \right)^{\theta_2} \\ &= 0. \end{aligned}$$

From this we obtain $a_{2n} = a_{2n+1} = a_{2n+2}$, which implies $a_{2n} = T_\alpha a_{2n} = T_\beta a_{2n}$. Thus a_{2n} is a common fixed point of T_α and T_β and hence of T.
Suppose $a_{2n} \neq a_{2n+1}$, then we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &= d(T_\alpha a_{2n}, T_\beta a_{2n+1}) \\ &\leq \theta \ d(a_{2n}, a_{2n+1})^{\theta_1} \left(\frac{[1 + d(a_{2n}, a_{2n+1})].d(a_{2n+1}, a_{2n+2})}{1 + d(a_{2n}, a_{2n+1})} \right)^{\theta_2} \\ &= \theta \ d(a_{2n}, a_{2n+1})^{\theta_1}.d(a_{2n+1}, a_{2n+2})^{\theta_2}. \end{aligned}$$

Since $\theta_1 + \theta_2 < 1$, we have

$$(d(a_{2n+1}, a_{2n+2}))^{1-\theta_2} \leq \theta \ d(a_{2n}, a_{2n+1})^{\theta_1} \leq \theta \ d(a_{2n}, a_{2n+1})^{1-\theta_2}.$$

This implies that

$$
\begin{aligned}
d(a_{2n+1}, a_{2n+2}) &\leq \theta^{\frac{1}{1-\theta_2}}\ d(a_{2n}, a_{2n+1}) \\
&\leq \theta\ d(a_{2n}, a_{2n+1}) \\
&. \\
&. \\
&\leq \theta^{2n}\ d(a_0, a_1).
\end{aligned}
$$

Letting $n \to +\infty$, we obtain $d(a_{2n+1}, a_{2n+2}) \to 0$. From Lemma 2.1, the sequence $\{a_n\}$ is Cauchy and converges in A, say to a^*.
Again from equation (4), we have

$$
\begin{aligned}
d(T_\alpha a^*, a_{2n+1}) &= d(T_\alpha a^*, T_\beta a_{2n}) \\
&\leq \theta\ d(a^*, a_{2n})^{\theta_1} \left(\frac{[1 + d(a_{2n}, a_{2n+1})].d(a_{2n+1}, a_{2n+2})}{1 + d(a_{2n}, a_{2n+1})} \right)^{\theta_2}.
\end{aligned}
$$

Letting $n \to +\infty$, we get $\lim_{n\to+\infty} d(T_\alpha a^*, a_{2n+1}) = 0$. Let $b_n = T_\alpha a^*$, for all $n \in \mathbb{N}$, then from the definition of GSMS, we have

$$\lim_{n\to+\infty} \sup d(a^*, T_\alpha a^*) \leq \phi(a^*) \lim_{n\to+\infty} \sup d(a^*, a_n).$$

This gives $a^* = T_\alpha a^*$, i.e., a^* is a fixed point of T_α and hence of T.
Uniqueness: Let $a, b \in Fix(T)$. Then $a = T_\alpha a$ and $b = T_\beta b$. From equation (2), we obtain

$$d(a, b) \leq \theta\ d(a, b),$$

which holds only if $d(a, b) = 0$, that is $a = b$. Hence the fixed point is unique. □

Example 2.3 *Let $A = [1, 2]$ and define the mapping $d : A \times A \to [0, +\infty)$ as*

$$d(a, b) = \begin{cases} ab & \text{if } a \neq b \\ 0 & \text{if } a = b. \end{cases}$$

Also, $\phi : A \to [1, +\infty)$ is defined as $\phi(a) = 1 + a$. Clearly, d satisfies condition (1) and (2). Now for condition (3), let $\{a_n\}$ and $\{b_n\}$ be distinct sequences in A such that $\lim_{n\to+\infty} d(a_n, b_n) = 0$, implies $\lim_{n\to+\infty} a_n b_n = 0$. WLOG assume that $\lim_{n\to+\infty} a_n = 0$ and $\lim_{n\to+\infty} b_n = u$ for some $u \in A$. Moreover, for $b \in A$, we have

$$\lim_{n\to+\infty} \sup d(b_n, b) = 0 \leq \phi(b) \lim_{n\to+\infty} \sup d(a_n, b).$$

Now we prove that (A, d, ϕ) is convex. Let $W(a, b; \alpha) = (1 - \alpha)a + \alpha b$, then if $u \neq W(a, b; \alpha)$

$$
\begin{aligned}
d(u, W(a, b; \alpha)) &= (1 - \alpha)a + \alpha b).u \\
&= (1 - \alpha)a.u + \alpha b.u \\
&= (1 - \alpha)d(u, a) + \alpha d(u, b).
\end{aligned}
$$

If $u = W(a, b; \alpha)$, then

$$
\begin{aligned}
d(u, W(a, b; \alpha)) &= 0 \\
&\leq (1 - \alpha)a.u + \alpha b.u \\
&= (1 - \alpha)d(u, a) + \alpha d(u, b).
\end{aligned}
$$

Thus, (A,d,ϕ,W) is a convex GSMS. Now, define a self-map T on A such that $Ta = 2-\frac{a}{2}$. In particular, taking $\alpha=\beta=\frac{1}{3}$ we get $T_\alpha = T_\beta = \frac{4}{3}$. Therefore, for all $a,b\in[1,2]$, we have

$$\begin{aligned} d(W(a,Ta,\alpha),W(b,Tb,\beta)) &= 0 \\ &\leq \theta\ (d(a,b))^{\theta_1}\left(\frac{[1+d(a,W(a,Ta,\alpha))d(b,W(b,Tb,\beta))]}{1+d(a,b)}\right)^{\theta_2}, \end{aligned}$$

for all $\theta,\theta_1,\theta_2\in[0,1)$ such that $\theta_1+\theta_2<1$. By Theorem 2.1, T has a unique fixed point, which is $\frac{4}{3}$.

Corollary 2.3 *Let T be an (α,θ)-enriched IDGRC on a complete convex GSMS (A,d,ϕ,W). If for the sequence $\{a_n\}$ defined as*

$$a_{n+1} = W(a_n,Ta_n,\alpha),$$

$a_0\in A$, $\lim_{n\to+\infty}\phi(a_n)$ exists. Then the sequence $\{a_n\}$ converges to the unique fixed point of T.

Proof: Taking $\alpha=\beta$ and following the proof of Theorem 2.2, we obtain the desired result. □

Corollary 2.4 *Let T be an IDGRC on a complete convex GSMS (A,d,ϕ,W). If for a Picard sequence $\{a_n\}$*

$$a_{n+1} = Ta_n,$$

$a_0\in A$, $\lim_{n\to+\infty}\phi(a_n)$ exists. Then the sequence $\{a_n\}$ converges to the unique fixed point of T.

Proof: Taking $\alpha=0=\beta$ and following the proof of Theorem 2.2, we obtain the desired result. □

3. Application

Now, we show the existence and uniqueness of solution of a Caputo Atangana-Baleanu fractional differential equation using fixed point results.

$$\begin{aligned} D_t^\alpha f(t) &= g(t,f(t)), t\in I=[0,1] \\ f(0) &= a. \end{aligned} \tag{3.1}$$

Here $g: I\times\mathbb{R}\to\mathbb{R}$ and $f: I\to\mathbb{R}$ are continuous functions satisfying $g(0,x(0))=0$, $\alpha\in(0,1)$ and a is a constant. Consider $X=C(I,\mathbb{R})$ to be the space of continuous function on the interval I and $d(x,y)=\sup_{t\in I}|x(t)-y(t)|^2$, where $\phi(x)=2+\sup_{t\in I}|x(t)|$. Let the mapping $W: X\times X\times[0,1]\to X$ be defined as $W(x,y;\lambda)=\lambda x+(1-\lambda)y$, for all $x,y\in X$. It is clear that (X,d,ϕ,W) is a complete convex GSMS.

Definition 3.1 *[3,19,26] Consider $f\in H^1(a,b)$ with $a<b$ and $\alpha\in[0,1]$. The Caputo Atangana-Baleanu fractional derivative of f of order α is defined by*

$$D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx.$$

Here $B(\alpha)$ is a normalizing positive function satisfying $B(0)=B(1)=1$ and E_α is the Mittage-Leffler function defined by $E_\alpha(y)=\sum_{n=0}^{+\infty}\frac{y^n}{\Gamma(\alpha n+1)}$. Then, the associated fractional integral is defined as

$$I_t^\alpha f(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\,{}_tI^\alpha f(t).$$

Here ${}_tI^\alpha$ is the left Riemann-Liouville fractional integral, which is defined as

$${}_tI^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)dx.$$

Proposition 3.1 *[1] For* $0 < \alpha < 1$

$$I_a^\alpha D_a^\alpha f(x) = f(x) - f(a)E_\alpha(\lambda(x-a)^\alpha) - \frac{\alpha}{1-\alpha} f(a)x^\alpha E_{\alpha,\alpha+1}(\lambda(x-a)^\alpha)$$
$$= f(x) - f(a).$$

Similarly, $I_b^\alpha D_b^\alpha f(x) = f(x) - f(b)$.

Theorem 3.1 *Let* $X = C(I, \mathbb{R})$ *and consider problem (3.1) as defined above. Suppose that*

$$|g(s, f_1(s)) - g(s, f_2(s))| \leq \left[\frac{B(\alpha)\Gamma(\alpha)}{1-\lambda}\left(\frac{1}{((1-\alpha)\Gamma(\alpha)+1)} - \frac{1}{4}\right)\right].|f_1(s) - f_2(s)|.$$

for all $f_1, f_2 \in C(I, \mathbb{R})$ *and* $t \in [0,1]$ *and* $\lambda \in [0,1)$*. Then, the problem (3.1) has a unique solution* $f(t) \in C([0,1], \mathbb{R})$.

Proof: From Proposition 3.1, on operating the Atangana–Baleanu integral to both sides of (3.1), we obtain

$$f(t) = a + I_t^\alpha g(t, f(t)).$$

Defining $T : X \to X$ by

$$(Tf)(t) = a + I_t^\alpha g(t, f(t)). \tag{3.2}$$

Consider a mapping $W : X \times X \times [0,1] \to X$, which is defined as $W(x, y; \lambda) = \lambda x + (1-\lambda)y$, for all $x, y \in X$. We know that f is a solution of problem (3.1) if $f \in C([0,1], \mathbb{R})$ is a fixed point of T.

$$\begin{aligned}
d(T_\lambda f_1, T_\lambda f_2) &= \sup_{t\in I} |(T_\lambda f_1)(t) - (T_\lambda f_2)(t)|^2 \\
&= \sup_{t\in I} |\lambda f_1(t) + (1-\lambda)(Tf_1)(t) - \lambda f_2(t) - (1-\lambda)(T_\lambda f_2)(t)|^2 \\
&= \sup_{t\in I} |\lambda(f_1(t) - f_2(t)) + (1-\lambda)((Tf_1)(t) - (Tf_2)(t))|^2 \\
&\leq \sup_{t\in I} \left[\lambda|f_1(t) - f_2(t)| + (1-\lambda)\sup_{t\in I}|(Tf_1)(t) - (Tf_2)(t)|\right]^2 \\
&\leq \sup_{t\in I} [\lambda|f_1(t) - f_2(t)| + (1-\lambda)|I_t^\alpha[g(s, f_1(s)) - g(s, f_2(s))]|]^2 \\
&\leq \sup_{t\in I} [\lambda|f_1(t) - f_2(t)| + (1-\lambda) \\
&\qquad \left|\frac{1-\alpha}{B(\alpha)}[g(t, f_1(t)) - g(t, f_2(t))] + \frac{\alpha}{B(\alpha)}{}_tI^\alpha[g(t, f_1(t)) - g(t, f_2(t))]\right|\Bigg]^2 \\
&\leq \sup_{t\in I} [\lambda|f_1(t) - f_2(t)| + (1-\lambda) \\
&\qquad \left(\frac{1-\alpha}{B(\alpha)}\right)\left(\frac{B(\alpha)\Gamma(\alpha)}{((1-\alpha)\Gamma(\alpha)+1)(1-\lambda)} - \frac{B(\alpha)\Gamma(\alpha)}{4(1-\lambda)}\right).|f_1(s) - f_2(s)| + \\
&\qquad \frac{\alpha}{B(\alpha)}\left[\frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}[g(t, f_1(t)) - g(t, f_2(t))]dx\right]\Bigg]^2 \\
&\leq \sup_{t\in I} [\lambda|f_1(t) - f_2(t)| + (1-\lambda) \\
&\qquad \left(\frac{1-\alpha}{B(\alpha)}\right)\left(\frac{B(\alpha)\Gamma(\alpha)}{((1-\alpha)\Gamma(\alpha)+1)(1-\lambda)} - \frac{B(\alpha)\Gamma(\alpha)}{4(1-\lambda)}\right).|f_1(s) - f_2(s)| + \\
&\qquad \frac{\alpha}{B(\alpha)}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\left\{\left(\frac{B(\alpha)\Gamma(\alpha)}{((1-\alpha)\Gamma(\alpha)+1)(1-\lambda)}\right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{B(\alpha)\Gamma(\alpha)}{4(1-\lambda)}\right).|f_1(s) - f_2(s)|dx\right\}\right]\Bigg]^2
\end{aligned}$$

$$\leq \left[\lambda + (1-\lambda)\left\{\left(\frac{(1-\alpha)\Gamma(\alpha)}{((1-\alpha)\Gamma(\alpha)+1)(1-\lambda)}\right) + \left(\frac{1}{6((1-\alpha)\Gamma(\alpha)+1)(1-\lambda)}\right) - \frac{(1-\alpha)\Gamma(\alpha)}{4(1-\lambda)} - \frac{1}{4(1-\lambda)}\right\}\right]^2 d(f_1, f_2)$$

$$\leq \left[1 - \frac{(1-\alpha)}{4} - \frac{1}{4}\right]^2 d(f_1, f_2)$$

$$\leq \left[\frac{1}{2} + \frac{\alpha}{4}\right]^2 d(f_1, f_2).$$

Clearly, $\theta = \left[\frac{1}{2} + \frac{\alpha}{4}\right]^2 \in (0,1)$, since $\alpha \in (0,1)$. Therefore, by Corollary 2.1, T has a unique fixed point, i.e., the fractional differential equation (3.1) has a unique solution. □

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References

1. Abdeljawad, T. & Baleanu, D. (2017) Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.* 10, 1098–1107.
2. Agarwal, R. P., O'Regan, D., & Sahu, D. R. (2009). Fixed point theory for Lipschitzian-type mappings with application. In *Topological Fixed Point Theory and Its Applications* (p. 6). Springer.
3. Atangana, A. & Baleanu, D. (2016). New fractional derivative with non-local and non-singular kernal, *Thermal Sci.* 20(2), 757–763.
4. Bakhtin, I.A. (1989). The contraction mapping principle in almost metric spaces. *Functional Analysis*, 30, 26-–37.
5. Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales. *Fundamenta Mathematicae*, 3, 133–181.
6. Berinde, V., & Pacurar, M. (2020). Approximating fixed points of enriched contractions in Banach spaces. *Journal of Fixed Point Theory and Applications*, 22, 1–10.
7. Berinde, V., & Pacurar, M. (2021). Existence and Approximation of Fixed Points of Enriched Contractions and Enriched ϕ-Contractions. *Symmetry*, 13, 498.
8. Berinde, V. (2019). Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces. *Carpathian Journal of Mathematics*, 35, 293–304.
9. Branciari, A. (2000). A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. *Publicationes Mathematicae*, 57(1–2), 31-–37.
10. Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, 1, 5–11.
11. Dass, B. K., & Gupta, S. (1975). An extension of Banach contraction principle through rational expressions. *Indian Journal of Pure and Applied Mathematics*, 6, 1455-1458.
12. Fréchet, M. (1906). Sur quelques points du calcul fonctionnel. *Rendiconti del Circolo Matematico di Palermo*, 22(1), 1–72.
13. Gangwar, A., Rawat, S., & Dimri, R.C. (2023). Solution of differential inclusion problem in controlled S-metric spaces via new multivalued fixed point theorem. *Journal of Analysis*, 31, 2459–2472.
14. Gangwar, A., Rawat, S., Isik, H., & Dimri, R.C. (2024). Enriched multivalued contractions on double controlled metric type spaces with an application. *Adv. Fixed Point Theory*, 14.
15. Jleli, M., & Samet, B. (2015). A generalized metric space and related fixed point theorems. *Fixed Point Theory and Applications*, 2015, 61.
16. Kamran, T., Samreen, M., & UL Ain, Q. (2017). A Generalization of b-metric space and some fixed point theorems. *Mathematics*, 5, 1–7.
17. Karapınar, E., & Khojasteh, F. (2022). Super Metric Spaces. *Filomat*, 36(10), 3545–3549.
18. Karapinar, E., & Fulga, A. (2022). Contraction in Rational Forms in the Framework of Super Metric Spaces. *Mathematics*, 10, 3077.

19. Kilbas, A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and application of fractional differential equations, North Holland Math. Stud.

20. Matthews, S.G. (1994). Partial metric topology. *Annals of the New York Academy of Sciences-Paper Edition*, 728, 183–197.

21. Rawat, S., Dimri, R.C. & Bartwal, A. (2022). F-Bipolar metric spaces and fixed point theorems with applications. *Journal of Mathematical and Computational Science*, 26(2), 184-195.

22. Rawat, S., Kukreti, S. & Dimri, R.C. (2022). Fixed point results for enriched ordered contractions in noncommutative Banach spaces. *J. Anal.* 30, 1555–1566.

23. Rawat, S., Bartwal, A. & Dimri, R.C. (2023). Approximation and existence of fixed points via interpolative enriched contractions. *Filomat*, 37(16), 5455–5467.

24. Rus, I.A. (1979). Principles and applications of the Fixed Point Theory; Editura Dacia: Cluj-Napoca, Romania.

25. Rus, I.A. (1991). Basic problems of the metric fixed point theory revisited (II). *Studia Universitatis Babeş-Bolyai Mathematica* , 36, 81–99.

26. Samko, S. G., Kilbas, A. A. & Marichev, O. (1993). Fractional integrals and derivatives: theory and applications, Yverdon: Gordon and Breach.

27. Takahashi, W. (1970). A convexity in metric space and nonexpansive mappings, I. *Kodai Mathematical Seminar Reports*, 22(2), 142–149.

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