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Stability and topological conjugacy for affine differential equations

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ABSTRACT: This paper discusses stability properties of affine autonomous ordinary differential equations and generalizes a classical result on topological conjugacy for hyperbolic linear autonomous equations to the affine case.

Key Words: affine differential equations, topological conjugacy.

Contents

1. Introduction

For linear autonomous differential equations, stability theory and classification with respect to topological conjugacy are classical topics in the theory of differential equations; see e.g. Robinson [7], Hirsch, Smale, and Devaney [6] or the recent lecture notes [4]. It is the purpose of this paper to expose analogous results for affine autonomous differential equations of the form

$$\dot{x} = Ax + a, \tag{1}$$

where $A \in \mathfrak{gl}(d, \mathbb{R})$ and $a \in A$. We characterize different stability notions for a corresponding equilibrium, which here, in contrast to the homogenous case $a = 0$, is, in general, different from the origin. Furthermore, we characterize when for two systems of the form (1) the associated affine flows are topologically conjugate. For hyperbolic matrices A (i.e., there are no eigenvalues on the imaginary axis) this is the case, iff the dimensions of the stable subspaces coincide.

In Section 2, we discuss basic properties of affine differential equations and via transformation to Jordan canonical form, their stability properties in relation to their eigenvalues. Section 3 characterizes topological conjugacy under the assumption that the matrix A is hyperbolic.

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2. Affine differential equation

In this section we prove that the real part of the eigenvectors determines the exponential behavior of the solutions of an affine differential equation, described by the Lyapunov exponents.

We begin by recalling some facts about affine differential equations. A differentiable function $x : \mathbb{R} \to \mathbb{R}^d$ such that $\dot{x}(t) = Ax(t) + a$ for all $t \in \mathbb{R}$ is called a solution of (1). The initial value problem for a linear differential equation $\dot{x} = Ax + a$ consists in finding, for a given initial value $x_0 \in \mathbb{R}^d$, a solution $x(\cdot, x_0)$ such that $x(0, x_0) = x_0$.

It is well known (see, e.g., Lecture 16 of Agarwal and Gupta [1]) that for each initial value problem given by $(A, a) \in \mathfrak{gl}(d, \mathbb{R}) \times \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$, the solution $x(\cdot, x_0)$ is unique and given by

$$x(t, x_0) = e^{At} x_0 + \int_0^t e^{A(t-s)} a ds.$$

The distinct eigenvalues of $A \in \mathfrak{gl}(d, \mathbb{R})$ will be denoted by $\mu_1, \mu_2, \ldots, \mu_r \in \mathbb{C}$. The real versions of the generalized eigenspaces are denoted by $E(A, \mu_k) \subset \mathbb{R}^d$ or simply E_k for $k = 1, \ldots, r \leq d$. A matrix $A \in \mathfrak{gl}(d, \mathbb{R})$ is similar to a matrix in real Jordan form denoted by $J_A^{\mathbb{R}}$. This means that there exists a matrix $T \in \mathrm{Gl}(d, \mathbb{R})$ such that $A = T^{-1} J_A^{\mathbb{R}} T$ and $J_A^{\mathbb{R}}$ is a block diagonal matrix,

$$J = \mathrm{blockdiag}(J_1, ..., J_l),$$

with real Jordan blocks given for a real eigenvalue λ by

$$J_i = \begin{pmatrix} \lambda & 0 & . & . & . & 0 \\ 1 & \lambda & & & & . \\ . & . & . & & & . \\ . & & . & . & & . \\ . & & & . & \lambda & 0 \\ 0 & & & & 1 & \lambda \end{pmatrix},$$

and for a complex conjugate pair $\mu, \bar{\mu} = \lambda \pm i\nu, \nu > 0$, of eigenvalues by

$$J_i = \begin{pmatrix} \begin{matrix} \lambda & -\nu \\ \nu & \lambda \end{matrix} & 0 & . & . & . & 0 \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} \lambda & -\nu \\ \nu & \lambda \end{matrix} & & & & . \\ . & . & . & & & . \\ . & & . & . & & . \\ . & & & . & \begin{matrix} \lambda & -\nu \\ \nu & \lambda \end{matrix} & 0 \\ 0 & & & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} \lambda & -\nu \\ \nu & \lambda \end{matrix} \end{pmatrix}.$$

See e.g. [4].

Note that for $A = T^{-1}J_A^{\mathbb{R}}T$ and $a \in \mathbb{R}^d$ one has

$$\begin{aligned} e^{At} + \int_0^t e^{A(t-s)} a ds &= T^{-1}e^{J_A^{\mathbb{R}}t}T + \int_0^t T^{-1}e^{J_A^{\mathbb{R}}t}T \cdot T^{-1}e^{J_A^{\mathbb{R}}(-s)}Ta\ ds \\ &= T^{-1}e^{J_A^{\mathbb{R}}t}T + \int_0^t T^{-1}e^{J_A^{\mathbb{R}}(t-s)}Ta\ ds. \end{aligned}$$

We illustrate this result by looking at the following example. Consider a Jordan block B of dimension $2m$ associated with the complex eigenvalue pair $\mu, \bar{\mu} = \lambda \pm i\nu$ of a matrix $A \in \mathfrak{gl}(d, \mathbb{R})$. Let

$$= \begin{pmatrix} \lambda & -\nu \\ \nu & \lambda \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \hat{D} = \begin{pmatrix} \cos\nu t & -\sin\nu t \\ \sin\nu t & \cos\nu t \end{pmatrix}.$$

Then for

$$B = \begin{pmatrix} D & I & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & \cdot & I \\ & & & & D \end{pmatrix} \text{ one has } e^{Bt} = e^{\lambda t} \begin{pmatrix} \hat{D} & t\hat{D} & \frac{t^2}{2!}\hat{D} & \cdot & \cdot & \frac{t^{m-1}}{(m-1)!}\hat{D} \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \frac{t^2}{2!}\hat{D} \\ & & & & \cdot & t\hat{D} \\ & & & & & \hat{D} \end{pmatrix}.$$

More explicitly, consider the solution $y(t, y_0)$ of

$$\dot{y}(t) = By(t) + b,\ y(0) = y_0,$$

where $y_0 = [y_1, z_1, \ldots, y_m, z_m]^t \in E(A, \mu, \bar{\mu})$ and $b = [a_1, b_1, \ldots, a_m, b_m]^t$.

Then for $j = 1, \ldots, m$ the j-th components given by

$$\begin{aligned} y_j(t, y_0) &= e^{\lambda t} \textstyle\sum_{k=j}^m \frac{t^{k-j}}{(k-j)!}(y_k \cos\nu t - z_k \sin\nu t) \\ &+ \textstyle\int_0^t e^{\lambda(t-s)} \sum_{k=j}^m \frac{(t-s)^{k-j}}{(k-j)!}[a_k \cos\nu(t-s) - b_k \sin\nu(t-s)]ds \end{aligned}$$

and

$$\begin{aligned} z_j(t, y_0) &= e^{\lambda t} \textstyle\sum_{k=j}^m \frac{t^{k-j}}{(k-j)!}(z_k \cos\nu t + y_k \sin\nu t) \\ &+ \textstyle\int_0^t e^{\lambda(t-s)} \sum_{k=j}^m \frac{(t-s)^{k-j}}{(k-j)!}[a_k \cos\nu(t-s) + b_k \sin\nu(t-s)]ds. \end{aligned}$$

For a better illustration we take $m = 2$. Then

$$\begin{aligned} y_1(t, y_0) &= e^{\lambda t}[y_1 \cos\nu t - z_1 \sin\nu t) + t(y_2 \cos\nu t - z_2 \sin\nu t)] \\ &+ \textstyle\int_0^t e^{\lambda(t-s)}[a_1 \cos\nu(t-s) - b_1 \sin\nu(t-s)]ds \\ &+ \textstyle\int_0^t e^{\lambda(t-s)}(t-s)[a_2 \cos\nu(t-s) - b_2 \sin\nu(t-s)]ds \end{aligned}$$

and

$$\begin{aligned} z_1(t, y_0) &= e^{\lambda t}[y_1 \cos\nu t + z_1 \sin\nu t + t(y_2 \cos\nu t + z_2 \sin\nu t)] \\ &+ \textstyle\int_0^t e^{\lambda(t-s)}[a_1 \cos\nu(t-s) + b_1 \sin\nu(t-s)]ds \\ &+ \textstyle\int_0^t e^{\lambda(t-s)}(t-s)[a_2 \cos\nu(t-s) + b_2 \sin\nu(t-s)]ds. \end{aligned}$$

Similarly,

$$y_2(t, y_0) = e^{\lambda t}[y_2\cos\nu t - z_2\sin\nu t] + \int_0^t e^{\lambda(t-s)}[a_2\cos\nu(t-s) - b_2\sin\nu(t-s)]ds$$

and

$$z_2(t, y_0) = e^{\lambda t}[y_2\cos\nu t + z_2\sin\nu t) + \int_0^t e^{\lambda(t-s)}[y_2\cos\nu(t-s) + z_2\sin\nu(t-s)]ds.$$

Then using mathematical software or direct computation, we get

$$\begin{aligned} y_1(t, y_0) &= e^{\lambda t}[f_1^y(t) + tg_1^y(t)] + C_1^y, \\ z_1(t, y_0) &= e^{\lambda t}(f_1^z(t) + tg_1^z(t)) + C_1^z, \\ y_2(t, y_0) &= e^{\lambda t} f_2^y(t) + C_2^y, \\ z_2(t, y_0) &= e^{\lambda t} f_2^z(t) + C_2^z; \end{aligned}$$

here for $i = 1, 2$, the functions $f_i^y, g_i^y, f_1^z, g_1^z$ are bounded $\mathbb{R}$ and C_i^y, f_i^z are constants.

We proceed to analyze stability properties.

Definition 2.1 *A point $e_0 \in \mathbb{R}^d$ is a fixed point of the affine differential equation $\dot{x}(t) = Ax(t) + a$ if $x(t, e_0) = e_0$ for all $t \in \mathbb{R}$.*

Recall that for the linear equation $\dot{x} = Ax$, every solution tends for $t \to \infty$ to the origin, if the spectrum $\sigma(A)$ is contained in the negative complex halfplane $\mathbb{C}_- := \{z \in \mathbb{C}, \mathrm{Re}z < 0\}$, i.e., if all eigenvalues of A have negative real parts. The following proposition gives an analogous result for affine differential equations.

Proposition 2.1 *Suppose A stable, that is, $\mathrm{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$. Then*

i) There exists a unique fixed point for $\dot{x}(t) = Ax(t) + a$,

ii) For all $x_0 \in \mathbb{R}^d$, $x(t, x_0) \to e_0$ if $t \to \infty$.

Proof: Since $0 \notin \sigma(A)$, the matrix A is invertible and the equation $0 = Ae_0 + a$ has the unique solution $e_0 = -A^{-1}a$. Since $e_0 = e^{At}e_0 + \int_0^t e^{A(t-s)}ads$, it follows for all $x_0 \in \mathbb{R}^d$ that

$$\begin{gathered} \|x(t, x_0) - e_0\| \\ = \|e^{At}x_0 + \int_0^t e^{A(t-s)}ads - e^{At}e_0 - \int_0^t e^{A(t-s)}ads\| \\ = \|e^{At}(x_0 - e_0)\| \to 0 \end{gathered}$$

if $t \to \infty$. □

Next we discuss, how fast the solutions approach the equilibrium and define Lyapunov exponents, which measure exponential growth rates.

Definition 2.2 *Let $x(\cdot, x_0)$ be a solution of the affine differential equation $\dot{x}(t) = Ax(t) + a$. Suppose that A is invertible and hence A has a unique fixed point e_0. The Lyapunov exponent for x_0 is defined as*

$$\lambda(x_0) = \lim\,\sup_{t\to\infty} \frac{1}{t} \ln\|x(t, x_0) - e_0\|.$$

The Lyapunov exponents are determined by the Jordan structure of the matrix A.

Theorem 2.1 *Suppose that A is invertible and denote by e_0 its unique fixed point. Define $L(\lambda_j)$ as the sum of the real generalized eigenspaces for all eigenvalues of A with real parts equal to λ_j. Then the Lyapunov exponent $\lambda(x_0)$ of a solution $x(\cdot, x_0)$ of $\dot{x}(t) = Ax(t) + a$ satisfies $\lambda(x_0) = \lim_{t\to\pm\infty} \frac{1}{t} \ln\|x(t, x_0) - e_0\| = \lambda_j$ if and only if $x_0 - e_0 \in L(\lambda_j)$.*

Proof: Recall that for any matrix A there is a matrix $T \in \mathrm{Gl}(d, \mathbb{R})$ such that $A = T^{-1} J_A^{\mathbb{R}} T$, where $J_A^{\mathbb{R}}$ is the real Jordan canonical form of A. Hence we can consider A in the real Jordan form. Then the assertions of the theorem follow from the solution formulas. We give an idea of these computations for a Jordan block corresponding to a complex-conjugate pair of eigenvalues with $m = 2$, as in the above. Take the above solutions $y_1(t, y_0), z_1(t, y_0), y_2(t, y_0), z_2(t, y_0)$ and note that

$$\|y(t, y_0)\| = \sqrt{y_1^2 + z_1^2 + y_2^2 + z_2^2} =$$

$$\sqrt{(e^{\lambda t}(f_1^y + t g_1^y) + C_1^y)^2 + (e^{\lambda t}(f_1^z + t g_1^z) + C_1^z)^2 + (e^{\lambda t} f_2^y + C_2^y)^2 + (e^{\lambda t} f_2^z + C_2^z)^2}.$$

Then isolating $(e^{\lambda t})^2$ inside the root, the last expression can be written as

$$\|y(t, y_0)\| = \sqrt{(e^{\lambda t})^2 f(t)}.$$

Hence

$$\frac{1}{t} \ln\|y(t, y_0)\| = \frac{1}{t} \ln\sqrt{(e^{\lambda t})^2 f(t)} = \frac{1}{t} \ln\sqrt{(e^{\lambda t})^2} + \frac{1}{t} \ln\sqrt{f(t)},$$

where $\frac{1}{t} \ln\sqrt{f(t)} \to 0$ for $t \to \infty$.

Therefore, $\lim_{t\to\infty} \frac{1}{t} \ln\|y(t, y_0)\| = \lambda$. By this computation, it is easy to see that $\lim_{t\to\pm\infty} \frac{1}{t} \ln\|y(t, y_0) - e_0\| = \lambda$. □

With the next lemma, some of our results in affine differential equation will be an immediate consequence of the correspondent result in the linear context.

Lemma 2.1 *Let $x(t, x_0), t \in \mathbb{R}$, be the solution of the system $\dot{x}(t) = Ax(t) + a$ and e_0 its fixed point. Then $x(t, x_0) - e_0, t \in \mathbb{R}$, is a solution of $\dot{x}(t) = Ax(t)$, that is, $\frac{d}{dt}[x(t, x_0) - e_0] = A[x(t, x_0) - e_0]$.*

Proof: Note that $\frac{d}{dt}[x(t, x_0) - e_0) = Ax(t, x_0) + a$. On the other hand, $0 = Ae_0 + a$. Then $a = -Ae_0$. Hence $Ax(t, x_0) + a = Ax(t, x_0) - Ae_0 = A[x(t, x_0) - e_0]$. □

As in case of linear differential equation (see [4]), in the following result we characterize asymptotic and exponential stability in terms of the eigenvalue of A.

Theorem 2.2 *For an affine differential equation $\dot{x}(t) = Ax(t) + a$ in $\mathbb{R}^d$ the following statements are equivalent:*
i) The fixed point $e_0 \in \mathbb{R}^d$ is asymptotically stable.
ii) The fixed point $e_0 \in \mathbb{R}^d$ is exponentially stable.
iii) All Lyapunov exponents (hence all real parts of the eigenvalues) are negative.
iv) The stable subspace L^- satisfies $L^- = \mathbb{R}^d$.

Proof: Take Φ as solution of the system $\dot{x}(t) = Ax(t) + a$. By above lemma, $\Phi(t, y_0) - e_0$ is a solution of the linear system $\dot{x}(t) = Ax(t)$, where $x - e_0$ is the initial value of the solution $\Phi(t, y_0) - e_0$. Then this theorem is an immediate consequence of Theorem 2.15 in [4]. □

3. Conjugacy for affine differential equations

In this section we study the affine differential equation $\dot{x}(t) = Ax(t) + a$, with $(A, a) \in \mathfrak{gl}(d, \mathbb{R}) \times \mathbb{R}^d$, from the point of view of dynamical systems, or flows in $\mathbb{R}^d$.

First we introduce the affine flow associated to this differential equation.

Lemma 3.1 *For $A \in \mathfrak{gl}(d, \mathbb{R})$ and $a \in \mathbb{R}^d$, the solutions of $\dot{x}(t) = Ax(t) + a$ form a continuous dynamical system with time set $\mathbb{R}$ and state space $M = \mathbb{R}^d$.*

Proof: The map $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\Phi(t, x) = x(t, x) = e^{At}x + \int_0^t e^{A(t-s)} a ds$$

has, as claimed, the following properties:
(i) $\Phi(0, x) = x$, for all $x \in \mathbb{R}^d$,
(ii) $\Phi(u + t, x) = \Phi(u, \Phi(t, x))$ for all $u, t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. In fact,

$$\Phi(u, \Phi(t, x)) = e^{A(u+t)}x + \int_0^t e^{A(u+t-s)} a ds + \int_0^u e^{A(u-s)} a ds.$$

But

$$\int_0^u e^{A(u-s)} a ds = \int_0^{u+t} e^{A(u+t-s)} a ds.$$

In fact, call $t - s = -v$ then $s = t + v$. Hence $ds = dv$ and, if $s = t$ then $v = 0$, and, if $s = u + t$ then $v = u$. This implies $\int_0^{u+t} e^{A(u+t-s)} a ds = \int_0^u e^{A(u-v)} a dv$ and therefore

$$\Phi(u + t, x) = \Phi(u, \Phi(t, x)).$$

(iii) $\Phi(t, x)$ is continuous by its definition. □

Recall that two C^k flows Φ and Ψ, $k \geq 0$, are C^k conjugate if there is a C^k map h with C^k inverse such that for $t \in \mathbb{R}$

$$h \circ \Phi(t, \cdot) = \Psi(t, h(\cdot)).$$

The flows are called topologically conjugate if $k = 0$, and they are called linearly conjugate, if the conjugation map h is linear.

Theorem 3.1 *Consider the dynamical systems* Φ *and* Ψ *associated with* $\dot{x}(t) = Ax(t) + a$ *and* $\dot{x}(t) = Bx(t) + b$*, respectively, where* $A, B \in \mathfrak{gl}(d, \mathbb{R})$ *and* $a, b \in \mathbb{R}^d$*. Assume that* A *and* B *are invertible, hence there are unique equilibria* e_A *and* e_B*, respectively. Then the following statements are equivalent:*

(i) the flows Φ *and* Ψ *are* C^k *conjugate for* $k \geq 1$*;*

(ii) the flows Φ *and* Ψ *are linearly conjugate;*

(iii) the flows Φ *and* Ψ *are affinely similar, that is,* $A = TBT^{-1}$ *and* $Ta = b$ *for some* $T \in \mathrm{Gl}(d, \mathbb{R})$*.*

Proof: The flows are given by

$$\Phi(t,x) = e^{At}x + \int_0^t e^{A(t-s)}a\ ds \text{ and } \Psi(t,x) = e^{Bt}x + \int_0^t e^{B(t-s)}b\ ds.$$

Hence, for a conjugation map h satisfies for all t and all x

$$h(e^{At}x + \int_0^t e^{A(t-s)}a\ ds) = e^{Bt}h(x) + \int_0^t e^{B(t-s)}b\ ds.$$

Observe that (ii) obviously implies (i). Next we prove that (iii) implies (ii). Note the equivalences

$$A = TBT^{-1} \Leftrightarrow e^{tA} = Te^{tB}T^{-1} \Leftrightarrow T^{-1}e^{tA} = e^{tB}T^{-1} \tag{2}$$

(here $\Longleftarrow$ is seen by differentiating and evaluating in $t = 0$). Then define a linear map $h : \mathbb{R}^d \to \mathbb{R}^d$ as $h(z) = T^{-1}z$. Then the conjugation property follows from

$$\begin{aligned} & h(e^{At}x + \int_0^t e^{A(t-s)}ads) = T^{-1}(e^{At}x + \int_0^t e^{A(t-s)}a\ ds) \\ = \ & T^{-1}e^{At}x + T^{-1}\int_0^t e^{A(t-s)}a\ ds = e^{Bt}T^{-1}x + \int_0^t T^{-1}e^{A(t-s)}a\ ds \\ = \ & e^{Bt}T^{-1}x + \int_0^t e^{B(t-s)}T^{-1}a\ ds = e^{Bt}h(x) + \int_0^t e^{B(t-s)}b\ ds. \end{aligned}$$

Next, supposing (ii) we prove that (iii) holds. By (ii) there is a linear conjugacy h such that for all t and all x

$$h(e^{At}x + \int_0^t e^{A(t-s)}a\ ds) = e^{Bt}h(x) + \int_0^t e^{B(t-s)}b\ ds. \tag{3}$$

Differentiating with respect to x, we find for all t

$$Dh(e^{At}x + \int_0^t e^{A(t-s)}a\ ds)e^{At} = e^{Bt}Dh(x). \tag{4}$$

Observing that h is linear, we see with $T^{-1} := Dh(0)$

$$T^{-1}e^{At} = Dh(0)e^{At} = e^{Bt}Dh(0) = e^{Bt}T^{-1} \tag{5}$$

and hence, by (2),

$$A = TBT^{-1}.$$

Inserting into (3), we find for all t and all x

$$T^{-1}(e^{At}x + \int_0^t e^{A(t-s)}a\ ds) = e^{Bt}T^{-1}x + \int_0^t e^{B(t-s)}b\ ds,$$

which, with (5), implies for all t

$$T^{-1}\int_0^t e^{A(t-s)}a\ ds = e^{Bt}T^{-1}x - T^{-1}e^{At}x + \int_0^t e^{B(t-s)}b\ ds = \int_0^t e^{B(t-s)}b\ ds.$$

Then using (5) again, one finds for all t

$$\begin{aligned} & e^{Bt}\int_0^t e^{-Bs}b\ ds = \int_0^t e^{B(t-s)}b\ ds = T^{-1}\int_0^t e^{A(t-s)}a\ ds = T^{-1}e^{At}\int_0^t e^{-As}a\ ds \\ = \ & e^{Bt}T^{-1}\int_0^t e^{-As}a\ ds = e^{Bt}\int_0^t T^{-1}e^{-As}a\ ds = e^{Bt}\int_0^t e^{-Bs}T^{-1}a\ ds. \end{aligned}$$

This implies that $e^{-Bt}b = e^{-Bt}T^{-1}a$ for all t and hence $b = T^{-1}a$.

Suppose that (i) holds. In order to show (iii), let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a C^k-conjugacy, $k \geq 1$. Thus for all $x \in \mathbb{R}^d$ and $t > 0$

$$h(\Phi(t,x)) = h(e^{At}x) = e^{Bt}h(x) = \Psi(h(x)).$$

Differentiating with respect to x we find as in (4)

$$Dh(e^{At}x + \int_0^t e^{A(t-s)}ads)e^{At} = e^{Bt}Dh(x).$$

Evaluating this at $x = e_0$ we get with $H := Dh(e_0)$

$$He^{At} = e^{Bt}H \text{ for all } t \in \mathbb{R}.$$

Differentiation with respect to t in $t = 0$ finally gives $HA = BH$. Then similar computations as in the proof above yield (ii). Since h is a diffeomorphism, the linear map $H = Dh(0)$ is invertible and hence defines a linear conjugacy. □

In particular, Theorem 3.1, shows a conjugacy relation for the dynamical system obtained when we replace the matrix A by its real Jordan matrix.

Corollary 3.1A *Consider $\dot{x}(t) = Ax(t) + a$, with $A \in \mathfrak{gl}(d,\mathbb{R})$, and let Φ its associated dynamical system. Let Ψ be the dynamical system for*

$$\dot{x} = J_A^{\mathbb{R}}x + T^{-1}a,$$

where $A = TJ_A^{\mathbb{R}}T^1, T \in Gl(d,\mathbb{R})$, is the associated matrix in real Jordan normal form. Then there is a linear conjugacy h for Φ and Ψ.

Proof: This follows from the equivalence of conditions (ii) and (iii) in Theorem 3.1. □

We note the following result on existence of an adapted norm.

Proposition 3.1 *Denote the dynamical system associated to $\dot{x} = Ax + a$, $A \in \mathfrak{gl}(d, \mathbb{R}), a \in \mathbb{R}^d$, by Φ, and assume that A is invertible. Then there exists a unique equilibrium $e_0 = -A^{-1}a$ and the following properties are equivalent:*

(i) there are a norm $\|\cdot\|_*$ on $\mathbb{R}^d$ and $\alpha > 0$ such that for all $x \in \mathbb{R}^d$

$$\|\Phi(t, x) - e_0\|_* \leq e^{-\alpha t}\|x - e_0\|_* \text{ for all } t \geq 0;$$

(ii) for every norm $\|\cdot\|$ on $\mathbb{R}^d$ there are $\alpha > 0$ and $C > 0$ such that for all $x \in \mathbb{R}^d$

$$\|\Phi(t, x) - e_0\| \leq Ce^{-\alpha t}\|x - e_0\| \text{ for all } t \geq 0;$$

(iii) for every eigenvalue λ of A one has $\mathrm{Re}\lambda < 0$.

Proof: The assertion for e_0 is immediate. Item (i) implies (ii), since all norms on $\mathbb{R}^d$ are equivalent. Items (ii) and (iii) are equivalent by Theorem 2.2. It remains to show that (ii) implies (i). Clearly, $\Phi(t, x_0) - e_0$ is the solution of the linear equation $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0 - e_0$. Hence Proposition 3.17 in [4] shows that there exist a norm $\|\cdot\|_*$ on $\mathbb{R}^d$ and $\alpha > 0$ satisfying (i). □

Next we prove existence of topological conjugacies for stable affine systems.

Proposition 3.2 *Consider the dynamical systems Φ and Ψ associated with $\dot{x}(t) = Ax(t) + a$ and $\dot{x}(t) = Bx(t) + b$, respectively, where $A, B \in \mathfrak{gl}(d, \mathbb{R})$ and $a, b \in \mathbb{R}^d$. If all eigenvalues of A and of B have negative real parts, then the flows Φ and Ψ are topologically conjugate.*

Proof: Recall that the flows Φ and Ψ are given by

$$\Phi(t, x) = e^{At}x + \int_0^t e^{A(t-s)}a ds \text{ and } \Psi(t, x) = e^{Bt}x + \int_0^t e^{B(t-s)}b ds.$$

If all eigenvalues of A and B have negative real parts, then by Proposition 3.19 in [4] there exists a homeomorphism h with $h(e^{At}x) = e^{Bt}h(x)$ for all t and all x. Furthermore, there are unique equilibria e_A and e_B, respectively.

Note that $\Phi(t, x) - e_A$ is the solution of $\dot{x}(t) = Ax(t)$ with initial value $x - e_A$; analogously, $\Psi(t, x) - e_B$ is the solution of $\dot{x}(t) = Bx(t)$ with initial value $x - e_B$ and it follows that

$$\Phi(t, x) - e_A = e^{At}(x - e_A) \text{ and } \Psi(t, x) - e_B = e^{Bt}(x - e_B). \tag{6}$$

The conjugation property of h implies that

$$h(e^{At}(x - e_A)) = e^{Bt}h(x - e_A) = e^{Bt}(h(x - e_A) + e_B - e_B).$$

Using (6) we can write this as

$$h(\Phi(t,x)-e_A)=\Psi(t,h(x-e_A)+e_B)-e_B. \tag{7}$$

Define $H(x)=h(x-e_A)+e_B$. Then by (7) this map satisfies the conjugation property $H(\Phi(t,x))=\Psi(t,H(x))$.

Since h is bijective, continuous, invertible and with continuous inverse, the same is true for H. Therefore H is a topological conjugacy. □

The next theorem presents the main result of this paper. It shows that for affine differential equations with hyperbolic matrices having the same dimension of the stable subspace, topological conjugacy follows.

Theorem 3.2 *Consider the dynamical systems Φ and Ψ associated with $\dot{x}=Ax+a$ and $\dot{x}=Bx+b$, respectively, where $A,B\in\mathfrak{gl}(d,\mathbb{R})$ and $a,b\in\mathbb{R}^d$. Suppose A and B are hyperbolic. Then Φ and Ψ are topologically conjugate if and only if the dimensions of the stable subspaces (and hence the dimensions of the unstable subspaces) of A and B agree.*

Proof: If A and B are hyperbolic then A and B has no eigenvalues with zero real part. Then we can decompose $\mathbb{R}^d$ as $\mathbb{R}^d=\mathbb{E}^s_A\oplus\mathbb{E}^u_A$ and $\mathbb{R}^d=\mathbb{E}^s_B\oplus\mathbb{E}^u_B$; here $\mathbb{E}^s_A$ and $\mathbb{E}^u_A$ denote the stable and unstable subspace associated with A and analogously for B. Denote the natural projections by $\pi^s_A:\mathbb{R}^d\to\mathbb{E}^s_A$ and $\pi^u_A:\mathbb{R}^d\to\mathbb{E}^u_A$ and analogously for B.

These stable and unstable subspaces are invariant under e^{At} and e^{Bt}, respectively. Consider the affine differential equations

$$\dot{x}=A|_{\mathbb{E}^s_A}x+\pi^s_A a \text{ in } \mathbb{E}^s_A,\ \dot{x}=A|_{\mathbb{E}^u_A}x+\pi^u_A a \text{ in } \mathbb{E}^u_A, \tag{8}$$

and

$$\dot{x}=B|_{\mathbb{E}^s_B}x+\pi^s_B b \text{ in } \mathbb{E}^s_B,\ \dot{x}=B|_{\mathbb{E}^u_B}x+\pi^u_B b \text{ in } \mathbb{E}^u_A \tag{9}$$

Note that solutions of $\dot{x}=Ax+a$ can uniquely be written as the sum of solutions of the equations in (8) and analogously for B.

As the stable subspaces have the same dimension, by Proposition 3.2 there is a conjugation for the stable systems

$$H^s:\mathbb{E}^s_A\to\mathbb{E}^s_B.$$

Inverting time one also finds a conjugation for the unstable systems

$$H^u:\mathbb{E}^u_A\to\mathbb{E}^u_B.$$

Hence we define a topological conjugation of Φ and Ψ as

$$H(x)=H^s(\pi^s(x))+H^u(\pi^u(x)).$$

□

Note that topological conjugacy of affine equations $\dot{x}=Ax+a$ is determined by the matrix $A\in\mathfrak{gl}(n,\mathbb{R})$; it is independent of the affine term $a\in\mathbb{R}^d$. As in case of linear differential equation, if $A\in\mathfrak{gl}(n,\mathbb{R})$ is hyperbolic and B is close enough to A, then the corresponding affine flows are topologically conjugate.

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References

1. Agarwal, R.P. and Gupta, R.C. *Essentials of Ordinary Differential Equations*, McGraw-Hill Book Company, (1991).
2. Aulbach, B. and Wanner, T. *Integral manifolds for Carathéodory type differential equations in Banach spaces*, in: Six Lectures on Dynamical Systems, B. Aulbach and F. Colonius, eds., World Scientific, 45-119, (1996).
3. Ayala, V., Colonius, F. and Kliemann, W. *On topological equivalence of linear flows with applications to bilinear control systems*, J. Dynam. Control Systems 13, 337-362, (2007).
4. Colonius, F. *Dynamics and Linear Algebra*, Notes of the IMA (Institute of Mathematics and Its Applications) PI Summer Program for Graduate Students: Linear Algebra and Applications, Iowa State University, Ames, Iowa (June 30-July 25, 2008).
5. Colonius, F. and Kliemann, W. *The Dynamics of Control*, Birkhäuser, Boston, (2000).
6. Hirsch, M.W., Smale, S. and Devaney, R.L. *Differential Equations, Dynamical Systems and an Introduction to Chaos*, Elsevier, (2004).
7. Robinson, C. *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos* 2nd. Edition, CRC Press, (1999).

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