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Local structural stability of actions of $\mathbb{R}^n$ on n-manifolds *†

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ABSTRACT: Let M^m be a compact m-manifold and $\varphi : \mathbb{R}^n \times M^m \to M^m$ a C^r, $r \geq 1$, action with infinitesimal generators of class C^r. We introduce the concept of transversally hyperbolic singular orbit for an action φ and explore this concept in its relations to stability. Our main result says that if $m = n$ and $\mathcal{O}_p$ is a compact singular orbit of φ that is transversally hyperbolic, then φ is C^1 locally structurally stable at $\mathcal{O}_p$.

Key Words: Group action, singular orbit, hyperbolicity, structural stability

Contents

1. Introduction

Let M (resp. N) denote a compact orientable m-manifold (resp. n-manifold) and $A^r(\mathbb{R}^n, M)$, $r \geq 1$, the space of C^r-actions of $\mathbb{R}^n$ on M with infinitesimal generators of class C^r and the topology defined by saying that two actions are C^1-close if its infinitesimal generators are C^1-close. Take $\varphi \in A^r(\mathbb{R}^n, M)$ and $p \in M$. The φ-orbit of p will be denoted by $\mathcal{O}_p(\varphi)$ or simply by $\mathcal{O}_p$. If $\dim \mathcal{O}_p < n$, then $\mathcal{O}_p$ is called a *singular orbit* of φ and when $\dim \mathcal{O}_p = 0$ p is called a *fixed point* of φ and $\mathcal{O}_p$ a point orbit. An action φ is called *singular* if every φ-orbit is singular. The possible topological types of the orbits of φ are $T^k \times \mathbb{R}^\ell$, with $0 \leq k + \ell \leq n$, where $T^k = S^1 \times \cdots \times S^1$ k-times. Very little is known about actions of $\mathbb{R}^n$, $n \geq 2$, compare to what is known when $n = 1$. Camacho, in [4], defined the concept of hyperbolic fixed point of an action φ and proved that if p is a hyperbolic fixed point of φ, then φ is locally C^1 structurally stable at p. Here we introduce the concept of transversally hyperbolic singular orbit of an action φ; this concept coincides with Camacho's definition of hyperbolic fixed point when

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$\mathcal{O}_p$ is a point orbit. Next, we explore this concept in the particular case $m = n$ and prove the following theorem:

Theorem 1.1 *If $\mathcal{O}_p$ is a transversally hyperbolic compact singular orbit of $\varphi \in A^r(\mathbb{R}^n, N)$, $r \geq 1$, then φ is locally C^1 structurally stable at $\mathcal{O}_p$.*

We also show, see Example 3.1, that Theorem 1.1 is not necessarily true when $n < m$. It is natural to ask if the reciprocal of Theorem 1.1 is true. In [1] we answered this question negatively in the case of real analytic actions. In fact, for each $n \geq 2$, we exhibited a family $\mathscr{C}_n \subset A^\omega(\mathbb{R}^n, N)$ of singular actions such that each $\varphi \in \mathscr{C}_n$ has a first integral and besides φ is C^1 structurally stable. But a compact singular orbit of $\varphi \in \mathscr{C}_n$ can never be transversally hyperbolic. With regard to global stability, it seems reasonable to conjeture that if every compact singular orbit of $\varphi \in A^r(\mathbb{R}^n, N)$ is transversally hyperbolic, then φ is C^1 structurally stable. Up to now, we can prove this conjecture for $n = 2$ and also for $n > 2$ in some particular cases. This topic will be considered in a future paper. The problem of characterizing the local structural stability of a compact singular orbit of a $\varphi \in A^r(\mathbb{R}^n, M)$ is far from been solved.

2. Transversally hyperbolic singular orbits

M will denote a closed connected and orientable differentiable manifold. A C^r-action of Lie group G on M is a C^r-map $\varphi : G \times M \to M$, $1 \leq r \leq \omega$, such that $\varphi(e, p) = p$ and $\varphi(gh, p) = \varphi(g, \varphi(h, p))$, for each $g, h \in G$ and $p \in M$, where e is the identity in G. $\mathcal{O}_p = \{\varphi(g, p);\ g \in G\}$ is called the *φ-orbit* of p. $G_p = \{g \in G;\ \varphi(g, p) = p\}$ is called the *isotropy group* of p. For each $p \in M$ the map $g \mapsto \varphi(g, p)$ induce an injective immersion of the homogeneous space G/G_p in M with image $\mathcal{O}_p$. When $G = \mathbb{R}^n$, the possible φ-orbits are injective immersions of $T^k \times \mathbb{R}^\ell$, $0 \leq k + l \leq n$, where $T^k = S^1 \times \cdots \times S^1$, k times.

For each $0 \leq i \leq n-1$ let $\mathrm{Sing}_i(\varphi) = \{p \in M; \dim \mathcal{O}_p = i\}$ and $\mathrm{Sing}(\varphi) = \cup_{i=0}^{n-1} \mathrm{Sing}_i(\varphi)$. If $p \in \mathrm{Sing}(\varphi)$, $\mathcal{O}_p$ is called a *singular orbit* and when $p \in \mathrm{Sing}_0(\varphi)$, $\mathcal{O}_p$ is also called a *point orbit* and p a *fixed point* by φ. We also write $p \in \mathrm{Sing}_i^c(\varphi)$, $i = 1, \ldots, n-1$, when $\mathcal{O}_p$ is a T^i-orbit. If $\mathrm{Sing}(\varphi) = M$, we call φ a *singular action*.

For each $w \in \mathbb{R}^n \setminus \{0\}$ φ induz a C^r-flow $(\varphi_w^t)_{t \in \mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw, p)$ and its corresponding C^{r-1}-vector field X_w is given by $X_w(p) = D_1\varphi(0, p) \cdot w$. If $\{w_1, \ldots, w_n\}$ is a base of $\mathbb{R}^n$ the associated vector fields $X_{w_1}, \ldots, X_{w_n}$ determine completely the action φ and are called a set of infinitesimal generators of φ. Note that $[X_{w_i}, X_{w_j}] = 0$ for any two of them. We denote by $X_1, \ldots, X_n$ the infinitesimal generators of φ associated to the canonical base of $\mathbb{R}^n$.

Denote by $A^r(\mathbb{R}^n, M)$ the set of C^r-actions, $r \geq 1$, of $\mathbb{R}^n$ on M such that their canonical infinitesimal generators are also C^r vector fields. Given two actions $\{\varphi; X_1, \ldots, X_n\}$ and $\{\psi; Y_1, \ldots, Y_n\}$ define $d_k(\varphi, \psi) = \max\limits_{1 \leq i \leq n} \|X_i - Y_i\|_k$. $A^r(\mathbb{R}^n, M)$ is a metric space and the corresponding topology is the C^k-topology.

The notions of topological equivalence and C^k structural stability that we use here for actions are the standard one's.

2.1. CAMACHO'S RESULTS ON HYPERBOLIC FIXED POINTS. In this subsection we give the definition of hyperbolic fixed point due to Camacho [4] and enunciate without proof his results that we shall use in this paper. Let E be a m-dimensional real vector space and $\mathrm{Aut}(E)$ the group of its linear automorphisms. Consider Lie groups G, H of the form $\mathbb{R}^k \times \mathbb{Z}^\ell$. A homomorphism $\varrho : G = \mathbb{R}^k \times \mathbb{Z}^\ell \to \mathrm{Aut}(E)$, is called a *linear action* of G on E. By definition $\mathrm{rank}(G) = k + \ell$.

Definition 2.1 A linear action ϱ is said to be *hyperbolic* if it satisfies the following properties:

(a) if $k+\ell = 1$, then for each $s \in G$, $s \neq 0$, all eigenvalues of $\varrho(s)$ have modulus different from 1;

(b) if $k + \ell \geq 2$, we give the definition by induction on $k + \ell$. Assume that we already defined hyperbolicity for linear actions of groups H such that $\mathrm{rank}(H) < k + \ell$. Then, ϱ is hyperbolic if:

(b.1) There exists a decomposition $E = \bigoplus_t E_t$, ϱ-invariant, such that ϱ is transitive on each connected component of $E_t \setminus \{0\}$ for each t.

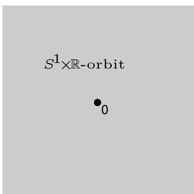
(b.2) The action $\chi_t = \varrho|_{G_v(\varrho)} : G_v(\varrho) \to \mathrm{Aut}(\bigoplus_{t' \neq t} E_{t'})$, $v \in E_t$ is hyperbolic for each t. This makes sense since from (b.1) $\mathrm{rank}(G_v(\varrho)) = \mathrm{rank}(G) - 1$.

A fixed point p of $\varphi \in A^r(\mathbb{R}^k \times \mathbb{Z}^\ell, M)$ is said to be *hyperbolic* if the induced linear action $\varrho : \mathbb{R}^k \times \mathbb{Z}^\ell \to \mathrm{Aut}(T_pM)$ given by $\varrho(g) = D\varphi_g(p)$ is hyperbolic.

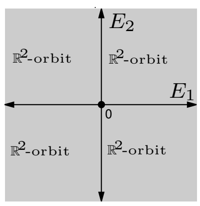
Example 2.2 Each linear action $\varrho : \mathbb{R}^2 \to \mathrm{Aut}(\mathbb{R}^2)$ is of the form $\varrho(t_1, t_2) = \exp(t_1 A_1 + t_2 A_2)$, where A_i, $i = 1, 2$, is a (2×2)-matrix and $A_1 A_2 = A_2 A_1$. Assume that ϱ is hyperbolic, then except for a linear change of coordinates, there are two cases:

(i) if $A_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}$, $i = 1, 2$, then $\alpha_1\beta_2 - \beta_1\alpha_2 \neq 0$. The orbit structure is like in Figure 1(a).

(ii) if $A_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \mu_i \end{pmatrix}$, $i = 1, 2$, then $\lambda_1\mu_2 - \mu_1\lambda_2 \neq 0$. The orbit structure is like in Figure 1(b) .



(a)



(b)

Figure 1:

We shall make use of the following results on hyperbolic fixed points whose proof can be found in [4].

2.1.1. Let p be a hyperbolic fixed point of an action $\varphi : (\mathbb{R}^k \times \mathbb{Z}^\ell) \times M^m \to M^m$ with $m \leq k+1$ and $\varrho : \mathbb{R}^k \times \mathbb{Z}^\ell \to \mathrm{Aut}(T_pM)$ the induced linear action of φ at p. Then,

(i) *there exists a neighborhood V of p, and a homeomorphism $h : V \to T_pM$ such that $h \circ \varphi_g = \varrho(g) \circ h$.*

(ii) *φ is C^1 locally structurally stable at p.*

For each $g \in \mathbb{R}^n$ put $\mathbb{R}^n(g) = \{tg; t \in \mathbb{R}\}$ e $\mathbb{R}^n_+(g) = \{tg; t > 0\}$. A *cone* on $\mathbb{R}^n$ is a set $\mathcal{C} = \cup_{g \in i(D)} \mathbb{R}^n_+(g)$, where $i : D \to \mathbb{R}^n - \{0\}$ is an affine embedding of a d-disk D, $0 \leq d \leq n$.

Let p be a hyperbolic fixed point of $\varphi \in A^r(\mathbb{R}^n, M^m)$. There exists a decomposition $T_pM^m = \bigoplus_t E_t$ invariant under the induced linear action $\varrho : \mathbb{R}^n \times T_pM^m \to T_pM^m$, where either E_t is straight line and $E_t - \{p\}$ is the union of two $\mathbb{R}$-orbits or E_t is a plane and $E_t - \{p\}$ is a $S^1 \times \mathbb{R}$-orbit of ϱ. The isotropy subgroup $G_t(v)$ of a point $v \in E_t - \{0\}$ does not depend on the v and $G_t = \mathbb{R}^{n-1}$ ($\mathbb{R}^{n-1} \times \mathbb{Z}$) if E_t is a straight line (a plane).

2.1.2. Let p a hyperbolic fixed point of $\varphi \in A^r(\mathbb{R}^n, M^m)$, $r \geq 1$, and $\varrho : \mathbb{R}^n \times T_pM^m \to T_pM^m$ the induced linear action. Let G be a closed subgroup of $\mathbb{R}^n$, $E_G = \mathrm{Fix}(\varrho|_G)$ and $V_G = \mathrm{Fix}(\varphi|_G)$. Then V_G is a C^r submanifold of M tangent to E_G em p and for any cone $\mathcal{C} \subset G - \cup_t G_t$ with $G \not\subset G_t$ the subsets

$$W^s_{\mathcal{C}}(V_G) = \{q \in M^m; \lim_{g \to \infty} \varphi(g,q) \in V_G\, , g \in \mathcal{C}\},$$
$$W^u_{\mathcal{C}}(V_G) = \{q \in M^m; \lim_{g \to \infty} \varphi(-g,q) \in V_G\, , g \in \mathcal{C}\}$$

are C^r-submanifolds that intersect transversally along V_G and also φ_h is normally hyperbolic in V_G, for every $h \in \mathcal{C}$.

It follows from 2.1.2 that there exist φ-invariant submanifolds V_t diffeomorphic to E_t and tangent to E_t at p, were $V_t = \mathrm{Fix}(\varphi|_{G_t})$.

2.2. TRANSVERSALLY HYPERBOLIC COMPACT SINGULAR ORBITS. Before giving the definition of transversally hyperbolic singular orbit we need two trivialization lemmas. Let $D^m_\varepsilon = \{(x_1, \ldots, x_m) \in \mathbb{R}^m;\ |x_i| < \varepsilon\}$, $\varepsilon > 0$, and $\frac{\partial}{\partial x_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ the constant vector field.

Lemma 2.3 (k-flow box) *Let $\varphi \in A^r(\mathbb{R}^k, M^m)$ with infinitesimal generators $X_1, \ldots, X_k$, and $\mathcal{O}_p$ a k-dimensional orbit. There exists a C^r- diffeomorphism $h : V_p \to D^m_\varepsilon$, where V_p is a neighborhood of p, such that $h_* X_i = \frac{\partial}{\partial x_i}$ in D^m_ε, for each $i = 1, \ldots, k$.*

Proof: Let $\rho : U \to U_0$ be a chart of M^m with $\rho(p) = 0$ and $Y_i = \rho_* X_i$, $i = 1, \ldots, k$. There exists a neighborhood $V_0 \subset U_0$ where the local flows Y_i^t define a local C^r-action $\phi : D^k_\tau \times V_0 \to U_0$ dada por $\phi(\tau_1, \ldots, \tau_k, x) = Y_1^{\tau_1} \circ \cdots \circ Y_k^{\tau_k}(x)$.

Let H be a subspace of $\mathbb{R}^m$ orthogonal to subspace generated by the vectors $Y_1(0),\dots,Y_k(0)$, $W_0 = H \cap V_0$ and $\psi : D^k_\tau \times W_0 \to U_0$ the restriction of ϕ. Take a base $\{e_1,\dots,e_m\}$ de $\mathbb{R}^m$ such that $\{e_1,\dots,e_k\}$ is the canonical base of $\mathbb{R}^k$ and $\{e_{k+1},\dots,e_m\}$ is a base of $\{0\}\times H$. Since $D\psi(0,0) : \mathbb{R}^k \times H \to \mathbb{R}^m$ is an isomorphism, there exists an $\varepsilon > 0$ such that the restriction of ψ to $D^m_\varepsilon = D^k_\varepsilon \times D^{m-k}_\varepsilon$ is a diffeomorphism onto its image. Put $V_p = \rho^{-1}(\psi(D^m_\varepsilon))$, then $h = \psi^{-1}\circ\rho$ is the desired chart. □

Remark 2.4 *Note that the diffeomorphism $h = h(\varphi) : V_p \to D^m_\varepsilon$ depends continuously on φ in the following sense: given $\eta > 0$, there exists $\delta > 0$ such that if $\widetilde{\varphi} \in A^r(\mathbb{R}^k, M)$ is δ C^1-close to φ, then $h(\widetilde{\varphi}) : \widetilde{V}_p \to D^m_\varepsilon$ is η C^1-close to $h(\varphi)$ in $V_p \cap \widetilde{V}_p$.*

A pair (V_p, h) as in Lemma 2.3 will be called a *k-flow box* at p. If $q \in \mathcal{O}_p$ with $q \neq p$, then there exists $u \in \mathbb{R}^k$ such that $X^1_u(p) = q$. We shall call $\gamma = \{X^t_u(p); 0 \le t \le 1\}$ an *arc of* φ in $\mathcal{O}_p$. By using Lemma 2.3 one can also prove:

Lemma 2.5 (Long k-flow box) *Let $\varphi \in A^r(\mathbb{R}^k, M)$, $\mathcal{O}_p$ a k-dimensional orbit of φ and $\gamma \subset \mathcal{O}_p$ an arc of φ in $\mathcal{O}_p$. Then, there exists k-flow box (V_γ, h), where V_γ is a neighborhood of γ.*

Let $\mathcal{O}_p$ be singular k-dimensional orbit of $\varphi \in A^r(\mathbb{R}^n, M^m)$ and G_p its isotropy group. Call G^0_p the connected component of G_p that contains the origin and let H be a k-dimensional subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = H \oplus G^0_p$. Let $\{w_1,\dots,w_n\}$ be a base of $\mathbb{R}^n$ such that $\{w_1,\dots,w_k\}$ is a base of H and $\{w_{k+1},\dots,w_n\}$ is a base of G^0_p, and $\{X_i = X_{w_i}; i = 1,\dots,n\}$ the corresponding set of infinitesimal generators. Note that $X_{k+1}(q) = \cdots = X_n(q) = 0$ for every $q \in \mathcal{O}_p$. We shall say that $X_1,\dots,X_n$ is a set of *infinitesimal generators adapted to* $\mathcal{O}_p$.

Applying Lemma 2.3 to the action φ restricted to H we obtain a chart $h : V_p \to D^m_\varepsilon$ of M^m such that if $(\theta, x) \in D^m_\varepsilon = D^k_\varepsilon \times D^{m-k}_\varepsilon$, then the vector fields X_i in this chart can be written

$$(*)\qquad \begin{aligned} X_i(\theta,x) &= \frac{\partial}{\partial\theta_i}, \quad i = 1,\dots,k \\ X_{k+i}(\theta,x) &= \sum_{j=1}^{k} a_{ji}(x)\frac{\partial}{\partial\theta_j} + \sum_{j=k+1}^{m} a_{ji}(x)\frac{\partial}{\partial x_j}, \quad i = 1,\dots,n-k \end{aligned}$$

A chart like above is called *adapted to $\mathcal{O}_p$ at p.* The vector fields

$$\widehat{X}_i = \sum_{j=k+1}^{m} a_{ji}(x)\frac{\partial}{\partial x_j}, \quad i = 1,\dots,n-k,$$

define a local action φ_T of $\mathbb{R}^{n-k}$ on D^{m-k}_ε having $0 \in D^{m-k}_\varepsilon$ as a fixed point. When p is a fixed point of φ then a chart adapted to $\mathcal{O}_p$ at p will be any chart of M which contains p. In this case $\widehat{X}_i = X_i$, $i = 1, \ldots, n$. It can be verified that φ_T has the following two properties:

(1) Although φ_T depends on the chart (V_p, h) which in turn depends on H, the fact that $0 \in D_\varepsilon^{m-k}$ be a hyperbolic fixed point of φ_T does not depend on the chart.

(2) If $q \in \mathcal{O}_p$ and $q \neq p$, the there exists a chart (V_p, h) adapted to $\mathcal{O}_p$ such that $q \in V_p$.

It follows from the two properties above that the following concept is well defined.

Definition 2.6 Let $\mathcal{O}_p$ be singular k-dimensional orbit of φ. $\mathcal{O}_p$ *is transversally hyperbolic* if there exist a chart adapted to $\mathcal{O}_p$ at p such that $0 \in D_\varepsilon^{m-k}$ is a hyperbolic fixed point of the action φ_T.

Remark 2.7 *Note that when* $k = n-1$, φ_T *is the local flow of the vector field*

$$\widehat{X}_n(x) = \sum_{j=n}^{m} a_{jn}(x)\frac{\partial}{\partial x_j}, \;\; x = (x_n, \ldots, x_m) \in D_1^{m-n+1}.$$

Therefore, $\mathcal{O}_p$ *is transversally hyperbolic if and only if* $0 \in D_\varepsilon^{m-n+1}$ *is a hyperbolic singularity of* $\widehat{X}_n$.

Remark 2.8 *Note that* $\{X_1, \ldots X_k, \widehat{X}_1, \ldots, \widehat{X}_{n-k}\}$ *define a local* $\mathbb{R}^n$*-action* $\widehat{\varphi}$ *on* D_ε^m *and that* $\mathcal{O}_{(\theta,x)}(\widehat{\varphi}) = \mathcal{O}_{(\theta,x)}(h \circ \varphi \circ h^{-1})$ *for each* $(\theta, x) \in D_\varepsilon^m$.

3. Local structural stability

Let $\mathcal{O}_p$ be a transversally hyperbolic compact singular orbit of $\varphi \in A^r(\mathbb{R}^n, M)$, $n < m$. It is not difficult to prove, from Remark 2.4, that $\mathcal{O}_p$ is C^1-persistent, i. e., given a neighborhood V of $\mathcal{O}_p$ there exists $\delta > 0$ such that if $d_1(\psi, \varphi) < \delta$, then ψ has a compact orbit $\mathcal{O}'$ diffeomorphic to $\mathcal{O}_p$ inside V. The following example shows that local structural stability is not a consequence of transversal hyperbolicity when $n < m$.

Example 3.1 Let $S^2 = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2+z^2 = 1\}$, $N = (0,0,1)$, $S = (0,0,-1), X_0 = x\partial/\partial x + y\partial/\partial y$ on $R^2 = \{(x,y,0) \in \mathbb{R}^3\}$, $P_N : R^2 \to S^2$ ($P_S : R^2 \to S^2$) the projection with focus in N (S) and X the tangent vector field to S^2 defined by

$$X(p) = \begin{cases} (P_N)_* X_0, & p \neq N; \\ 0, & p = N. \end{cases}$$

It is clear that X is the meridian vector field on S^2 and that in a neighborhood of S (N) using the coordinate system P_N^{-1} (P_S^{-1}) $X = x\partial/\partial x + y\partial/\partial y$ ($X = -x\partial/\partial x - y\partial/\partial y$). Now, consider on $\mathbb{R} \times S^2$ the vector fields $X_1 = \partial/\partial t$ and $X_2(t,p) = X(p)$ and the diffeomorphism

$$\Phi : \mathbb{R} \times S^2 \to \mathbb{R} \times S^2, \; \Phi(t,p) = (t-1, p)$$

It is clear that $\Phi_* X_1 = X_1$ and $\Phi_* X_2 = X_2$. Thus X_1 and X_2 induce vector fields Y_1 and Y_2 on $S^1 \times S^2$, the quotient manifold of $\mathbb{R} \times S^2$ under the action of $\mathbb{Z}$ generated by Φ, such that $[Y_1, Y_2] = 0$. Call φ the action of $\mathbb{R}^2$ on $S^1 \times S^2$ with infinitesimal generators Y_1, Y_2, $\mathcal{O}_S$ ($\mathcal{O}_N$) the S^1-orbit of φ induced by $\mathbb{R} \times \{S\}$ ($\mathbb{R} \times \{N\}$). By construction $\mathcal{O}_S$ is a transversally hyperbolic compact singular orbit of φ surrounded by cylindrical orbits. If instead of Φ we consider the diffeomorphism Φ_α given by $\Phi_\alpha(t,p) = (t-1, R_\alpha(p))$, where R_α is a small rotation of S^2 leaving the z-axis fixed and of an irrational angle α we obtain an action φ_α C^1-close to φ which is not topologically equivalent to φ in any neighborhood of $\mathcal{O}_S$. Thus φ is not locally C^1 structurally stable at $\mathcal{O}_S$.

Let $\mathcal{O}_p$ be a compact singular orbit of $\varphi \in A^r(\mathbb{R}^n, N)$ and $\mathcal{O}$ a φ-orbit such that $\mathrm{cl}(\mathcal{O}) \supset \mathcal{O}_p$. Then $G_{\mathcal{O}}$, the isotropy group of $\mathcal{O}$, is a subgroup of G_p. When $\mathcal{O}_p$ is transversally hyperbolic one can obtain additional information about the relation between $G_{\mathcal{O}}$ and G_p.

Proposition 3.2 *Let $\mathcal{O}_p$ be a transversally hyperbolic T^k-orbit, $0 \le k < n$, of $\varphi \in A^r(\mathbb{R}^n, N)$. Then, there exist a linear k-subspace H of $\mathbb{R}^n$ transversal to G_p and $\Gamma \subset H$ isomorphic to $\mathbb{Z}^k$ such that*

1. *$\Gamma^k = H \cap G_p$ is isomorphic to $\mathbb{Z}^k$;*
2. *Γ is a subgroup of Γ^k;*
3. *$\Gamma = G_{\mathcal{O}} \cap H$ for each orbit $\mathcal{O}$ with $\mathcal{O}_p \subset \mathrm{cl}(\mathcal{O})$.*

Lemma 3.3 *Let $\mathcal{O}_p$ be a T^k-orbit, $0 \le k < n$, of $\varphi \in A^r(\mathbb{R}^n, N)$ and $\mathcal{O}$ a $T^{k+l} \times \mathbb{R}^{\ell-k-l}$-orbit, $k + l < \ell \le n$, such that $\mathcal{O}_p \subset \mathrm{cl}(\mathcal{O})$. Then, $G_{\mathcal{O}} \cap G_p^0$ is isomorphic to $\mathbb{R}^{n-\ell} \times \mathbb{Z}^l$.*

Proof: It is known that G_p and $G_{\mathcal{O}}$ are isomorphic to $\mathbb{R}^{n-k} \times \mathbb{Z}^k$ and $\mathbb{R}^{n-\ell} \times \mathbb{Z}^{k+l}$, respectively and that $G_{\mathcal{O}} \subset G_p$ and $G_{\mathcal{O}}^0 \subset G_p^0$. Let $\{u_1 \dots, u_{n-\ell}, v_1, \dots, v_{k+l}\}$ be a set of generators of the group $G_{\mathcal{O}}$ such that $\{u_1 \dots, u_{n-\ell}\}$ is a base of $G_{\mathcal{O}}^0$. We will show that $G_p^0 \cap \{v_1, \dots, v_{k+l}\}$ has exactly l elements, but this implies that $G_{\mathcal{O}} \cap G_p^0$ is isomorphic to $\mathbb{R}^{n-\ell} \times \mathbb{Z}^l$, which is the desired conclusion. In fact, assume that there exists $0 < k' \le k$ such that $\{v_1, \dots, v_{l+k'}\} \subset G_p^0$ and let ξ be the action of $\mathbb{R}^{k-k'}$ given by the vector fields X_{v_i}, $i = l+k'+1, \dots, l+k$. Let $q \in \mathcal{O}_p$ such that $q \notin \mathcal{O}_p(\xi)$. Since $\mathcal{O}_p \subset \mathrm{cl}(\mathcal{O})$, there are sequences $\{p_j \in \mathcal{O}; j \in \mathbb{N}\}$ and $\{t_{ij} \in [0,1]; i = 1, 2 \dots, k+l \text{ and } j \in \mathbb{N}\}$ such that

$$\lim_{j\to\infty} p_j = p \quad \text{and} \quad \lim_{j\to\infty} \varphi\Big(\sum_{i=1}^{k+l} t_{ij} v_i, p_j\Big) = q.$$

For each $i = 1, \dots, k+l$, we can assume, extracting a subsequence if necessary, that $t_{ij} \to t_i \in [0,1]$. Then

$$q = \varphi\Big(\sum_{i=1}^{k+l} t_i v_i, p\Big) = \varphi\Big(\sum_{i=l+k'+1}^{l+k} t_i v_i\, , \varphi\Big(\sum_{i=1}^{l+k'} t_i v_i, p\Big)\Big) = \varphi\Big(\sum_{i=l+k'+1}^{k+l} t_i v_i, p\Big)$$

which contradicts the fact that $q \notin \mathcal{O}_p(\xi)$. □

Let $\mathcal{O}_p$ be a transversally hyperbolic T^k-orbit, $0 \le k < n$, of $\varphi \in A^r(\mathbb{R}^n, N)$ and $\mathcal{O}_i$, $i = 1, \dots, m$, n-dimensional orbits such that $\mathrm{cl}(\mathcal{O}_i) \supset \mathcal{O}_p$. It follows from Definition 2.6 that there exists $s \in \{0, \dots, n-k-1\}$ such that $\mathcal{O}_i$, $i = 1, \dots, m$, is homeomorphic to $T^{k+s} \times \mathbb{R}^{n-k-s}$. If G_i is the isotropy group of $\mathcal{O}_i$, $i = 1, \dots, m$, then $G_p \cong \mathbb{R}^{n-k} \times \mathbb{Z}^k$ and $G_i \cong \mathbb{Z}^{k+s}$, $i = 1, \dots, m$.

Lemma 3.4 *Let $\mathcal{O}_p$ be a transversally hyperbolic T^k-orbit, $0 \le k < n$. There exists a linear k-subspace H of $\mathbb{R}^n$ transversal to G_p, such that $H \cap G_p$ is a subgroup of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^k$, and if $\mathcal{O}$ is a $T^{k+l} \times \mathbb{R}^{\ell-k-l}$-orbit, $k+l < \ell \le n$, with $\mathrm{cl}(\mathcal{O}) \supset \mathcal{O}_p$, then $G_{\mathcal{O}} \cap H$ is a subgroup of $H \cap G_p$ isomorphic to $\mathbb{Z}^k$.*

Proof: Let W_i be the linear subspace of $\mathbb{R}^n$ generated by G_i. Then $\dim W_i = k + s$ and W_i is transversal to G_p^0. We first show that $W_1 = W_i$, $i = 2, \dots, m$. Assume that there exists $i \in \{2, \dots, m\}$ such that $W_1 \ne W_i$ and choose $u_1 \in G_1 \setminus G_i$, $u_2 \in G_i \setminus G_1$ such that $w = u_1 - u_2 \in G_p^0$. Let $X_w, X_{u_1}, X_{u_2} \in \mathfrak{X}^r(N)$ be the associated vector fields. Then $X_w = X_{u_1} - X_{u_2}$ or equivalently $X_w^t = X_{u_1}^t \circ X_{u_2}^{-t}$. Take infinitesimal generators $X_1, \dots, X_n$ of φ adapted to $\mathcal{O}_p$ so that $X_n = X_w$ and a chart (V, h) adapted to $\mathcal{O}_p$ at p. It follows from $X_{u_1}^1|_{\mathcal{O}_p} = id = X_{u_2}^1|_{\mathcal{O}_i}$ that $DX_{u_i}^1(p) = id$, $i = 1, 2$. Thus, $DX_w^1(p) = id$, which is equivalent to $DX_w(p) = 0$. However, this contradicts the fact that $\mathcal{O}_p$ is transversally hyperbolic and proves that $W_1 = W_i = W$, $i = 2, \dots, m$. By Lemma 3.3 $W \cap G_p^0$ is isomorphic to $\mathbb{R}^s$. By taking H as a k-subspace of W such that $W = H \oplus (W \cap G_p^0)$, it is easy to check - for each orbit $\mathcal{O}$ with $\mathcal{O}_p \subset \mathrm{cl}(\mathcal{O})$ - that $G_{\mathcal{O}} \cap H$ is a subgroup of $H \cap G_p$ isomorphic to $\mathbb{Z}^k$. □

Under the same hypotheses of Lemma 3.4, we obtain:

Corollary 3.5 *Let $\{u_1, \dots, u_k\}$ be a set of generators of $H \cap G_{\mathcal{O}}$. If $\mathcal{O} \subset \mathrm{cl}(\mathcal{O}_i)$, then there exist $n_1^i, \dots, n_k^i \in \mathbb{N}$ such that $\{n_1^i u_1, \dots, n_k^i u_k\}$ is a set of generators of $H \cap G_i$.*

Remark 3.6 Assume that $\mathrm{cl}(\mathcal{O}) \supset \mathcal{O}_p$ and that $\mathcal{O} \subset \mathrm{cl}(\mathcal{O}_i)$ for some $i = 1, \dots, m$. Let $\{u_1, \dots, u_k, u_{k+1}, \dots, u_{k+l}, \dots, u_\ell\}$ be a linearly independent subset of $\mathbb{R}^n$ such that $\{u_1, \dots, u_k\}$ is a set of generators of $H \cap G_{\mathcal{O}}$ and the vector fields $\{X_{u_1}, \dots, X_{u_\ell}\}$ are linearly independent on $\mathcal{O}$. Consider a segment L that is transversal to $\mathcal{O}$ at q_0 and such that at least one connected component of $L \setminus \{q_0\}$ is contained in $\mathcal{O}_i$ and call $\psi = \psi(\varphi)$ the locally free C^1 action of $\mathbb{R}^\ell$ with infinitesimal generators $\{X_{u_1}, \dots, X_{u_\ell}\}$ restricted to $U = \{X_{u_1}^{t_1} \circ \dots \circ X_{u_\ell}^{t_\ell}(q); q \in L \text{ and } (t_1, \dots, t_\ell) \in \mathbb{R}^\ell\}$. φ satisfies:

1. $\mathcal{O}$ is a ψ-orbit, its isotropy group $G_{\mathcal{O}}(\psi)$ is generated by $\{u_1, \dots, u_k, u_{k+1}, \dots, u_{k+l}\}$ and $H \cap G_{\mathcal{O}}(\psi) = H \cap G_{\mathcal{O}}$;
2. there exists $0 \le l' \le \ell - k$ such that $\mathcal{O}_q(\psi)$ is homeomorphic to $T^{k+l'} \times \mathbb{R}^{\ell-k-l'}$ and $H \cap G_q(\psi) = H \cap G_i$ for each $q \in U \cap \mathcal{O}_i$;

Lemma 3.7 *There exist a neighborhood V_0 of q_0 in U and C^r functions $\nu_i : V_0 \to \mathbb{R}^\ell$, $i = 1, \ldots, k$, such that $\nu_i(q_0) = u_i$ and $H \cap G_q(\psi) = H \cap G_i$ is generated by $\{\nu_1(q), \ldots, \nu_k(q)\}$ for each $q \in V_0 \cap \mathcal{O}_i$;*

Proof: Let $h : V_{q_0} \subset V \to D_\varepsilon^{\ell+1}$ with $h(q_0) = 0$, be a ℓ-flow box at q_0. Let $D_i = D_i(\varepsilon) = \{(x_1, \ldots, x_{\ell+1}) \in D_\varepsilon^{\ell+1}; x_i = 0\}$ and $\Sigma_i = \Sigma_i(\varepsilon) = h^{-1}(D_i)$. The functions $\tau_i : V_{q_0} \to (-\varepsilon, \varepsilon)$ given by $\tau_i(q) = -x_i(q)$, where $h(q) = (x_1(q), \ldots, x_{\ell+1}(q))$, are such that $X_i^{\tau_i(q)}(q) \in \Sigma_i$, for $i = 1, \ldots, \ell$. We know that $X_{u_i}^1(q_0) = q_0$, $i = 1, \ldots, k$. Therefore, there exists $0 < \delta < \varepsilon$ such that $X_{u_i}^1(\Sigma_i(\delta)) \subset V_{q_0}$, $i = 1, \ldots, k$. Let $\Sigma_{q_0} = \Sigma_{q_0}(\delta) = \cap_{i=1}^{n-1} \Sigma_i(\delta)$. Σ_{q_0} is a transversal section to $\mathcal{O}$ at q_0. For each $i = 1, \ldots, k$, consider the function $w_i : \Sigma_{q_0} \to \mathbb{R}^\ell$ given by

$$\nu_i(q) = \sum_{j=1}^{i-1} \tau_j(X_{u_i}^1(q))u_j + (1 + \tau_i(X_{u_i}^1(q)))u_i + \sum_{j=i+1}^{\ell} \tau_j(X_{u_i}^1(q))u_j \,. \tag{3.1}$$

It can be verified that every orbit of $X_{\nu_i(q)}$ inside $\mathcal{O}_q(\psi)$, $q \in \Sigma_{q_0} \cap \mathcal{O}_i$, is periodic of period one and $\nu_i(q_0) = u_i$, $i = 1, \ldots, k$. We can extend the functions ν_i to the open set $V_0 = \cup_{q \in \Sigma_{q_0}} (\mathcal{O}_q(\psi) \cap V_{q_0})$ by defining $\nu_i(q) = \nu_i(\Sigma_{q_0} \cap \mathcal{O}_q(\psi))$. Thus, $H \cap G_q(\psi)$ is generated by $\{\nu_1(q), \ldots, \nu_k(q)\}$ for each $q \in V_0 \cap \mathcal{O}_i$ and the proof is completed. □

Proof of Proposition 3.2. Let H be given by Lemma 3.4. The result follows from Corollary 3.5 and Lemma 3.7. □

Proof of Theorem 1.1. There exist a neighborhood $\mathcal{V}_\varphi$ of φ and a neighborhood V of $\mathcal{O}_p$ such that every $\psi \in \mathcal{V}_\varphi$ has an unique T^k-orbit $\mathcal{O}(\psi)$ in V which is transversally hyperbolic. We can assume without lost of generality that $p \in \mathcal{O}(\psi)$, or in other words, that $\mathcal{O}(\psi) = \mathcal{O}_p(\psi)$. Let $\{v_1, \ldots, v_k\}$ be a set of generators of $\Gamma^k(\varphi)$ ($\Gamma^k(\varphi)$ was defined in Proposition 3.2) and $\{X_1, \ldots, X_n\}$ a set of infinitesimal generators of φ adapted to $\mathcal{O}_p$ such that $X_i = X_{v_i}$, $i = 1, \ldots, k$. Take a chart $h : V_p \to D_\varepsilon^n$ adapted to $\mathcal{O}_p$ at p with $h(p) = 0$ and let $\Sigma = h^{-1}(D_\varepsilon^{n-k})$.

Let $\{Y_1, \ldots, Y_n\}$ be a set of infinitesimal generators of $\psi \in \mathcal{V}_\varphi$ adapted to $\mathcal{O}_p(\psi)$. Since $(\Sigma \cap V) \cap \mathrm{Fix}(\psi_T) = \{p\} = (\Sigma \cap V) \cap \mathrm{Fix}(\varphi_T)$, we have that $\Gamma^k(\psi) = H \cap G_p(\psi)$ is isomorphic to $\mathbb{Z}^k$. Let $\{\widetilde{v}_1, \ldots, \widetilde{v}_k\}$ be a set of generators of $\Gamma^k(\psi)$ such that $Y_i = Y_{\widetilde{v}_i}$ is close to X_i for $i = 1, \ldots, k$. It follows that $\Gamma(\psi)$ is generated by $\{n_1\widetilde{v}_1, \ldots, n_k\widetilde{v}_k\}$.

By reducing the size of $\mathcal{V}_\varphi$ and V, if necessary, by Remark 2.4 and (ii) of 2.1.1, there exist a neighborhood Σ_0 of p in Σ and a topological equivalence $g : \Sigma_0 \to \Sigma \cap V$ between $h^{-1} \circ \varphi_{_T} \circ h$ and $h^{-1} \circ \psi_{_T} \circ h$ at p. Let us consider the C^1 actions $\varphi_0, \psi_0 : \mathbb{R}^k \times N \to N$ defined by $\varphi_0(t; q) = \varphi(t_1 n_1 v_1, \ldots, t_k n_k v_k; q)$ and $\psi_0(t_1, \ldots, t_k; q) = \psi(t_1 n_1 \widetilde{v}_1, \ldots, t_k n_k \widetilde{v}_k; q)$, where $t = (t_1, \ldots, t_k)$. Note that the φ_0-orbits (resp. ψ_0-orbits) by points in Σ_0 (resp. $\Sigma \cap V$) are diffeomorphic to T^k and transversal to Σ_0 (resp. $\Sigma \cap V$). The open sets $S(\varphi) = \cup_{q \in \Sigma_0} \mathcal{O}_q(\varphi_0)$ and $S(\psi) = \cup_{q \in \Sigma \cap V} \mathcal{O}_q(\psi_0)$ are neighborhoods of $\mathcal{O}_p(\varphi)$ and $\mathcal{O}_p(\psi)$, respectively. If $q \in S(\varphi)$, there exists $t \in [0,1]^k$ such that $\varphi_0(t, q) \in \Sigma_0$. The map $H : S(\varphi) \to$

$S(\psi)$ defined by

$$F(q) = \psi_0(-t, g(\varphi_0(t, q)))).$$

is a topological equivalence between φ and ψ. □

References

1. J. L. ARRAUT and C. A. MAQUERA, *Structural stability of singular actions of* $\mathbb{R}^n$ *having a first integral*, Submitted for publication, (2003).
2. J. L. ARRAUT and C. A. MAQUERA, *On the orbit structure of* $\mathbb{R}^n$*-actions on n-manifolds*, Qual. Theory Dyn. Syst. **4** no. 2 (2003), 169–180..
3. C. CAMACHO e A. LINS NETO, *Teoria Geométrica das folheações*, Projeto Euclides, IMPA, (1979).
4. C. CAMACHO, On $\mathbb{R}^k \times \mathbb{Z}^\ell$ - actions. *Dynamical Systems, Editor M. M. Peixoto, Academic Press*, (1973), pg.23-70.
5. C. CAMACHO, *Morse-Smale* $\mathbb{R}^2$ *- Actions on two Manifolds*, Dynamical Systems, Editor M. M. Peixoto, Academic Press, (1973), 71–75.
6. P. SAD, *Centralizers of Vector Fields*, Topology, **18**, (1979), 97–104
7. N. SALDANHA, C^1 *Stability of Compact Actions of* $\mathbb{R}^n$ *of codimension one*, Comment. Math. Hel., **69**, (1994), 431–446.

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