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On a class of topological groups

Dinamérico P. Pombo Jr.

ABSTRACT: A topological group is an SNS-group if its identity element possesses a fundamental system of neighborhoods formed by normal subgroups. In this paper we prove the existence of initial SNS-topologies, from which we derive that the class of SNS-groups is closed under the formation of products and projective limits, and we prove the existence of final SNS-topologies, from which we derive that the class of SNS-groups is closed under the formation of free products and inductive limits.

Key Words: topological groups, linearly topologized groups.

Contents

1. Introduction

Although the class of linearly topologized groups (a linearly topologized group is an abelian topological group whose identity element possesses a fundamental system of neighborhoods formed by subgroups) is quite rich in examples, there exist important non-abelian SNS-groups which occur, for example, in Galois Theory. This fact has motivated us to write the present paper, where basic results concerning SNS-groups have been established.

The main purpose of the paper is the discussion of the fundamental constructions in the class of SNS-groups. In this regard, it is shown that this class is closed under the formation of products, projective limits, free products and inductive limits. Moreover, we emphasize the universal properties satisfied by the objects which have been constructed. It should also be mentioned that, although part of the results presented here are well-known in the mathematical folklore, the ones concerning topological free products and topological inductive limits are possibly new.

Throughout this paper G is an arbitrary group whose identity element is denoted by e, unless otherwise specified. In the whole paper, the operation on a group has been written multiplicatively.

Definition 1. A topological group (G, τ) is said to be an *SNS-group*, and τ is said to be an *SNS-topology* on G, if e admits a fundamental system of τ-neighborhoods consisting of normal subgroups of G.

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In the case where G is an abelian group, to say that τ is an SNS-topology on G is equivalent to saying that τ is a linear topology on G; we refer to [5] and [6] for the study of such topologies.

It is easily seen that the bilateral completion ([9], §5) of a separated SNS-group is an SNS-group.

Example 2. The discrete topology and the chaotic topology on a group G are SNS-topologies on G.

Example 3. If (E, τ) is a linearly topologized module ([9], Definition 31.4), then the underlying additive topological group is an SNS-group.

Example 4. If (E, τ) is a non-archimedean topological vector space over a non-trivially valued division ring ([7], Definition 3.2), then the underlying additive topological group is an SNS-group in view of Theorem 3.7 of [7].

Example 5. Let A be a non-empty set and (G, τ) an SNS-group. If $\mathcal{F}(A, G)$ is the group of all mappings from A into G and τ_A is the topology of uniform convergence on $\mathcal{F}(A, G)$, then $(\mathcal{F}(A, G), \tau_A)$ is an SNS-group.

In fact, it is easily seen that τ_A is a group topology on $\mathcal{F}(A, G)$. Moreover, if $\mathcal{V}$ is a fundamental system of τ-neighborhoods of e consisting of normal subgroups of G, then the sets

$$\{f \in \mathcal{F}(A, G); f(A) \subset V\} \quad (V \in \mathcal{V})$$

form a fundamental system of τ_A-neighborhoods of the identity element of $\mathcal{F}(A, G)$ consisting of normal subgroups of $\mathcal{F}(A, G)$.

Proposition 6. If $\mathcal{B}$ is a filter base on a group G consisting of normal subgroups of G, then there exists a unique SNS-topology on G for which $\mathcal{B}$ is a fundamental system of neighborhoods of e.

Proof. Follows immediately from Corollary 1.5 of [9].

Example 7. Let G be an abelian group and let p be a positive prime number. It is easily seen that $\mathcal{B} = \{p^n G; n \in \mathbb{N}\}$ is a filter base on G consisting of subgroups of G (observe that $p^m G \cap p^n E = p^n G$ if $n \geq m$). By Proposition 6, there exists a unique SNS-topology on G for which $\mathcal{B}$ is a fundamental system of neighborhoods of e, called the p-adic topology on G.

Example 8. Let F, G and H be three abelian groups and let e_F (resp. e_G, e_H) be the identity element of F (resp. G, H). Let $B \colon F \times G \to H$ be a $\mathbb{Z}$-bilinear mapping.

For each finite subset $\{y_1, \ldots, y_m\}$ of G put

$$U_{\{y_1,\ldots,y_m\}} = \{x \in F;\ B(x, y_i) = e_H \text{ for } i = 1, \ldots, m\}.$$

It is easily seen that the set $\mathcal{B}$ formed by all the sets $U_{\{y_1,\ldots,y_m\}}$ is a filter base on F consisting of subgroups of F. By Proposition 6, there exists a unique SNS-topology on F for which $\mathcal{B}$ is a fundamental system of neighborhoods of e_F.

Similarly, for each finite subset $\{x_1, \ldots, x_m\}$ of F put

$$V_{\{x_1,\ldots,x_m\}} = \{y \in G;\ B(x_i, y) = e_H \text{ for } i = 1, \ldots, m\}.$$

Then all the sets $V_{\{x_1,\dots,x_m\}}$ constitute a fundamental system of neighborhoods of e_G for a unique SNS-topology on G.

Let us mention two basic examples of $\mathbb{Z}$-bilinear mappings. For F, G and H as above, let $\mathrm{Hom}(F,G)$ (resp. $\mathrm{Hom}(G,H)$, $\mathrm{Hom}(F,H)$) be the abelian group of all group homomorphisms from F into G (resp. G into H, F into H). Then the mappings

$$(x,u) \in F \times \mathrm{Hom}(F,G) \mapsto u(x) \in G$$

and

$$(u,v) \in \mathrm{Hom}(F,G) \times \mathrm{Hom}(G,H) \mapsto v \circ u \in \mathrm{Hom}(F,H)$$

are $\mathbb{Z}$-bilinear.

Example 9. Let N be a Galois extension of a field K and Γ the Galois group of N over K. For each intermediate field $K \subset L \subset N$ which is a finite Galois extension of K, let $g(L)$ be the Galois group of N over L; then $g(L)$ is a normal subgroup of Γ and the set $\mathcal{B}$ formed by all the sets $g(L)$ constitutes a filter base on Γ (see Appendix II of [3] for the details). By Proposition 6, there exists a unique SNS-topology on Γ for which $\mathcal{B}$ is a fundamental system of neighborhoods of the identity element 1_N of Γ.

2. Initial and final SNS-topologies

Now let us begin the discussion of initial SNS-topologies. **Theorem 10**. Let $((G_i,\tau_i))_{i\in I}$ be a non-empty family of SNS-groups, G a group and, for each $i \in I$, let $u_i : G \to G_i$ be a group homomorphism. If τ is the initial topology on G for the family $((G_i,\tau_i),u_i)_{i\in I}$ ([2], p.28, Proposition 4), then (G,τ) is an SNS-group.

Proof. By Theorem 1.9 of [9], (G,τ) is a topological group. For each $i \in I$ let $\mathcal{V}_i$ be a fundamental system of τ_i-neighborhoods of the identity element e_i of G_i consisting of normal subgroups of G_i. For each finite subset $\{i_1,\dots,i_m\}$ of I and for each $V_{i_1} \in \mathcal{V}_{i_1},\dots,V_{i_m} \in \mathcal{V}_{i_m}$, consider the set

$$u_{i_1}^{-1}(V_{i_1}) \cap \dots \cap u_{i_m}^{-1}(V_{i_m}),$$

which is a normal subgroup of G. Since all these sets constitute a fundamental system of τ-neighborhoods of e, then (G,τ) is an SNS-group, as was to be shown.

Remark 11. Under the conditions of Theorem 10, if τ_i is the chaotic topology on G_i for all $i \in I$, then τ is the chaotic topology on G.

Example 12. Let X be a non-empty set and $\mathcal{A}$ a set of non-empty subsets of X. Let (G,τ) be an SNS-group and $\mathcal{F}(X,G)$ the group of all mappings from X into G. For each $A \in \mathcal{A}$ consider the group homomorphism

$$u_A : f \in \mathcal{F}(X,G) \mapsto f|_A \in \mathcal{F}(A,G).$$

By Theorem 10, $\mathcal{F}(X,G)$ endowed with the initial topology $\tau_{\mathcal{A}}$ for the family $((\mathcal{F}(A,G),\tau_A),u_A)_{A\in\mathcal{A}}$ is an SNS-group ($(\mathcal{F}(A,G),\tau_A)$ being as in Example 5). $\tau_{\mathcal{A}}$ is the topology of $\mathcal{A}$-convergence on $\mathcal{F}(X,G)$.

Corollary 13. Let (H,θ) be an SNS-group, G a group and $u : G \to H$ a group homomorphism. If τ is the inverse image of θ under u, then (G,τ) is an SNS-group.

In particular, every subgroup of an SNS-group is an SNS-group under the induced topology.
Proof. Follows immediately from Theorem 10.
Example 14. Let (X,τ) be a non-empty topological space, $\mathcal{A}$ a set of non-empty subsets of X, (G,θ) an SNS-group and $\mathcal{C}(X,G)$ the subgroup of $\mathcal{F}(X,G)$ consisting of all continuous mappings from (X,τ) into (G,θ). Then, in view of Corollary 13, $\mathcal{C}(X,G)$ is an SNS-group under the topology induced by $\tau_{\mathcal{A}}$ ($\tau_{\mathcal{A}}$ being as in Example 12).
Corollary 15. If $((G_i,\tau_i))_{i\in I}$ is a non-empty family of SNS-groups, G is the product group $\prod\limits_{i\in I} G_i$ and τ is the product topology $\prod\limits_{i\in I}\tau_i$ on G, then (G,τ) is an SNS-group.
Proof. Follows immediately from Theorem 10.
Remark 16. Let $((G_i,\tau_i))_{i\in I}$ and (G,τ) be as in Corollary 15 and, for each $i\in I$, let $\mathrm{pr}_i\colon G\to G_i$ be the projection on the i-th factor. Then (G,τ) satisfies the following universal property: for each SNS-group (H,θ), the mapping

$$u\in \mathrm{Hom}_c(H,G)\mapsto (\mathrm{pr}_i\circ u)_{i\in I}\in\prod_{i\in I}\mathrm{Hom}_c(H,G_i)$$

is a group isomorphism, where $\mathrm{Hom}_c(H,G)$ is the group of all continuous group homomorphisms from (H,θ) into (G,τ), $\mathrm{Hom}_c(H,G_i)$ is the group of all continuous group homomorphisms from (H,θ) into (G_i,τ_i) for all $i\in I$ and $\prod\limits_{i\in I}\mathrm{Hom}_c(H,G_i)$ is the corresponding product group.
Corollary 17. If $(\tau_i)_{i\in I}$ is a non-empty family of SNS-topologies on a group G and $\tau=\sup\limits_{i\in I}\tau_i$, then (G,τ) is an SNS-group.
Proof. Follows immediately from Theorem 10.
Definition 18. Let $((G_i,\tau_i),u_{ij})_{i\in I}$ be a projective system of SNS-groups (this means that I is a non-empty set endowed with a partial order $\leq$, (G_i,τ_i) is an SNS-group for all $i\in I$, $u_{ij}\colon(G_j,\tau_j)\to(G_i,\tau_i)$ is a continuous group homomorphism for $i,j\in I$ with $i\leq j$, $u_{ii}=1_{G_i}$ for all $i\in I$ and $u_{ik}=u_{ij}\circ u_{jk}$ for $i,j,k\in I$ with $i\leq j\leq k$).

$$\begin{array}{ccc} (G_k,\tau_k) & \xrightarrow{u_{ik}} & (G_i,\tau_i)\\ u_{jk}\searrow & & \nearrow u_{ij}\\ & (G_j,\tau_j) & \end{array}$$

For each $i\in I$ let $\mathrm{pr}_i\colon G=\prod\limits_{i\in I}G_i\to G_i$ be the projection on the i-factor. Put

$$\varprojlim G_i=\{x\in G;(u_{ij}\circ\mathrm{pr}_j)(x)=\mathrm{pr}_i(x)\text{ for all }i,j\in I\text{ with }i\leq j\},$$

which is a subgroup of the product group $G=\prod\limits_{i\in I}G_i$ (note that $\varprojlim G_i=G$ if $\leq$ is the equality relation). For each $i\in I$ let $u_i=\mathrm{pr}_i|(\varprojlim G_i)$, which is called the *canonical group homomorphism* from $\varprojlim G_i$ into G_i. If τ is the initial topology on $\varprojlim G_i$ for the family $((G_i,\tau_i),u_i)_{i\in I}$, which makes $\varprojlim G_i$ an SNS-group in view of Theorem 10, then $(\varprojlim G_i, \tau)$ is said to be the *topological projective limit* of the system $((G_i, \tau_i), u_{ij})_{i \in I}$.

Remark 19. The projective limit topology for the system $((G_i, \tau_i), u_{ij})_{i \in I}$ is an SNS-topology since, by definition ([2], p.51), it is precisely the topology τ considered in Definition 18.

Proposition 20. Let $(\varprojlim G_i, \tau)$ be the topological projective limit of the projective system $((G_i, \tau_i), u_{ij})_{i \in I}$ of SNS-groups. Then the topology θ on $\varprojlim G_i$ induced by the product topology $\prod_{i \in I} \tau_i$ coincides with τ.

Proof. Let $H = \varprojlim G_i$, and let pr_i and u_i be as in Definition 18. For each $i \in I$ let $\mathcal{V}_i$ be the set of all τ_i-neighborhoods of the identity element e_i of G_i. If $i_1, \ldots, i_m \in I$, $V_{i_1} \in \mathcal{V}_{i_1}, \ldots, V_{i_m} \in \mathcal{V}_{i_m}$, we have

$$\begin{aligned} &\left[\mathrm{pr}_{i_1}^{-1}(V_{i_1}) \cap \cdots \cap \mathrm{pr}_{i_m}^{-1}(V_{i_m})\right] \cap H \\ &= \left(\mathrm{pr}_{i_1}^{-1}(V_{i_1}) \cap H\right) \cap \cdots \cap \left(\mathrm{pr}_{i_m}^{-1}(V_{i_m}) \cap H\right) \\ &= u_{i_1}^{-1}(V_{i_1}) \cap \cdots \cap u_{i_m}^{-1}(V_{i_m}). \end{aligned}$$

Consequently, $\theta = \tau$, as asserted.

Example 21. Let p be a positive prime number. Let $\mathbb{Z}$ be the additive group of integers and, for each positive integer m, let G_m be the quotient group $\mathbb{Z}/p^m\mathbb{Z}$ endowed with the discrete topology τ_m ((G_m, τ_m) is an SNS-group by Example 2). For $m \leq n$ let $u_{mn} \colon G_n \to G_m$ be the canonical group homomorphism, which is obviously continuous from (G_n, τ_n) into (G_m, τ_m). Then $((G_m, \tau_m), u_{mn})_{m \in \mathbb{N}^*}$ is a projective system of compact SNS-groups, and hence we can consider the topological projective limit $(\varprojlim G_m, \tau)$ of this system. Since $\varprojlim G_m$ is closed in $\left(\prod_{m \in \mathbb{N}^*} G_m, \prod_{m \in \mathbb{N}^*} \tau_m\right)$ and $\left(\prod_{m \in \mathbb{N}^*} G_m, \prod_{m \in \mathbb{N}^*} \tau_m\right)$ is compact, it follows from Proposition 20 that $(\varprojlim G_m, \tau)$ is compact. If $\mathbb{Z}_p$ is the linearly topologized ring of p-adic integers ([8], p.11), then the underlying additive topological group is $(\varprojlim G_m, \tau)$.

The topological projective limit $(\varprojlim G_i, \tau)$ of the projective system $((G_i, \tau_i), u_{ij})_{i \in I}$ of SNS-groups satisfies the following universal property:

Proposition 22. Let (H, θ) be an SNS-group and, for each $i \in I$, let $\alpha_i \colon (H, \theta) \to (G_i, \tau_i)$ be a continuous group homomorphism such that $u_{ij} \circ \alpha_j = \alpha_i$ for $i \leq j$. Then there exists a unique continuous group homomorphism $u \colon (H, \theta) \to (\varprojlim G_i, \tau)$ such that $\alpha_i = u_i \circ u$ for all $i \in I$ (u_i being as in Definition 18).

$$\begin{array}{ccc} (G_j, \tau_j) & \xrightarrow{u_{ij}} & (G_i, \tau_i) \\ \alpha_j \nwarrow & & \nearrow \alpha_i \\ & (H, \theta) & \end{array} \qquad \begin{array}{ccc} (H, \theta) & \xrightarrow{u} & (\varprojlim G_i, \tau) \\ \alpha_i \searrow & & \swarrow u_i \\ & (G_i, \tau_i) & \end{array}$$

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Proof. Let $y \in H$ be arbitrary. It is clear that $u(y) = (\alpha_i(y))_{i \in I}$ is the unique element of $\prod_{i \in I} G_i$ satisfying $\alpha_i(y) = \mathrm{pr}_i(u(y))$ for all $i \in I$. Moreover, $u(y) \in \varprojlim G_i$ because $(u_{ij} \circ \alpha_j)(y) = \alpha_i(y)$ for $i \leq j$. It then follows that the mapping $u \colon H \to \varprojlim G_i$ so defined is the unique group homomorphism such that $\alpha_i = u_i \circ u$ for all

$i \in I$. Finally, $u\colon (H, \theta) \to (\varprojlim G_i, \tau)$ is continuous because $u_i \circ u\colon (H, \theta) \to (G_i, \tau_i)$ is continuous for all $i \in I$.

Corollary 23. Let $((G_i, \tau_i), u_{ij})_{i \in I}$ and $((H_i, \theta_i), v_{ij})_{i \in I}$ be two projective systems of SNS-groups, and let $(\varprojlim G_i, \tau)$ and $(\varprojlim H_i, \theta)$ be the corresponding topological projective limits. For each $i \in I$ let $\beta_i\colon (G_i, \tau_i) \to (H_i, \theta_i)$ be a continuous group homomorphism such that $v_{ij} \circ \beta_j = \beta_i \circ u_{ij}$ for $i \leq j$. Then there exists a unique continuous group homomorphism $u\colon (\varprojlim G_i, \tau) \to (\varprojlim H_i, \theta)$ such that $v_i \circ u = \beta_i \circ u_i$ for all $i \in I$, where $u_i\colon \varprojlim G_i \to G_i$ and $v_i\colon \varprojlim H_i \to H_i$ are the canonical group homomorphisms ($i \in I$).

$$
\begin{array}{ccc}
(G_j, \tau_j) & \xrightarrow{\beta_j} & (H_j, \theta_j) \\
u_{ij} \downarrow & & \downarrow v_{ij} \\
(G_i, \tau_i) & \xrightarrow[\beta_i]{} & (H_i, \theta_i)
\end{array}
\qquad\qquad
\begin{array}{ccc}
(\varprojlim G_i, \tau) & \xrightarrow{u} & (\varprojlim H_i, \theta) \\
u_i \downarrow & & \downarrow v_i \\
(G_i, \tau_i) & \xrightarrow[\beta_i]{} & (H_i, \theta_i)
\end{array}
$$

Proof. Put $\alpha_i = \beta_i \circ u_i$ for $i \in I$; then α_i is a continuous group homomorphism from $(\varprojlim G_i, \tau)$ into (H_i, θ_i). Since

$$v_{ij} \circ \alpha_j = (v_{ij} \circ \beta_j) \circ u_j = (\beta_i \circ u_{ij}) \circ u_j = \beta_i \circ (u_{ij} \circ u_j) = \beta_i \circ u_i = \alpha_i$$

for $i \leq j$, Proposition 22 guarantees the existence of a unique continuous group homomorphism $u\colon (\varprojlim G_i, \tau) \to (\varprojlim H_i, \theta)$ such that $\alpha_i = v_i \circ u$ for all $i \in i$. This completes the proof.

Now let us turn to the discussion of final SNS-topologies.

Theorem 24. Let $((G_i, \tau_i))_{i \in I}$ be a non-empty family of SNS-groups and let G be a group. For each $i \in I$ let $u_i\colon G_i \to G$ be a group homomorphism. Then there exists a unique SNS-topology τ on G which is final for the family $((G_i, \tau_i), u_i)_{i \in I}$, in the following sense: for every SNS-group (H, θ) and for every group homomorphism $u\colon G \to H$, we have that $u\colon (G, \tau) \to (H, \theta)$ is continuous if and only if $u \circ u_i\colon (G_i, \tau_i) \to (H, \theta)$ is continuous for all $i \in I$.

Proof. For each $i \in I$ let $\mathcal{V}_i$ be the set of all τ_i-neighborhoods of the identity element e_i of G_i. Put

$$\mathcal{B} = \{U \subset G; U \text{ is a normal subgroup of } G \text{ and } u_i^{-1}(U) \in \mathcal{V}_i \text{ for all } i \in I\}.$$

Clearly, $\mathcal{B}$ is a filter base on G. Let τ be the SNS-topology on G for which $\mathcal{B}$ is a fundamental system of τ-neighborhoods of e (Proposition 6). By construction, $u_i\colon (G_i, \tau_i) \to (G, \tau)$ is continuous for all $i \in I$.

We claim that τ is final for the family $((G_i, \tau_i), u_i)_{i \in I}$. Indeed, let (H, θ) be an SNS-group and let $u\colon G \to H$ be a group homomorphism. If $u\colon (G, \tau) \to (H, \theta)$ is continuous, then $u \circ u_i\colon (G_i, \tau_i) \to (H, \theta)$ is continuous for all $i \in I$. Conversely, assume that $u \circ u_i\colon (G_i, \tau_i) \to (H, \theta)$ is continuous for all $i \in I$, and let V be a θ-neighborhood of the identity element f of H which is a normal subgroup of H. Then $u^{-1}(V)$ is a normal subgroup of G and $u_i^{-1}(u^{-1}(V)) = (u \circ u_i)^{-1}(V) \in \mathcal{V}_i$ for all $i \in I$; thus $u^{-1}(V) \in \mathcal{B}$. Therefore $u\colon (G, \tau) \to (H, \theta)$ is continuous, proving our claim.

In order to prove the uniqueness, let $\tilde{\tau}$ be an SNS-topology on G such that $u_i\colon (G_i,\tau_i) \to (G,\tilde{\tau})$ is continuous for all $i \in I$, and consider the identity mapping $1_G\colon (G,\tau) \to (G,\tilde{\tau})$. Since $1_G \circ u_i\colon (G_i,\tau_i) \to (G,\tilde{\tau})$ is continuous for all $i \in I$, it follows that $1_G\colon (G,\tau) \to (G,\tilde{\tau})$ is continuous. Thus τ is the finest SNS-topology on G which makes all the u_i continuous, and hence the uniqueness is established. This completes the proof.

Remark 25. Under the conditions of Theorem 24, if τ_i is the discrete topology on G_i for all $i \in I$, then τ is the discrete topology on G.

Corollary 26. Every non-empty family of SNS-topologies on a group G admits an infimum in the partially ordered set of all SNS-topologies on G.

Proof. Follows immediately from Theorem 24.

Corollary 27. Let (G,τ) be an SNS-group, H a normal subgroup of G and $\pi\colon G \to G/H$ the canonical surjection. Then the quotient topology τ' on G/H coincides with the final SNS-topology τ'' for the pair $((G,\tau),\pi)$.

Proof. It is easily seen that $(G/H,\tau')$ is an SNS-group and that τ' is finer than τ''. Therefore $\tau' = \tau''$, as asserted.

Definition 28. Let $((G_i,\tau_i))_{i\in I}$ be a non-empty family of SNS-groups and let G be the free product of the family $(G_i)_{i\in I}$ ([1]; [4], p.24). For each $i \in I$ let $u_i\colon G_i \to G$ be the canonical group homomorphism. If τ is the final SNS-topology on G for the family $((G_i,\tau_i),u_i)_{i\in I}$, (G,τ) is said to be the *topological free product* of the family $((G_i,\tau_i))_{i\in I}$.

Remark 29. Let $((G_i,\tau_i))_{i\in I}$, G, u_i and τ be as in Definition 28. Then (G,τ) satisfies the following universal property: for each SNS-group (H,θ), the mapping

$$u \in \mathrm{Hom}_c(G,H) \mapsto (u \circ u_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}_c(G_i,H)$$

is a group isomorphism, where $\mathrm{Hom}_c(G,H)$ is the group of all continuous group homomorphisms from (G,τ) into (H,θ), $\mathrm{Hom}_c(G_i,H)$ is the group of all continuous group homomorphisms from (G_i,τ_i) into (H,θ) for all $i \in I$ and $\prod_{i\in I} \mathrm{Hom}_c(G_i,H)$ is the corresponding product group.

Before we proceed let us establish an auxiliary result which is probably known:

Proposition 30. Let T be a non-empty subset of a group G. Then the normal subgroup of G generated by T (that is, the smallest normal subgroup of G containing T) is the set N of all finite products of elements of the form $g\,t\,g^{-1}$, where $g \in G$ and t is either an element of T or the inverse of an element of T.

Proof. Obviously, $e \in N$ and $ab^{-1} \in N$ for all $a,b \in N$. Let $g \in G$ and $a = g_1\,t_1\,g_1^{-1}\,g_2\,t_2\,g_2^{-1}\ldots g_m\,t_m\,g_m^{-1} \in N$. Then $g\,a\,g^{-1} = (gg_1)t_1\,(gg_1)^{-1}\,(gg_2)t_2\,(gg_2)^{-1}\ldots (gg_m)t_m\,(gg_m)^{-1} \in N$. Thus N is a normal subgroup G, which clearly contains T. Finally, it is clear that any normal subgroup of G containing T also contains N.

Proposition 31. Let $(G_i,u_{ji})_{i\in I}$ be an inductive system of groups and let G be the free product of the family $(G_i)_{i\in I}$. Let T be the subset of G formed by all elements of the form $u_i(x_i)(u_j \circ u_{ji})(x_i^{-1})$ $(i \le j, x_i \in E_i)$, u_i being as in Definition 28, and let N be the normal subgroup of G generated by T. Let $\varinjlim G_i$ be the quotient group G/N, $\pi\colon G \to \varinjlim G_i$ the canonical surjection and, for each

$i \in I$, let $v_i = \pi \circ u_i$ be the *canonical group homomorphism.* Then $v_j \circ u_{ji} = v_i$ for $i \leq j$. Moreover, if H is a group and, for each $i \in I$, $\alpha_i \colon G_i \to H$ is a group homomorphism such that $\alpha_j \circ u_{ji} = \alpha_i$ for $i \leq j$, then there exists a unique group homomorphism $u \colon \varinjlim G_i \to H$ such that $\alpha_i = u \circ v_i$ for all $i \in I$.

$$\begin{array}{ccc} G_i & \xrightarrow{v_i} & \varinjlim G_i \\ u_i \searrow & & \nearrow \pi \\ & G & \end{array} \qquad \begin{array}{ccc} G_i & \xrightarrow{u_{ji}} & G_j \\ \alpha_i \searrow & & \swarrow \alpha_j \\ & H & \end{array} \qquad \begin{array}{ccc} \varinjlim G_i & \xrightarrow{u} & H \\ v_i \nwarrow & & \nearrow \alpha_i \\ & G_i & \end{array}$$

Proof. Firstly, let us verify that $v_j \circ u_{ji} = v_i$ for $i \leq j$. In fact, for all $x_i \in E_i$,

$$\begin{aligned} v_i(x_i)\big(\,(v_j \circ u_{ji})\,(x_i)\big)^{-1} &= (\pi \circ u_i)(x_i)\big((\pi \circ u_j \circ u_{ji})(x_i)\big)^{-1} \\ &= (\pi \circ u_i)(x_i)\big((\pi \circ u_j \circ u_{ji})(x_i^{-1})\big) = \pi\big(u_i(x_i)(u_j \circ u_{ji})(x_i^{-1})\big) \\ &= N. \end{aligned}$$

Now, let $\bar{u} \colon G \to H$ be the unique group homomorphism such that $\bar{u} \circ u_i = \alpha_i$ for all $i \in I$. We claim that $N \subset \mathrm{Ker}(\bar{u})$. In fact, let f be the identity element of H. For $i \leq j$ and $x_i \in E_i$, we have

$$\begin{aligned} &\bar{u}\big(u_i(x_i)(u_j \circ u_{ji})(x_i^{-1})\big) = (\bar{u} \circ u_i)(x_i)\bar{u}\big((u_j \circ u_{ji})(x_i^{-1})\big) \\ &\quad = \alpha_i(x_i)(\bar{u} \circ u_j)(u_{ji}(x_i^{-1})) = \alpha_i(x_i)(\alpha_j \circ u_{ji})(x_i^{-1}) \\ &\quad = \alpha_i(x_i)\alpha_i(x_i^{-1}) = f. \end{aligned}$$

Therefore it follows from Proposition 30 that $N \subset \mathrm{Ker}(\bar{u})$.

By the isomorphism theorem, there exists a unique group homomorphism $u \colon \varinjlim G_i \to H$ such that $\bar{u} = u \circ \pi$. Moreover,

$$u \circ v_i = (u \circ \pi) \circ u_i = \bar{u} \circ u_i = \alpha_i$$

for all $i \in I$.

Finally, let $v \colon \varinjlim G_i \to H$ be a group homomorphism such that $v \circ v_i = \alpha_i$ for all $i \in I$, and put $\bar{v} = v \circ \pi$. Then

$$\bar{v} \circ u_i = v \circ (\pi \circ u_i) = v \circ v_i = \alpha_i$$

for all $i \in I$. Consequently, $\bar{v} = \bar{u}$, and hence $v = u$. This completes the proof.

Definition 32. Let $((G_i, \tau_i), u_{ji})_{i \in I}$ be an inductive system of SNS-groups (this means that I is a non-empty set endowed with a partial order $\leq$, (G_i, τ_i) is an SNS-group for all $i \in I$, $u_{ji} \colon (G_i, \tau_i) \to (G_j, \tau_j)$ is a continuous group homomorphism for $i, j \in I$ with $i \leq j$, $u_{ii} = 1_{G_i}$ for all $i \in I$ and $u_{ki} = u_{kj} \circ u_{ji}$ for $i, j, k \in I$ with $i \leq j \leq k$). Let $\varinjlim G_i$ and v_i $(i \in I)$ be as in Proposition 31. If τ is the final SNS-topology on $\varinjlim G_i$ for the family $((G_i, \tau_i), v_i)_{i \in I}$, then $(\varinjlim G_i, \tau)$ is said to be the *topological inductive limit* of the system $((G_i, \tau_i), u_{ji})_{i \in I}$.

Proposition 33. Let $(\varinjlim G_i, \tau)$ be the topological inductive limit of the inductive system $((G_i, \tau_i), u_{ji})_{i \in I}$ of SNS-groups and let $(G, \tilde{\tau})$ be the topological free product of the family $((G_i, \tau_i))_{i \in I}$. Then the quotient topology $\approx\!\!\tau$ on $G/N (= \varinjlim G_i)$ coincides with τ, N being as in Proposition 31.

Proof. Let π, u_i and v_i be as in Proposition 31. Since $v_i\colon (G_i,\tau_i)\to(\varinjlim G_i,\overset{\approx}{\tau})$ is continuous for all $i\in I$, $\overset{\approx}{\tau}$ is coarser than τ.

Conversely, let U be a normal subgroup of $\varinjlim G_i$ such that $v_i^{-1}(U)$ is a τ_i-neighborhood of the identity element of G_i for all $i\in I$ (U is a basic τ-neighborhood of the identity element of $\varinjlim G_i$). Since $v_i=\pi\circ u_i$, $u_i^{-1}(\pi^{-1}(U))$ is a τ_i-neighborhood of the identity element of G_i for all $i\in I$. Therefore the normal subgroup $\pi^{-1}(U)$ of G is a $\tilde{\tau}$-neighborhood of the identity element of G, and hence U is a $\overset{\approx}{\tau}$-neighborhood of the identity element of $\varinjlim G_i$. Thus τ is coarser than $\overset{\approx}{\tau}$, and the equality $\overset{\approx}{\tau}=\tau$ is established.

The topological inductive limit $(\varinjlim G_i,\tau)$ of the inductive system $((G_i,\tau_i),u_{ji})_{i\in I}$ of SNS-groups satisfies the following universal property:

Proposition 34. Let (H,θ) be an SNS-group and, for each $i\in I$, let $\alpha_i\colon (G_i,\tau_i)\to(H,\theta)$ be a continuous group homomorphism such that $\alpha_j\circ u_{ji}=\alpha_i$ for $i\le j$. Then there exists a unique continuous group homomorphism $u\colon(\varinjlim G_i,\tau)\to(H,\theta)$ such that $\alpha_i=u\circ v_i$ for all $i\in I$ (v_i being as in Proposition 31).

$$\begin{array}{ccc} (G_i,\tau_i) & \xrightarrow{u_{ji}} & (G_j,\tau_j)\\ \alpha_i\searrow & & \swarrow\alpha_j\\ & (H,\theta) & \end{array}\qquad\qquad \begin{array}{ccc} (\varinjlim G_i,\tau) & \xrightarrow{u} & (H,\theta)\\ v_i\nwarrow & & \nearrow\alpha_i\\ & (G_i,\tau_i) & \end{array}$$

Proof. By Proposition 31, there exists a unique group homomorphism $u\colon \varinjlim G_i\to H$ such that $\alpha_i=u\circ v_i$ for all $i\in I$. Moreover, since $u\circ v_i\colon (G_i,\tau_i)\to(H,\theta)$ is continuous for all $i\in I$, then $u\colon(\varinjlim G_i,\tau)\to(H,\theta)$ is continuous. This completes the proof.

Corollary 35. Let $((G_i,\tau_i),u_{ji})_{i\in I}$ and $((H_i,\theta_i),v_{ji})_{i\in I}$ be two inductive systems of SNS-groups, and let $(\varinjlim G_i,\tau)$ and $(\varinjlim H_i,\theta)$ be the corresponding topological inductive limits. For each $i\in I$ let $\beta_i\colon(G_i,\tau_i)\to(H_i,\theta_i)$ be a continuous group homomorphism such that $v_{ji}\circ\beta_i=\beta_j\circ u_{ji}$ for $i\le j$. Then there exists a unique continuous group homomorphism $u\colon(\varinjlim G_i,\tau)\to(\varinjlim H_i,\theta)$ such that $u\circ v_i=w_i\circ\beta_i$ for all $i\in I$, where $v_i\colon G_i\to\varinjlim G_i$ and $w_i\colon H_i\to\varinjlim H_i$ are the canonical group homomorphisms ($i\in I$).

$$\begin{array}{ccc} (G_i,\tau_i) & \xrightarrow{\beta_i} & (H_i,\theta_i)\\ u_{ji}\downarrow & & \downarrow v_{ji}\\ (G_j,\tau_j) & \xrightarrow[\beta_j]{} & (H_j,\theta_j) \end{array}\qquad\qquad \begin{array}{ccc} (\varinjlim G_i,\tau) & \xrightarrow{u} & (\varinjlim H_i,\theta)\\ v_i\uparrow & & \uparrow w_i\\ (G_i,\tau_i) & \xrightarrow[\beta_i]{} & (H_i,\theta_i) \end{array}$$

Proof. For each $i\in I$ put $\alpha_i=w_i\circ\beta_i$; then α_i is a continuous group homomorphism from (G_i,τ_i) into $(\varinjlim H_i,\theta)$. Since

$$\alpha_j\circ u_{ji}=w_j\circ(\beta_j\circ u_{ji})=w_j\circ(v_{ji}\circ\beta_i)=(w_j\circ v_{ji})\circ\beta_i=w_i\circ\beta_i=\alpha_i$$

for $i\le j$, Proposition 34 guarantees the existence of a unique continuous group homomorphism $u\colon(\varinjlim G_i,\tau)\to(\varinjlim H_i,\theta)$ such that $\alpha_i=u\circ v_i$ for all $i\in I$. This completes the proof.

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Dinamérico P. Pombo Jr.
Instituto de Matemática,
Universidade Federal Fluminense
Rua Mário Santos Braga, s/nº., Niterói
CEP. 24020-140- RJ- Brazil
E-mail address: dpombo@terra.com.br