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A remark on the geometry of the Gowers space *

Luis A R Grados

ABSTRACT: Let G_p be the Gowers complex space of characteristic p, B_p be the unitary closed ball and S_p be the unitary sphere of G_p. Then, any $x \in B_p$ can be written in a unique form as the sum of an element of the torus and an element of the unitary open ball of the Gowers space of characteristic $p + k$, for some $k \in \mathbb{N}$, which permit us to show that B_p does not have complex extreme points.

Key Words: Gowers, complex space, extreme points.

Contents

1. Introduction

Let G_1 be the pre-dual of the Lorentz sequence space $d(\{\frac{1}{n}\}, 1)$ studied by Gowers in [3]. This space has attracted the attention of some authors in recent papers, but as Gowers was the first who observed that this space is useful in the study of problems related to norm attaining operators, see [1], we usually call this space, Gowers space. We study in [4] the Gowers space with characteristic p, where $p \in \mathbb{N}, p \geq 1$. When $p = 1$ this is the Gowers space. The existence of real or complex extreme points of the unit ball of a Banach space is connected to several problems in functional analysis. Recently, the existence of complex extreme points of the unit ball of a Banach space has received the attention of several mathematicians as it is connected with important problems such as the Maximum Modulus Theorem. Gowers showed in [3] that the closed unit ball B_1 of G_1 lacks real extreme points and used this fact to solve a norm attaining operator problem. Latter, Acosta and Payá used the fact that G_1 lacks real extreme points to show in [1] that there is no Bishop Phelps theorem for multilinear mappings. It is natural to ask if B_1 and, more generally B_p, has complex extreme points. In this note we are going to show that B_1 lacks complex extreme points and, as a consequence, we will get that the Banach algebra of all complex valued functions defined on B_1 which are bounded on B_1 and holomorphic in the interior of B_1, lacks to have peak points.

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2. The Gowers space with characteristic p and the non-existence of complex extreme points in its closed unit ball

Definition 2.1 *Let E be a complex Banach space. A point $x \in E$ such that $\| x \|= 1$ is called a complex extreme point of B_E if $y \in E$ verifying $\| x + \lambda y \| \leq 1$ for all $\lambda \in \mathbb{C}$ such that $| \lambda | \leq 1$, implies $y = 0$.*

Definition 2.2 *Let $\mathcal{A}$ be a function algebra on a metric space K. A point $x \in K$ is called a* **peak point** *for $\mathcal{A}$ if there is some $f \in \mathcal{A}$ such that $f(x) = 1$ and $| f(y) | < 1$ for all $y \neq x$.*

In this section we will introduce the Gowers space with characteristic p , to be denoted by G_p, and we will show that the set of complex extreme points of its closed unit ball is empty.

Definition 2.3 *If we fix $p \in N$, for each complex sequence $z = (z_j)_{j=1}^{\infty}$ we may define*

$$\phi_{p,n}(z) = \sup_{|J|=n} \frac{\sum\limits_{j\in J} | z_j |}{\sum\limits_{j=1}^{n} \frac{1}{p+j-1}}$$

where $J \subset N$ and $| J |$ denotes the cardinal of the set J. We denote by G_p the complex Banach space of the complex sequences $z = (z_j)_{j=1}^{\infty}$ such that $\lim_{n\to\infty}\phi_{p,n}(z) = 0$, endowed with the norm given by

$$\| z \|_p = \sup_{n\in N} \phi_{p,n}(z), \ \forall z \in G_p.$$

It is easy to check that $G_p \subset c_0$ as a set and $\{e_i\}$ is a Schauder's basis in G_p (where $e_i = (\delta_{ij})_{j=1}^{\infty}$ for all $i \in N$). We remark that $\| e_i \|_p = p$ for all $i \in N$. Let S_p and B_p denote, respectively, the unit sphere $\{z \in G_p : \| z \|_p = 1\}$ and the closed unit ball $\{z \in G_p : \| z \|_p \leq 1\}$. An element $z = (z_i)_{i=1}^{\infty} \in G_p$ is called a finite vector if there exists $N \in \mathbb{N}$ such that $z_i = 0 \quad \forall i > N$. If $z = (z_i)_{i=1}^{\infty} \in G_p$ is a finite vector, the set of all j such that $z_j \neq 0$ is called the support of z and is denoted by $supp(z)$.

We denote by C_k^{∞} the set of all combinations $(i_1, ..., i_k)$ of k elements of $\mathbb{N}$ satisfying $i_1 < i_2 < ... < i_k$. For each $\sigma = (i_1, ..., i_k) \in C_k^{\infty}$, define $\sigma^* = \{i_1, ..., i_k\}$, Q_σ the complement of the set σ^* with respect to $\mathbb{N}$. and $C = \cup_{k=1}^{\infty} C_k^{\infty}$.

Definition 2.4 *Let $\sigma \in C_k^{\infty}$. The polydisc in G_p associated to σ, is the set D_p^{σ} of all $z = (z_j)_{j=1}^{\infty} \in G_p$ such that*

(d1) $z_j = 0 \quad \forall j \notin \sigma^*$

(d2) $\sum\limits_{j\in\gamma^*} | z_j | \leq \sum\limits_{j=1}^{t} \frac{1}{p+j-1}$ *for all $\gamma \in C_t^{\infty}$ satisfying $\gamma^* \subset \sigma^*$.*

Definition 2.5 *: Let $\sigma \in C_k^\infty$. The torus in G_p associated to σ is the set T_p^σ of all $z = (z_j)_{j=1}^\infty \in G_p$ such that*

$(t1)$ $z_j \neq 0$ *if and only if* $j \in \sigma^*$

$(t2)$ $\sum_{j\in\sigma^*} \mid z_j \mid = \sum_{j=1}^{k} \frac{1}{p+j-1}$

$(t3)$ $\sum_{j\in\gamma^*} \mid z_j \mid \leq \sum_{j=1}^{t} \frac{1}{p+j-1}$ *for all $\gamma \in C_t^\infty$ satisfying $\gamma^* \subset \sigma^*$.*

We call any vector belonging to a torus a toroidal vector. We remark that clearly $T_p^\sigma \subset S_p$ and $D_p^\sigma \subset B_p$ for all $\sigma \in C_k^\infty$.

The set σ^* is called the support of the T_p^σ.

Let $P_\sigma : G_p \to G_p$ be the bounded linear operator defined by

$$P_\sigma(z) = \sum_{j\in\sigma^*} z_j \, e_j \quad \forall z = (z_i)_{i=1}^\infty \in G_p.$$

For each $\sigma \in C_k^\infty$, we call P_σ the σ-projection. It is clear that the dimension of the image of the $\sigma-$projection is k where $k =\mid \sigma^* \mid$. We say that σ^* is the support of P_σ and in this case we write $\sigma^* = suppP_\sigma$.

Lemma 2.1 *Given any $x \in S_p$, let $k = max\,\{\mid J \mid : \sum_{j\in J} \mid x_j \mid = \sum_{j=1}^{|J|} \frac{1}{p+j-1}\}$. There exists a unique $\sigma \in C_k^\infty$ such that $P_\sigma(x) \in T_p^\sigma$ and $\| x - P_\sigma(x) \|_{p+k} < 1$.*

Proof: Given $x = (x_1, x_2, .., x_k, ...) \in S_p$, take $N > 0$ such that

$$\mid \phi_n(x) \mid < \frac{1}{3}, \quad \text{for all } n \geq N.$$

We have that

$$1 = \| x \|_p = max_{n\leq N-1} \quad \phi_n(x)$$

and the set

$$X = \{\mid J \mid : \sum_{j\in J} \mid x_j \mid = \sum_{j=1}^{|J|} \frac{1}{p+j-1}\}$$

is finite and non-empty.

Let $k = max\ X$ and $\sigma = (j_1, j_2, ..., j_k) \in C_k^\infty$ such that

$$\sum_{j\in\sigma^*} \mid x_j \mid = \sum_{j=1}^{k} \frac{1}{p+j-1}.$$

Let $\sigma_1^* = \{\ j \in \sigma^* :\ x_j \neq 0\ \}$. Clearly $|\sigma_1^*| = q \leq k$ and from

$$\sum_{j=1}^{k} \frac{1}{p+j-1} = \sum_{j\in\sigma^*} \mid x_j \mid = \sum_{j\in\sigma_1^*} \mid x_j \mid \leq \sum_{j=1}^{q} \frac{1}{p+j-1}$$

we infer that $q = k$. So, $x_j \neq 0$ for all $j \in \sigma^*$ and then $P_\sigma(x) \in T_p^\sigma$.

Let us show that $\| x - P_\sigma(x) \|_{p+k} < 1$. . In fact, if $\| x - P_\sigma(x) \|_{p+k} \geq 1$ we can get $\mu \in C_t^\infty$ so that $\mu^* \cap \sigma^* = \emptyset$ and

$$\sum_{j\in\mu^*} \mid x_j \mid \geq \sum_{j=1}^{t} \frac{1}{p+k+j-1}.$$

Thus $\mid \mu^* \cup \sigma^* \mid = k + t > k$ and

$$\sum_{j\in\mu^*\cup\sigma^*} \mid x_j \mid = \sum_{j\in\sigma^*} \mid x_j \mid + \sum_{j\in\mu^*} \mid x_j \mid \geq \sum_{j=1}^{k} \frac{1}{p+j-1} + \sum_{j=1}^{t} \frac{1}{p+k+j-1}$$

$$= \sum_{j=1}^{k+t} \frac{1}{p+j-1}.$$

Hence $\| x \|_p > 1$ provided that $\| x - P_\sigma(x) \|_{p+k} > 1$ (which contradicts $x \in S_p$) or $k + t \in X$ provided that $\| x - P_\sigma(x) \|_{p+k} = 1$ (which contradicts the maximality of k). This proves that $\| x - P_\sigma(x) \|_{p+k} < 1$.

Finally, we claim that σ is unique. Indeed, let $\tau \in C$ such that $\mid \tau \mid = k$ and $\sum_{j\in\tau^*} \mid x_j \mid = \sum_{j=1}^{k} \frac{1}{p+j-1}$. If $\tau^* \cap \sigma^* = \emptyset$ we have

$$\sum_{j\in\tau^*} \mid x_j \mid + \sum_{j\in\sigma^*} \mid x_j \mid = \sum_{j=1}^{k} \frac{1}{p+j-1} + \sum_{j=1}^{k} \frac{1}{p+j-1}$$

$$> \sum_{j=1}^{k} \frac{1}{p+j-1} + \sum_{j=1}^{k} \frac{1}{p+k+j-1} = \sum_{j=1}^{2k} \frac{1}{p+j-1}$$

and this contradicts that $x \in S_p$. If $\tau^* \cap \sigma^* \neq \emptyset$, let $\zeta \in C$ such that $\zeta^* = \tau^* \cap \sigma^*$ and let $m = \mid \zeta^* \mid$. If $\tau^* - \sigma^* \neq \emptyset$, take $\alpha \in C$ so that $\alpha^* = \tau^* - \sigma^*$. Evidently $\mid \alpha^* \mid = k - m > 0$ and by the definition of k we have

$$\sum_{j\in\sigma^*\cup\alpha^*} \mid x_j \mid < \sum_{j=1}^{2k-m} \frac{1}{p+j-1};$$

$$\sum_{j\in\sigma^*\cup\alpha^*} \mid x_j \mid = \sum_{j\in\sigma^*} \mid x_j \mid + \sum_{j\in\alpha^*} \mid x_j \mid = \sum_{j=1}^{k} \frac{1}{p+j-1} + \sum_{j\in\alpha^*} \mid x_j \mid$$

and consequently

$$\sum_{j\in\alpha^*} \mid x_j \mid < \sum_{j=k+1}^{2k-m} \frac{1}{p+j-1}.$$

From the above inequality and from

$$\sum_{j=1}^{k} \frac{1}{p+j-1} = \sum_{j\in\tau^*} \mid x_j \mid = \sum_{j\in\alpha^*} \mid x_j \mid + \sum_{j\in\zeta^*} \mid x_j \mid$$

$$< \sum_{j=k+1}^{2k-m} \frac{1}{p+j-1} + \sum_{j=1}^{m} \frac{1}{p+j-1},$$

we infer

$$\sum_{j=1}^{k} \frac{1}{p+j-1} < \sum_{j=k+1}^{2k-m} \frac{1}{p+j-1} + \sum_{j=1}^{m} \frac{1}{p+j-1},$$

from which it follows that $\sum_{j=m+1}^{k} \frac{1}{p+j-1} < \sum_{j=k+1}^{2k-m} \frac{1}{p+j-1}$.

So, if we consider $\tau^* \cap \sigma^* \neq \emptyset$ and $\tau^* - \sigma^* \neq \emptyset$ it leads to a contradiction since $p+m+j < p+k+j$ for all $j = 0, 1, ..., k-m-1$. Thus, $\gamma^* - \sigma^* = \emptyset$ and we have $\gamma = \sigma$ as $\gamma^* \cap \sigma^* \neq \emptyset$ and $\mid \gamma^* \mid = \mid \sigma^* \mid$. This proves the uniqueness of σ. □

Proposition 2.1 *The closed unit ball B_p of G_p lacks complex extreme points.*

Proof: It is known that the set of all complex extreme points of B_p is a subset of S_p. Given $x = (x_j)_{j=1}^{\infty} \in B_p$, by Lemma 2.1 and the definition of k, there exists $\sigma \in C_k^{\infty}$ such that $y = P_\sigma(x) \in T_p^\sigma$ and if $z = x - P_\sigma(x) \in G_p$ then $x = y + z$, $supp(y) \cap supp(z) = \emptyset$ and $\| z \|_{p+k} < 1$.

Observing that $G_{p+k}^\sigma = \{ w = (w_j)_{j=1}^{\infty} \in G_{p+k} : w_j = 0 \ \forall j \in \sigma^* \}$, $B_{p+k}^\sigma = G_{p+k}^\sigma \cap B_{p+k}$ and $S_{p+k}^\sigma = G_{p+k}^\sigma \cap S_{p+k}$. As $z \in B_{p+k}^\sigma - S_{p+k}^\sigma$, z can not be a complex extreme point of B_{p+k}^σ and so there exists $v = (v_j)_{j=1}^{\infty} \in G_{p+k}^\sigma$, $v \neq 0$, such that $z + \lambda v \in B_{p+k}^\sigma$ for all $|\lambda| \leq 1$.

Clearly $x + \lambda v \in B_p$, for all $\lambda \in C$ such that $|\lambda| \leq 1$. So B_p lacks complex extreme points. □

Remark 2.1 *Let $A_b(B_E)$ be the Banach space of all complex valued functions defined on the closed unit ball B_E of a Banach space E which are bounded on B_E and holomorphic in the interior of B_E. It is clear that $A_b(B_E)$ is a Banach algebra when equipped with the norm $\| f \| = sup_{x\in B_E} \mid f(x) \mid$. Using theorem 4 of [2] we get that the set of peak points of the algebra $A_b(B_p)$ is empty as a consequence of the above result .*

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Luis A. R. Grados
Departmento de Matemática,
Universidade Estadual de Ponta Grossa,
Brasil
E-mail address: romerogrados@yahoo.com