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On some low separation axioms in bitopological spaces

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ABSTRACT: Recently, in 2003, Caldas et al. [1] introduced the notions of Λ_δ-T_i and Λ_δ-R_i topological spaces for $i = 1, 2$ as a version of the known notions of T_i and R_i ($i = 1, 2$) topological spaces [8] and [2]. In this paper, we extend Λ_δ-T_i and Λ_δ-R_i to bitopological spaces for $i = 1, 2$ and define the notions of pairwise Λ_δ-T_i and Λ_δ-R_i bitopological spaces for $i = 1, 2$. In this context, we study some of the fundamental properties of such spaces. Moreover, we investigate their relationship to some other known separation axioms.

Key Words: bitopological spaces, δ-open sets, δ-closure, pairwise Λ_δ-R_0, pairwise Λ_δ-R_1.

Contents

1. Introduction

The concept of bitopological spaces was introduced by Kelly [3] in 1963. In 1943, Shanin [8] introduced the separation axioms R_0 and R_1 in topological spaces. Murdeshwar and Naimpally [[5], Definition 6.9 and Definition 6.11] offered the notions pairwise R_0 and pairwise R_1 bitopological spaces. Recently, Caldas et al. [1] introduced the notions of Λ_δ-T_0, Λ_δ-T_1, Λ_δ-R_0 and Λ_δ-R_1 topological spaces as a version of the known notions of R_0 and R_1 topological spaces. In this paper, we offer the pairwise versions of Λ_δ-T_0, Λ_δ-T_1, Λ_δ-R_0 and Λ_δ-R_1 in bitopological spaces and investigate their fundamental properties.

Throughout the present paper, the space (X, τ_1, τ_2) always means a bitopological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a bitopological space (X, τ_1, τ_2), τ_i-$Cl(A)$ (resp. τ_i-$Int(A)$) denotes the closure (resp. interior) of A with respect to τ_i for $i = 1, 2$. The complement of A is denoted by $A^c (= X \setminus A)$.

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2. (τ_i, τ_j)-(Λ, δ)-open sets and associated separation axioms

Definition 1. *A subset A of a space (X, τ_1, τ_2) is said to be τ_i-Λ_δ-set if $A = \tau_i$-$\Lambda_\delta(A)$ where τ_i-$\Lambda_\delta(A) = \cap\{G \in \delta(X, \tau_i)/A \subset G\}$, $i = 1, 2$.*
In what follows, by a space we mean a bitopological space.

Definition 2. *Let A be a subset of a space (X, τ_1, τ_2).*
(i) A is called a (τ_i, τ_j)-(Λ, δ)-closed set if $A = T \cap C$, where T is a τ_i-Λ_δ-set and C is a τ_j-δ-closed set where $i, j = 1, 2$ and $i \neq j$. The complement of a (τ_i, τ_j)-(Λ, δ)-closed set is called (τ_i, τ_j)-(Λ, δ)-open. We denoted the collection of all (τ_i, τ_j)-(Λ, δ)-open sets (resp. (τ_i, τ_j)-(Λ, δ)-closed sets) by $\Lambda_\delta O(X, \tau_i, \tau_j)$ (resp. by $\Lambda_\delta C(X, \tau_i, \tau_j)$).
(ii) A point $x \in (X, \tau_1, \tau_2)$ is called a (τ_i, τ_j)-(Λ, δ)-cluster point of A if for every (τ_i, τ_j)-(Λ, δ)-open set U of (X, τ_1, τ_2) containing x, $A \cap U \neq \emptyset$. The set of all (τ_i, τ_j)-(Λ, δ)-cluster points is called the (τ_i, τ_j)-(Λ, δ)-closure of A and is denoted by $A^{(\Lambda,\delta)(\tau_i,\tau_j)}$.

Lemma 2.1. *Let A and B be subsets of a space (X, τ_1, τ_2). For the (τ_i, τ_j)-(Λ, δ)-closure where $i, j = 1, 2$ and $i \neq j$, the following properties hold.*
(1) $A \subset A^{(\Lambda,\delta)(\tau_i,\tau_j)}$.
(2) $A^{(\Lambda,\delta)(\tau_i,\tau_j)} = \cap\{F/A \subset F$ and F is (τ_i, τ_j)-(Λ, δ)-closed$\}$.
(3) If $A \subset B$, then $A^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset B^{(\Lambda,\delta)(\tau_i,\tau_j)}$.
(4) A is (τ_i, τ_j)-(Λ, δ)-closed if and only if $A = A^{(\Lambda,\delta)(\tau_i,\tau_j)}$.
(5) $A^{(\Lambda,\delta)(\tau_i,\tau_j)}$ is (τ_i, τ_j)-(Λ, δ)-closed.

Lemma 2.2. *Let A be a subset of a space (X, τ_1, τ_2). Then the following hold.*
(1) If A_k is (τ_i, τ_j)-(Λ, δ)-closed for each $k \in I$, then $\cap_{k\in I} A_k$ is (τ_i, τ_j)-(Λ, δ)-closed where $i, j = 1, 2$ and $i \neq j$.
(2) If A_k is (τ_i, τ_j)-(Λ, δ)-open for each $k \in I$, then $\cup_{k\in I} A_k$ is (τ_i, τ_j)-(Λ, δ)-open, where $i, j = 1, 2$ and $i \neq j$.

Definition 3. *Let (X, τ_1, τ_2) be a space, $A \subset X$. Then the (τ_i, τ_j)-Λ_δ-kernel of A, denoted by (τ_i, τ_j)-$\Lambda_\delta Ker(A)$, is defined to be the set*
(τ_i, τ_j)-$\Lambda_\delta Ker(A) = \cap\{G \in \Lambda_\delta O(X, \tau_i, \tau_j)/A \subset G\}$. where $i, j = 1, 2$ and $i \neq j$.

Lemma 2.3. *For any two subsets A, B of a space (X, τ_1, τ_2),*
(1) $A \subset B$ implies (τ_i, τ_j)-$\Lambda_\delta Ker(A) \subset (\tau_i, \tau_j)$-$\Lambda_\delta Ker(B)$, where $i, j = 1, 2$ and $i \neq j$.
(2) (τ_i, τ_j)-$\Lambda_\delta Ker((\tau_i, \tau_j)$-$\Lambda_\delta Ker(A)) = (\tau_i, \tau_j)$-$\Lambda_\delta Ker(A)$, where $i, j = 1, 2$ and $i \neq j$.

Lemma 2.4. *For any two points x, y of a space (X, τ_1, τ_2) , $y \in (\tau_i, \tau_j)$-$\Lambda_\delta Ker(\{x\})$ if and only if $x \in \{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. Let $y \notin (\tau_i, \tau_j)$-$\Lambda_\delta Ker(\{x\})$. Then there exists a (τ_i, τ_j)-(Λ, δ)-open set V containing x such that $y \notin V$. Hence $x \notin \{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. The converse is similarly shown.

Proposition 2.5. *If* (X,τ_1,τ_2) *is a space and* $A \subset X$*, then* (τ_i,τ_j)-$\Lambda_\delta Ker(A) = \{x \in X/\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap A \neq \emptyset\}$*, where* $i,j = 1,2$ *and* $i \neq j$*.*

Proof. Let $x \in (\tau_i,\tau_j)$-$\Lambda_\delta Ker(A)$ and suppose that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap A = \emptyset$. Then $x \notin X \backslash \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ which is a (τ_i,τ_j)-(Λ,δ)-open set containing A. This is impossible, since $x \in (\tau_i,\tau_j)$-$\Lambda_\delta Ker(A)$. Consequently, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap A \neq \emptyset$. Next, let $x \in X$ such that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap A \neq \emptyset$ and suppose that $x \notin (\tau_i,\tau_j)$-$\Lambda_\delta Ker(A)$. Then there exists a (τ_i,τ_j)-(Λ,δ)-open set U containing A and $x \notin U$. Let $y \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap A$. Hence U is a (τ_i,τ_j)-(Λ,δ)-neigbourhood of y which does not contain x. By this contradiction $x \in (\tau_i,\tau_j)\Lambda_\delta Ker(A)$.

Definition 4. *A space* (X,τ_1,τ_2) *is called*
(i) pairwise Λ_δ-T_0 *if for each pair of distinct points in* X*, there is a* (τ_i,τ_j)-(Λ,δ)-*open set containing one of the points but not the other, where* $i,j = 1,2$ *and* $i \neq j$*.*
(ii) pairwise Λ_δ-T_1 *if for each pair of distinct points* x *and* y *in* X*, there is a* (τ_i,τ_j)-(Λ,δ)-*open* U *in* X *containing* x *but not* y *and a* (τ_j,τ_i)-(Λ,δ)-*open set* V *in* X *containing* y *but not* x *, where* $i,j = 1,2$ *and* $i \neq j$*.*
(iii) pairwise Λ_δ-T_2 *if for each pair of distinct points* x *and* y *in* X*, there exist a* (τ_i,τ_j)-(Λ,δ)-*open set* U *and* (τ_j,τ_i)-(Λ,δ)-*open set* V *such that* $x \in U$*,* $y \in V$ *and* $U \cap V = \emptyset$*, where* $i,j = 1,2$ *and* $i \neq j$*.*

Remark 2.6. *If a space* (X,τ_1,τ_2) *is pairwise* Λ_δ-T_i *, then it is pairwise* Λ_δ-T_{i-1}*,* $i = 1,2$*.*

Theorem 2.7. *A space* (X,τ_1,τ_2) *is pairwise* Λ_δ-T_0 *if and only if for each pair of distinct points* x*,* y *of* X*,* $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$*, where* $i,j = 1,2$ *and* $i \neq j$*.*

Proof. Sufficiency: Suppose that $x,y \in X$, $x \neq y$ and $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. Let z be a point of X such that $z \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ but $z \notin \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. We claim that $x \notin \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. For it, if $x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ then $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, where $i,j = 1,2$ and $i \neq j$. And this contradicts the fact that $z \notin \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. Consequently, x belongs to the (τ_j,τ_i)-(Λ,δ)-open set $[\{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}]^c$ to which y does not belong.

Necessity: Let (X,τ_1,τ_2) be a pairwise Λ_δ-T_0 space and x, y be any two distinct points of X. There exists a (τ_i,τ_j)-(Λ,δ)-open set G containing x or y, say x but not y. Then G^c is a (τ_i,τ_j)-(Λ,δ)-closed set which does not contain x but contains y. Since $\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ is the smallest (τ_j,τ_i)-(Λ,δ)-closed set containing y (Lemma 2.1), $\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset G^c$, and so $x \notin \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. Consequently, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, where $i,j = 1,2$ and $i \neq j$.

Theorem 2.8. *A bitopological space* (X,τ_1,τ_2) *is pairwise* Λ_δ-T_1 *if and only if the singletons are* (τ_i,τ_j)-(Λ,δ)-*closed sets, where* $i,j = 1,2$ *and* $i \neq j$*.*

Proof. Suppose that (X,τ_1,τ_2) is pairwise Λ_δ-T_1 and x be any point of X. Let $y \in \{x\}^c$. Then $x \neq y$ and so there exists a (τ_i,τ_j)-(Λ,δ)-open set U_y such that $y \in U_y$ but x$\notin U_y$. Consequently, $y \in U_y \subset \{x\}^c$ i.e., $\{x\}^c = \bigcup\{U_y/y \in \{x\}^c\}$ which is (τ_i,τ_j)-(Λ,δ)-open.

Conversely, suppose that $\{p\}$ is (τ_i, τ_j)-(Λ, δ)-closed for every $p \in X$, where $i, j = 1, 2$ and $i \neq j$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence $\{x\}^c$ is a (τ_i, τ_j)-(Λ, δ)-open set containing y but not containing x. Similarly $\{y\}^c$ is a (τ_j, τ_i)-(Λ, δ)-open set containing x but not y. Therefore, X is a pairwise Λ_δ-T_1 space.

Definition 5. *A space (X, τ_1, τ_2) is pairwise Λ_δ-symmetric if for x and y in X, $x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ implies $y \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$, where $i, j = 1, 2$ and $i \neq j$.*

Definition 6. *A subset A of a space (X, τ_1, τ_2) is called a (τ_i, τ_j)-Λ_δ-generalized closed set (briefly (τ_i, τ_j)-Λ_δ-g-closed) if $A^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset U$ whenever $A \subset U$ and U is (τ_i, τ_j)-(Λ, δ)-open, where $i, j = 1, 2$ and $i \neq j$.*

Lemma 2.9. *Every (τ_i, τ_j)-(Λ, δ)-closed set is (τ_i, τ_j)-Λ_δ-g-closed, where $i, j = 1, 2$ and $i \neq j$.*

Remark 2.10. *The converse of Lemma 2.9 is not true as shown in the following example.*

Example 2.11. *Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{c\}$ is (τ_1, τ_2)-Λ_δ-g-closed but not (τ_1, τ_2)-(Λ, δ)-closed.*

Theorem 2.12. *A space (X, τ_1, τ_2) is pairwise Λ_δ-symmetric if and only if $\{x\}$ is (τ_i, τ_j)-Λ_δ-g-closed for each $x \in X$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. Assume that $x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ but $y \notin \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. This implies that the complement of $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ contains y. Therefore, the set $\{y\}$ is a subset of the complement of $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. This implies that $\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ is a subset of the complement of $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. Now the complement of $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset D \in \Lambda_\delta O(X, \tau_i, \tau_j)$, but $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ is not a subset of D. This means that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ and the complement of D are not disjoint. Let $y \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap (X \setminus D)$. Now we have $x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ which is a subset of the complement of D and $x \notin D$. But this is a contradiction.

Corollary 2.13. *If a space (X, τ_1, τ_2) is pairwise Λ_δ-T_1 space, then it is pairwise Λ_δ-symmetric.*

Proof. In a pairwise Λ_δ-T_1 space singleton sets are (τ_i, τ_j)-(Λ, δ)-closed by Theorem 2.8, and therefore, (τ_i, τ_j)-Λ_δ-g-closed by Lemma 2.9. By Theorem 2.12, the space is pairwise Λ_δ-symmetric.

Remark 2.14. *The converse of Corollary 2.13 is not true as shown in the following example.*

Example 2.15. *Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{b, c\}, X\}$. Then X is pairwise Λ_δ-symmetric but not pairwise Λ_δ-T_1 since X and $\emptyset$ are the only (τ_i, τ_j)-(Λ, δ)-open sets for $i, j = 1, 2$ and $i \neq j$.*

Corollary 2.16. *For a space (X, τ_1, τ_2) the following are equivalent:*
(1) (X, τ_1, τ_2) is pairwise Λ_δ-symmetric and pairwise Λ_δ-T_0;
(2) (X, τ_1, τ_2) is pairwise Λ_δ-T_1.

Proof. By Corollary 2.13 and Remark 2.6, it suffices to prove only (1) $\Rightarrow$ (2). Let $x \neq y$ and by pairwise Λ_δ-T_0, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in \Lambda_\delta O(X, \tau_i, \tau_j)$. Then $x \notin \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. Therefore, we have $y \notin \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. There exists a $G_2 \in \Lambda_\delta O(X, \tau_j, \tau_i)$ such that $y \in G_2 \subset \{x\}^c$. Therefore, (X, τ_1, τ_2) is a pairwise Λ_δ-T_1 space.

Theorem 2.17. *For a pairwise Λ_δ-symmetric space (X, τ_1, τ_2) the following are equivalent:*
(1) (X, τ_1, τ_2) is pairwise Λ_δ-T_0;
(2) (X, τ_1, τ_2) is pairwise Λ_δ-T_1.

Proof. (1) $\Rightarrow$ (2) : Follows from Corollary 2.16.
(2) $\Rightarrow$ (1) : Follows from Remark 2.6.

3. Pairwise Λ_δ-R_0 spaces

Definition 7. *A space (X, τ_1, τ_2) is pairwise R_0 [5] if for each τ_i-open set G, $x \in G$ implies τ_j-$Cl(\{x\}) \subset G$, where $i, j = 1, 2$ and $i \neq j$.*

Definition 8. *A space (X, τ_1, τ_2) is a pairwise Λ_δ-R_0 if for each (τ_i, τ_j)-(Λ, δ)-open set G, $x \in G$ implies $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset G$, where $i, j = 1, 2$ and $i \neq j$.*

Proposition 3.1. *In a space (X, τ_1, τ_2) the following statements are equivalent:*
(1) (X, τ_1, τ_2) is pairwise Λ_δ-R_0;
(2) for any (τ_i, τ_j)-(Λ, δ)-closed set F and a point $x \notin F$, there exists $U \in \Lambda_\delta O(X, \tau_j, \tau_i)$ such that $x \notin U$ and $F \subset U$ for $i, j = 1, 2$ and $i \neq j$;
(3) for any (τ_i, τ_j)-(Λ, δ)-closed set F and $x \notin F$, then $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \cap F = \emptyset$, for $i, j = 1, 2$ and $i \neq j$.

Proof. (1) $\Rightarrow$ (2): Let F be a (τ_i, τ_j)-(Λ, δ)-closed set and $x \notin F$. Then by (1), $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset X \setminus F$, where $i, j = 1, 2$ and $i \neq j$. Let $U = X \setminus \{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, then $U \in \Lambda_\delta O(X, \tau_j, \tau_i)$ and also $F \subset U$ and $x \notin U$.
(2) $\Rightarrow$ (3): Let F be a (τ_i, τ_j)-(Λ, δ)-closed set and a point $x \notin F$. Then by (2), there exists $U \in \Lambda_\delta O(X, \tau_j, \tau_i)$ such that $F \subset U$ and $x \notin U$, where $i, j = 1, 2$ and $i \neq j$. Since $U \in \Lambda_\delta O(X, \tau_j, \tau_i)$, $U \cap \{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} = \emptyset$. Then $F \cap \{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} = \emptyset$, where $i, j = 1, 2$ and $i \neq j$.
(3) $\Rightarrow$ (1): Let $G \in \Lambda_\delta O(X, \tau_i, \tau_j)$ and $x \in G$. Now $X \setminus G$ is (τ_i, τ_j)-(Λ, δ)-closed and $x \notin X \setminus G$. By (3), $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \cap (X \setminus G) = \emptyset$ and hence $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset G$, where $i, j = 1, 2$ and $i \neq j$. Therefore, the space (X, τ_1, τ_2) is pairwise Λ_δ-R_0.

Proposition 3.2. *A space (X, τ_1, τ_2) is pairwise Λ_δ-R_0 if and only if for each pair x, y of distinct points in X, $\{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)} = \emptyset$ or $\{x, y\} \subset \{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$.*

Proof. Let (X, τ_1, τ_2) be pairwise Λ_δ-R_0. Suppose that $\{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)} \neq \emptyset$ and $\{x, y\}$ is not a subset of $\{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$. Let $s \in \{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ and $x \notin \{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$. Then $x \notin \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ and $x \in X \setminus \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)} \in \Lambda_\delta O(X, \tau_2, \tau_1)$. But $\{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)}$ is not a subset of $X \setminus \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ since this is a contradiction. Hence for each pair x, y of distinct points in X, $\{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)} = \emptyset$ or $\{x, y\} \subset \{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$. Conversely, let U be a (τ_1, τ_2)-(Λ, δ)-open set and $x \in U$. Suppose that $\{x\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ is not a subset of U. So there is a point $y \in \{x\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ such that $y \notin U$ and $\{y\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap U = \emptyset$. Since $X \setminus U$ is (τ_1, τ_2)-(Λ, δ)-closed and $y \in X \setminus U$. Hence $\{x, y\}$ is not a subset of $\{y\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{x\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ and thus $\{y\}^{(\Lambda,\delta)(\tau_1,\tau_2)} \cap \{x\}^{(\Lambda,\delta)(\tau_2,\tau_1)} \neq \emptyset$.

Theorem 3.3. *In a space (X, τ_1, τ_2), the following statements are equivalent:*
(1) (X, τ_1, τ_2) is pairwise Λ_δ-R_0;
(2) For any $x \in X$, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} = (\tau_j, \tau_i)$-$\Lambda_\delta Ker(\{x\}$, for $i, j = 1, 2$ and $i \neq j$;
(3) For any $x \in X$, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset (\tau_j, \tau_i)$-$\Lambda_\delta Ker(\{x\}$, for $i, j = 1, 2$ and $i \neq j$;
(4) For any $x, y \in X$, $y \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ if and only if $x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, for $i, j = 1, 2$ and $i \neq j$;
(5) For any (τ_i, τ_j)-(Λ, δ)-closed set F, $F = \cap\{G/Gisa(\tau_i, \tau_j)$-$(\Lambda, \delta)$-open set and $F \subset G\}$, for $i, j = 1, 2$ and $i \neq j$;
(6) For any (τ_i, τ_j)-(Λ, δ)-open set G, $G = \cup\{F/F$ is a (τ_i, τ_j)-(Λ, δ)-closed set and $F \subset G\}$, for $i, j = 1, 2$ and $i \neq j$;
(7) For every $A \neq \emptyset$ and each $G \in \Lambda_\delta O(X, \tau_i, \tau_j)$ such that $A \cap G \neq \emptyset$, there exists a (τ_j, τ_i)-(Λ, δ)-closed set F such that $F \subset G$ and $A \cap F \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$. Then by Lemma 2.4 and Proposition 3.2, $y \in (\tau_j, \tau_i)$-$\Lambda_\delta Ker(\{x\}) \Leftrightarrow x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \Leftrightarrow y \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. Hence $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} = (\tau_j, \tau_i)$-$\Lambda_\delta Ker(\{x\})$, where $i, j = 1, 2$ and $i \neq j$.
(2) $\Rightarrow$ (3): Straightforward.
(3) $\Rightarrow$ (4): For any $x, y \in X$, if $y \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$, then $y \in (\tau_j, \tau_i)$-$\Lambda_\delta Ker(\{x\})$ by (3). Then by Lemma 2.4, $x \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, for $i, j = 1, 2$ and $i \neq j$. The converse follows by the same token.
(4) $\Rightarrow$ (5): Let F be a (τ_i, τ_j)-(Λ, δ)-closed set and $H = \cap\{G/G$ *is a* (τ_j, τ_i)-(Λ, δ)-*open set and* $F \subset G\}$. Clearly $F \subset H$. Let $x \notin F$. Then for any $y \in F$, we have that $\{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset F$. Hence follows that $x \notin \{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. Now by (4), $x \notin \{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ implies $y \notin \{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. There exists a (τ_j, τ_i)-(Λ, δ)-open set G_y such that $y \in G_y$ and $x \notin G_y$. Let $G = \bigcup_{y \in F}\{G_y/G_y$ *is* (τ_j, τ_i)-(Λ, δ)-*open*, $y \in G_y$ *and* $x \notin G_y\}$. Thus, there exists a (τ_j, τ_i)-(Λ, δ)-open set G such that $x \notin G$ and $F \subset G$. Hence, $x \notin H$. Therefore, $F = H$.
(5) $\Rightarrow$ (6): Straightforward.
(6) $\Rightarrow$ (7): Let $A \neq \emptyset$ and G be a (τ_i, τ_j)-(Λ, δ)-open set and $x \in A \cap G$. By (6), $G = \bigcup\{F/$ *is* (τ_i, τ_j)-(Λ, δ)-*closed and* $F \subset G\}$. It follows that there is a (τ_i, τ_j)-(Λ, δ)-closed set F such that $x \in F \subset G$. Hence $A \cap F \neq \emptyset$.
(7) $\Rightarrow$ (1): Let G be a (τ_i, τ_j)-(Λ, δ)-open set and $x \in G$, then $\{x\} \cap G \neq \emptyset$. Therefore by (7), there exists a (τ_j, τ_i)-(Λ, δ)-closed F such that $x \in F \subset G$ and $\{x\} \cap F \neq \emptyset$, which implies $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset G$, where $i, j = 1, 2$ and $i \neq j$. Therefore, (X, τ_1, τ_2)

is pairwise Λ_δ-R_0.

Remark 3.4. *Let* (X,τ_1,τ_2) *be a space. Then for each* $x\in X$, *let bi-*$\{x\}^{(\Lambda,\delta)}$ $=\{x\}^{(\Lambda,\delta)(\tau_1,\tau_2)}\cap\{x\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$ *and bi-*$\Lambda_\delta Ker(\{x\})=(\tau_1,\tau_2)$-$\Lambda_\delta Ker(\{x\})\cap(\tau_2,\tau_1)$-$\Lambda_\delta Ker(\{x\})$.

Proposition 3.5. *If a space* (X,τ_1,τ_2) *is pairwise* Λ_δ-R_0 *then for each pair of distinct points* $x,y\in X$, *either bi-*$\{x\}^{(\Lambda,\delta)}=$ *bi-*$\{y\}^{(\Lambda,\delta)}$ *or bi-*$\{x\}^{(\Lambda,\delta)}\cap$*bi-*$\{y\}^{(\Lambda,\delta)}=\emptyset$.

Proof. Let (X,τ_1,τ_2) be a pairwise Λ_δ-R_0 space. Suppose that *bi-*$\{x\}^{(\Lambda,\delta)}\neq$ *bi-*$\{y\}^{(\Lambda,\delta)}$ and *bi-*$\{x\}^{(\Lambda,\delta)}\cap$ *bi-*$\{y\}^{(\Lambda,\delta)}\neq\emptyset$. Let $s\in$ *bi-*$\{x\}^{(\Lambda,\delta)}\cap$ *bi-*$\{y\}^{(\Lambda,\delta)}$ and $x\notin$ *bi-*$\{y\}^{(\Lambda,\delta)}=\{y\}^{(\Lambda,\delta)(\tau_1,\tau_2)}\cap\{y\}^{(\Lambda,\delta)(\tau_2,\tau_1)}$. Then $x\notin\{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ where $i,j=1,2$ and $i\neq j$. and $x\in X\setminus\{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}\in\Lambda_\delta O(X,\tau_i,\tau_j)$, where $i,j=1,2$ and $i\neq j$. But $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ is not a subset of $X\setminus\{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ since $s\in$ *bi-*$\{x\}^{(\Lambda,\delta)}\cap$ *bi-*$\{y\}^{(\Lambda,\delta)}$. Thus (X,τ_1,τ_2) is not a pairwise Λ_δ-R_0 space which is a contradiction to our assumption. Hence we have either *bi-*$\{x\}^{(\Lambda,\delta)}=$ bi-$\{y\}^{(\Lambda,\delta)}$ or *bi-*$\{x\}^{(\Lambda,\delta)}\cap$ *bi-*$\{y\}^{(\Lambda,\delta)}=\emptyset$.

Theorem 3.6. *In a space* (X,τ_1,τ_2), *the following properties are equivalent:*
(1) (X,τ_1,τ_2) *is pairwise* Λ_δ-R_0;
(2) For any (τ_i,τ_j)-(Λ,δ)*-closed set* $F\subset X$, $F=(\tau_j,\tau_i)$-$\Lambda_\delta Ker(F)$, *where* $i,j=1,2$ *and* $i\neq j$;
(3) For any (τ_i,τ_j)-(Λ,δ)*-closed set* $F\subset X$ *and* $x\in F$, (τ_j,τ_i)-$\Lambda_\delta Ker(\{x\})\subset F$, *where* $i,j=1,2$ *and* $i\neq j$;
(4) For any $x\in X$, (τ_j,τ_i)-$\Lambda_\delta Ker(\{x\})\subset\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$, *where* $i,j=1,2$ *and* $i\neq j$.

Proof. (1) $\Rightarrow$ (2): Let F be (τ_i,τ_j)-(Λ,δ)-closed and $x\notin F$. Then $X\setminus F$ is (τ_i,τ_j)-(Λ,δ)-open containing x. Since (X,τ_1,τ_2) is pairwise Λ_δ-R_0, $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)}\subset X\setminus F$, where $i,j=1,2$ and $i\neq j$. Therefore, $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)}\cap F=\emptyset$ and by Proposition 2.5, $x\notin(\tau_j,\tau_i)$-$\Lambda_\delta Ker(F)$. Hence $F=(\tau_j,\tau_i)$-$\Lambda_\delta Ker(F)$, where $i,j=1,2$ and $i\neq j$.
(2) $\Rightarrow$ (3): Let F be a (τ_i,τ_j)-(Λ,δ)-closed set containing x. Then $\{x\}\subset F$ and (τ_j,τ_i)-$\Lambda_\delta Ker(\{x\})\subset(\tau_j,\tau_i)$-$\Lambda_\delta Ker(F)$. From (2), it follows that (τ_j,τ_i)-$\Lambda_\delta Ker(\{x\})\subset F$, where $i,j=1,2$ and $i\neq j$.
(3) $\Rightarrow$ (4): Since $x\in\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ and $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$ is (τ_i,τ_j)-(Λ,δ)-closed in X, by (3) it follows that (τ_j,τ_i)-$\Lambda_\delta Ker(F)\subset\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$, where $i,j=1,2$ and $i\neq j$.
(4) $\Rightarrow$ (1): It follows from Theorem 3.3.

4. Pairwise Λ_δ-R_1 spaces

Definition 9. *A space* (X,τ_1,τ_2) *is said to be pairwise* Λ_δ-R_1 *if for each* $x,y\in X$, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}\neq\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, *there exist disjoint sets* $U\in\Lambda_\delta O(X,\tau_j,\tau_i)$ *and* $V\in\Lambda_\delta O(X,\tau_i,\tau_j)$ *such that* $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)}\subset U$ *and* $\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}\subset V$, *where* $i,j=1,2$ *and* $i\neq j$.

Proposition 4.1. *If* (X,τ_1,τ_2) *is pairwise* Λ_δ-R_1 , *then it is pairwise* Λ_δ-R_0.

Proof. Suppose that (X, τ_1, τ_2) is pairwise Λ_δ-R_1 . Let U be a (τ_i, τ_j)-(Λ, δ)-open set and $x \in U$. Then for each point $y \in X \setminus U$, $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \neq \{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)}$. Since (X, τ_1, τ_2) is pairwise Λ_δ-R_1, there exists a (τ_i, τ_j)-(Λ, δ)-open set U_y and a (τ_j, τ_i)-(Λ, δ)-open set V_y such that $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset U_y$, $\{y\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset V_y$ and $U_y \cap V_y = \emptyset$, where $i, j = 1, 2$ and $i \neq j$. Let $A = \bigcup\{V_y \mid y \in X \setminus U\}$. Then $X \setminus U \subset A$, $x \notin A$ and A is a (τ_j, τ_i)-(Λ, δ)-open set. Therefore, $\{x\}^{(\Lambda,\delta)(\tau_j,\tau_i))} \subset X \setminus A \subset U$. Hence (X, τ_1, τ_2) is pairwise Λ_δ-R_0.

Proposition 4.2. *A space (X, τ_1, τ_2) is pairwise Λ_δ-R_1 if and only if for every pair of points x and y of X such that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, there exists a (τ_i, τ_j)-(Λ, δ)-open set U and (τ_j, τ_i)-(Λ, δ)-open set V such that $x \in V$, $y \in U$ and $U \cap V = \emptyset$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. Suppose that (X, τ_1, τ_2) is pairwise Λ_δ-R_1. Let x, y be points of X such that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, where $i, j = 1, 2$ and $i \neq j$. Then there exist a (τ_i, τ_j)-(Λ, δ)-open set U and a (τ_j, τ_i)-(Λ, δ)-open set V such that $x \in \{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset V$ and $y \in \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset U$. On the other hand, suppose that there exists a (τ_i, τ_j)-(Λ, δ)-open set U and (τ_j, τ_i)-(Λ, δ)-open set V such that $x \in V$, $y \in U$ and $U \cap V = \emptyset$, where $i, j = 1, 2$ and $i \neq j$. Since every pairwise Λ_δ-R_1 space is pairwise Λ_δ-R_0, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset V$ and $\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset U$. Hence the claim.

Proposition 4.3. *A pairwise Λ_δ-R_0 space (X, τ_1, τ_2) is pairwise Λ_δ-R_1 if for each pair of points x and y of X such that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} = \emptyset$, there exist disjoint sets $U \in \Lambda_\delta O(X, \tau_i, \tau_j)$ and $V \in \Lambda_\delta O(X, \tau_j, \tau_i)$ such that $x \in U$ and $y \in V$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. It follows directly from Definition 8 and Proposition 3.5.

Theorem 4.4. *In a space (X, τ_1, τ_2), the following statements are equivalent:*
(1) (X, τ_1, τ_2) is pairwise Λ_δ-R_1;
(2) For any two distinct points $x, y \in X$, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$ implies that there exist a (τ_i, τ_j)-(Λ, δ)-closed set F_1 and a (τ_j, τ_i)-(Λ, δ)-closed set F_2 such that $x \in F_1$, $y \in F_2$, $x \notin F_2$, $y \notin F_1$ and $X = F_1 \cup F_2$, where $i, j = 1, 2$ and $i \neq j$.

Proof. (1) $\Rightarrow$ (2): Suppose that (X, τ_1, τ_2) is pairwise Λ_δ-R_1. Let $x, y \in X$ such that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$. By Proposition 4.2, there exist disjoint sets $V \in \Lambda_\delta O(X, \tau_i, \tau_j)$ and $U \in \Lambda_\delta O(X, \tau_j, \tau_i)$ such that $x \in U$ and $y \in V$, where $i, j = 1, 2$ and $i \neq j$. Then $F_1 = X \setminus V$ is a (τ_i, τ_j)-(Λ, δ)-closed set and $F_2 = X \setminus U$ is a (τ_j, τ_i)-(Λ, δ)-closed set such that $x \in F_1$, $x \notin F_2$, $y \notin F_1$, $y \in F_2$ and $X = F_1 \cup F_2$, where $i, j = 1, 2$ and $i \neq j$.
(2) $\Rightarrow$ (1): Let $x, y \in X$ such that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \neq \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)}$, where $i, j = 1, 2$ and $i \neq j$. Hence for any two distinct points x, y of X, $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \cap \{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} = \emptyset$, where $i, j = 1, 2$ and $i \neq j$. Then by Proposition 3.2, (X, τ_1, τ_2) is pairwise Λ_δ-R_0. By (2), there exists a (τ_i, τ_j)-(Λ, δ)-closed set F_1 and a (τ_j, τ_i)-(Λ, δ)-closed set F_2 such that $X = F_1 \cup F_2$, $x \in F_1$, $y \in F_2$, $x \notin F_2$ and $y \notin F_1$. Therefore, $x \in X \setminus F_2 = U \in \Lambda_\delta O(X, \tau_j, \tau_i)$ and $y \in X \setminus F_1 = V \in \Lambda_\delta O(X, \tau_i, \tau_j)$ which implies that $\{x\}^{(\Lambda,\delta)(\tau_i,\tau_j)} \subset U$, $\{y\}^{(\Lambda,\delta)(\tau_j,\tau_i)} \subset V$ and $U \cap V = \emptyset$, where $i, j = 1, 2$ and $i \neq j$. Hence (X, τ_1, τ_2) is pairwise Λ_δ-R_0.

References

1. M. Caldas and S. Jafari, On some low separation axioms in topological space, Houston Journal of Math. 29(2003), 93-104.

2. A. S. Davis, Indexed systems of neighbourhoods for general topological spaces, Amer. Math. Monthly 68(1961), 886-893.

3. J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. (3)(13)(1963),71-89.

4. A. S. Mashhour, F. H. Khedr and S. N. El-Deeb, Five separation axioms in bitopological spaces, Bull. Fac. Sci. Assiut. Univ. 11(1) (1982), 53-67.

5. M. G. Murdeshwar and S. A. Naimpally, R_1-topological spaces, Canad. Math. Bull. 9(1966), 521-523.

6. S. A. Naimpally, On R_0-topological spaces, Ann. Univ. Sci. Budapest Eötvös Sect. Math. 10(1967), 53-54.

7. T. M. Nour, A note on five separation axioms in bitopological spaces, Indian J. Pure and Appl. Math. 26(7) (1995), 669-674.

8. N. A. Shanin, On separation in topological spaces, Dokl. Akad. Nauk. SSSR, 38 (1943), 110-113.

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