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A Characterization for Discrete Quantum Group [1]

Ming Liu, Lining Jiang and Guosheng Zhang

ABSTRACT: Based on the work of A.Van Daele, E.G.Effros and Z.J.Ruan on multiplier Hopf algerba and discrete quantum group, this paper states that discrete quantum group $(A,\ \Delta)$ is exactly the set $\{(\omega\otimes\iota)\Delta(a)|a\in A,\omega\in A^*\}$, where A^* is the space of all reduced functionals on A. Furthermore, this paper characterizes $(A,\ \Delta)$ as an algebraic quantum group with a standard *-operation and a special element $z\in A$ such that $(1\otimes a)\Delta(z)=\Delta(z)(a\otimes 1)$ $(\forall a\in A)$.

Key Words: discrete quantum group, reduced functional, cointegral.

Contents

1. Introduction

Let G be a discrete group. If G is finite, one can define the pointwise product $(fg)(p)=f(p)g(p)$ and the natural *-operation $f^*(p)=\overline{f(p)}$ on $C(G)$, the space of all complex functions on G, to make it into a unital *-algebra, where $f,g\in C(G), p\in G$. Furthermore, under the structure maps

$$\Delta(f)(p,q)=f(pq),\varepsilon(f)=f(e),S(f)(p)=f(p^{-1}),$$

$C(G)$ becomes a Hopf *-algebra. Here $C(G)\otimes C(G)$ is identified with the algebra of complex functions on $G\times G$ in the obvious way. If G is infinite, this is no longer possible. One then consider $K(G)$, the space of all complex functions with finite support on G. It is easy to check that $K(G)$ is a *-algebra and the range of Δ is not in $K(G)\otimes K(G)$ any more. However for any $f,g\in K(G)$ we have that $\Delta(f)(1\otimes g)$ and $\Delta(f)(g\otimes 1)$ are both in $K(G)\otimes K(G)$. This leads to the concept of multiplier Hopf *-algebras [1].

Let A be an algebra with a non-degenerate product, with or without identity. A multiplier of A is a pair (ρ_1,ρ_2) of linear maps from A to itself satisfying for all $a,b\in A$,

$$\rho_1(ab)=\rho_1(a)b,\quad \rho_2(ab)=a\rho_2(b),\quad \rho_2(a)b=a\rho_1(b).$$

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The set of all multipliers of A, denoted by $M(A)$, is made into a unital algebra in a natural way and called the multiplier algebra of A. There is a natural embedding of A into $M(A)$. Furthermore, $M(A)$ can be characterized as the largest algebra with identity in which A sits as an essential two-sided ideal. It's customary to alter (ρ_1, ρ_2) by an auxiliary object m by treating ρ_1 and ρ_2 as left and right multiplication, i.e., $\rho_1(a) = ma$, $\rho_2(a) = am$. Then to show $m \in M(A)$, it suffices to verify that A is a two-sided ideal of $\{m \mid a(mb) = (am)b, \forall a, b \in A\}$.

A comultiplication on A is defined as a homomorphism Δ: $A \longrightarrow M(A \otimes A)$ satisfying
i) $\Delta(A)(1 \otimes A)$ and $(A \otimes 1)\Delta(A)$ are subspaces of $A \otimes A$;
ii) Δ is coassociative in the following sense: $\forall a, b, c \in A$,

$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c).$$

Now let $(A,\ \Delta)$ be a pair of an algebra A with a non-degenerate product and a comultiplication Δ on A. If the maps T_1 and T_2 defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \quad T_2(a \otimes b) = (a \otimes 1)\Delta(b) \quad (a, b \in A)$$

are bijective, we call $(A,\ \Delta)$ a multiplier Hopf algebra, and call it regular if $(A,\ \Delta^{\circ p})$ is also a multiplier Hopf algebra (or equivalently if the antipode S is bijective from A to A). In fact, if $(A,\ \Delta)$ is a regular multiplier Hopf algebra, $a, b \in A$, then

$$\Delta(a)(1 \otimes b), \quad (a \otimes 1)\Delta(b)$$

$$\Delta(a)(b \otimes 1), \quad (1 \otimes a)\Delta(b)$$

all belong to $A \otimes A$. When a multiplier Hopf algebra has also a standard *-operation [2], it is called a multiplier Hopf *-algebra. It is clear that $K(G)$ described above is a multiplier Hopf *-algebra.

As we have known, a non-zero linear functional φ(resp.ψ) on a regular multiplier Hopf algebra $(A,\ \Delta)$ is called left (resp.right) integral if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ (resp.$(\psi \otimes \iota)\Delta(a) = \psi(a)1$) for all $a \in A$, where 1 denotes the identity in $M(A)$. In general such integrals do not always exist. Moreover, the left and right integrals need not be the same one even if they both exist. A regular multiplier Hopf algebra with a left (and hence a right) integral is called an algebraic quantum group. The paper will study a special class of algebraic quantum group (see [7]), namely, discrete quantum group, which was studied firstly as a dual of compact quantum group in [3]. A discrete quantum group is defined as a multiplier Hopf *-algebra $(A,\ \Delta)$ where A is a direct sum of full matrix algebras ([4-9]). More specifically, let $(A,\ \Delta)$ be an algebraic quantum group with a standard *-operation and a non-zero element $z \in A$ such that $\forall a \in A$, $(1 \otimes a)\Delta(z) = \Delta(z)(a \otimes 1)$, and A^* the space of all reduced functionals on A (see Definition 2.2). Then by Proposition 2.1 one can see that $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ are in $M(A)$, where $a \in A, \omega \in A^*$. Using the two types of multipliers, if $(A,\ \Delta)$ is a discrete quantum group, then

$$A = \{(\omega \otimes \iota)\Delta(a) | a \in A, \omega \in A^*\} = \{(\iota \otimes \omega)\Delta(a) | a \in A, \omega \in A^*\}.$$

From this, the paper gives a characterization of a discrete quantum group, as follows a discrete quantum group coincides with an algebraic quantum group with a standard *-operation and a non-zero element $z \in A$ such that $\forall a \in A$,

$$(1 \otimes a)\Delta(z) = \Delta(z)(a \otimes 1).$$

Throughout this paper, all algebras will be algebras over the complex field $\mathbb{C}$ and ι denotes the identity map. For general results on multiplier Hopf algebras theory, we refer the reader to [1, 10]. We shall use their notations, so we will use $m, \Delta, \varepsilon, S$ for the multiplication, the comultiplication, the counit and the antipode, respectively.

2. Characterization for Discrete Quantum Group

Let $(A,\ \Delta)$ be a regular multiplier Hopf algebra and A' the space of all linear functionals on A. Using $\forall a \in A$ and $\omega \in A'$, one can construct a multiplier of A.

For any $b \in A$, it is clear that $(\omega\otimes\iota)(\Delta(a)(1\otimes b)) \in A$ and $(\omega\otimes\iota)((1\otimes b)\Delta(a)) \in A$. That's to say, there exist maps ρ_1 and ρ_2 from A to itself defined by

$$\rho_1(b) = (\omega \otimes \iota)(\Delta(a)(1 \otimes b)),$$

$$\rho_2(b) = (\omega \otimes \iota)((1 \otimes b)\Delta(a)).$$

These are well defined because both $\Delta(a)(1 \otimes b)$ and $(1 \otimes b)\Delta(a)$ are in $A \otimes A$, and one can apply $\omega \otimes \iota$ mapping $A \otimes A$ to $A \otimes \mathbb{C}$, which is naturally identified with A itself.

Proposition 2.1 *Let $(A,\ \Delta)$ be a regular multiplier Hopf algebra and ρ_1, ρ_2 as defined above. Then $(\rho_1, \rho_2) \in M(A)$.*

Proof To prove that (ρ_1, ρ_2) is a multiplier of A, it suffices to prove $\rho_2(c)b = c\rho_1(b)$, for all $b, c \in A$.

$$\begin{aligned} c\rho_1(b) &= c((\omega \otimes \iota)(\Delta(a)(1 \otimes b))) \\ &= (\omega \otimes \iota)((1 \otimes c)(\Delta(a)(1 \otimes b))) \\ &= (\omega \otimes \iota)(((1 \otimes c)\Delta(a))(1 \otimes b)) \\ &= (\omega \otimes \iota)((1 \otimes c)\Delta(a))b \\ &= \rho_2(c)b. \end{aligned}$$

Thus $(\rho_1, \rho_2) \in M(A)$. ■

In the following, by $(\omega \otimes \iota)\Delta(a)$ we denote the multiplier (ρ_1, ρ_2). Similarly, put

$$\eta_1(b) = (\iota \otimes \omega)(\Delta(a)(b \otimes 1)),$$
$$\eta_2(b) = (\iota \otimes \omega)((b \otimes 1)\Delta(a)).$$

Then $(\eta_1, \eta_2) \in M(A)$ and can be written as $(\iota \otimes \omega)\Delta(a)$.

In general $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ are not in A. Indeed, consider the algebra A generated by $\{e_p, b | p \in \mathbb{Z}\}$ subject to

$$e_p e_q = \delta(p, q)e_p, \quad be_p = e_{p+1}b.$$

Then [8] A is a regular multiplier Hopf algebra with a comultiplication Δ on A defined by:

$$\Delta(e_p) = \sum_{k\in\mathbb{Z}} e_k \otimes e_{p-k},$$
$$\Delta(b) = a \otimes b + b \otimes a^{-1}.$$

Here $\lambda \in \mathbb{C} \setminus \{0\}$ and $a = \sum_{k\in\mathbb{Z}} \lambda^k e_k \in M(A)$. Notice that these infinite sums are well-defined in the "strict topology" on $M(A)$ (i.e., when one multiplies with elements of the algebra, one gets finite sums). Then for $\omega \in A^{'}$,

$$(\omega \otimes \iota)\Delta(e_p) = \sum_{k\in\mathbb{Z}} \omega(e_k)e_{p-k} \in M(A) - A,$$
$$(\omega \otimes \iota)\Delta(b) = \omega(a)b + \omega(b)a^{-1} \in M(A) - A.$$

Based on the example, it is natural to ask under which condition for a multiplier Hopf algebra the elements $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ are in A. With the help of reduced functional, whose definition is given below, one can answer the question and furthermore give a characterization for a discrete quantum group.

Definition 2.2 *Let (A, Δ) be a multiplier Hopf algebra and A' the space of all linear functionals on A. $\forall a \in A$ and $f \in A'$, one can define the left and right action of a on f, respectively: $\forall x \in A$*

$$af(x) := f(xa); \quad fa(x) := f(ax).$$

Furthermore, for $a, b \in A$ and $f \in A'$, one can define the functional afb by

$$afb(x) = f(bxa),$$

and call it a reduced functional on A.

In the following, by A^* we denote the set of all reduced functionals on A and by A_1, A_2 we denote the space $\{(\omega \otimes \iota)\Delta(a)|a \in A, \omega \in A^*\}$ and the space$\{(\iota \otimes \omega)\Delta(a)|a \in A, \omega \in A^*\}$, respectively. And we will denote them by $\widetilde{A}$ when $A_1 = A_2$. Furthermore, one can find that both A_1 and A_2 are subsets of A. Indeed, for all $a, b, c \in A, f \in A^{'}$,

$$(afb \otimes \iota)\,\Delta(c) = (f \otimes \iota)\left((b \otimes 1)\Delta(c)(a \otimes 1)\right),$$

$$(\iota \otimes afb)\,\Delta(c) = (\iota \otimes f)\left((1 \otimes b)\Delta(c)(1 \otimes a)\right).$$

For an algebraic quantum group (A, Δ),

$$\hat{A} = \{a\varphi|a \in A\} = \{\varphi a|a \in A\}$$

is defined as the dual of A, where φ is the left integral on A[7]. Now for $f, g \in A'$, notice that the functional $f \otimes g : A \otimes A \to \mathbb{C}$ can be extended uniquely to a functional $f \otimes g : M(A \otimes A) \to \mathbb{C}$ ([1]), put

$$f \star g(a) := (f \otimes g)\Delta(a),$$

then $f \star g \in A'$ is well defined.

Lemma 2.3 *Let (A, Δ) be an algebraic quantum group and $\hat{A}$ as defined above. Set*

$$\hat{A}_0 = \{\omega \in A^{'} | (\omega \otimes \iota)\Delta(a) \in A, (\iota \otimes \omega)\Delta(a) \in A, \forall a \in A\}.$$

Then (1) $\hat{A} \subset \hat{A}_0$;

(2) $\hat{A}_0$ is a unital associative algebra;

(3) $M(\hat{A}) = \hat{A}_0$.

Proof (1) For $a, b \in A$, since

$$\begin{aligned}(\iota \otimes b\varphi)\Delta(a) &= (\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) \\ &= (\iota \otimes \varphi)(\sum_{(a)} a_{(1)} \otimes a_{(2)} b) \\ &= \sum_{(a)} a_{(1)} \varphi(a_{(2)} b) \in A,\end{aligned}$$

where $\Delta(a)(1 \otimes b) = \sum_{(a)} a_{(1)} \otimes a_{(2)} b$ (this is possible for a regular multiplier Hopf algebra). Therefore $(\iota \otimes b\varphi)\Delta(a)$, and similarly $(b\varphi \otimes \iota)\Delta(a) \in A$. This means $b\varphi \in \hat{A}_0$ and $\hat{A} \subset \hat{A}_0$.

(2) Now for any $\omega_1, \omega_2 \in \hat{A}_0$, we have

$$((\omega_1 \star \omega_2) \otimes \iota)\Delta(a) = (\omega_2 \otimes \iota)\Delta((\omega_1 \otimes \iota)\Delta(a)),$$

$$(\iota \otimes (\omega_1 \star \omega_2))\Delta(a) = (\iota \otimes \omega_1)\Delta((\iota \otimes \omega_2)\Delta(a)).$$

In fact, $\forall a, b \in A$, $(\omega_1 \otimes \iota)\,\Delta\,(a) \in A$ and $(\omega_2 \otimes \iota)\,\Delta\,((\omega_1 \otimes \iota)\,\Delta\,(a)\,(b \otimes 1)) \in A$. Furthermore $\forall c \in A$,

$$\begin{aligned}c(\omega_2 \otimes \iota)\Delta((\omega_1 \otimes \iota)(\Delta(a)(b \otimes 1))) &= (\omega_2 \otimes \iota)((1 \otimes c)\Delta((\omega_1 \otimes \iota)\sum_{(a)} a_{(1)} b \otimes a_{(2)})) \\ &= \sum_{(a)} \omega_1(a_{(1)} b)(\omega_2 \otimes \iota)((1 \otimes c)\Delta(a_{(2)})) \\ &= \sum_{(a)} \omega_1(a_{(1)} b)\omega_2(a_{(2)}) c a_{(3)} \\ &= c\sum_{(a)} (\omega_1 \otimes \omega_2 \otimes \iota)(a_{(1)} b \otimes a_{(2)} \otimes a_{(3)}) \\ &= c\sum_{(a)} (\omega_1 \otimes \omega_2 \otimes \iota)(a_{(1)} b \otimes \Delta(a_{(2)})) \\ &= c\sum_{(a)} (\omega_1 \otimes \omega_2 \otimes \iota)(\Delta(a_{(1)} b) \otimes a_{(2)}) \\ &= c((\omega_1 \star \omega_2) \otimes \iota)(\Delta(a)(b \otimes 1)),\end{aligned}$$

for the last second equation we use the coassociativity of the comultiplication Δ. For the arbitrariness of b, one can get

$$c(\omega_2 \otimes \iota)\Delta((\omega_1 \otimes \iota)\Delta(a)) = c((\omega_1 \star \omega_2) \otimes \iota)\Delta(a),$$

and thus

$$((\omega_1 \star \omega_2) \otimes \iota)\Delta(a) = (\omega_2 \otimes \iota)\Delta((\omega_1 \otimes \iota)\Delta(a)).$$

For the second formula, we have the similar discussion.

Furthermore $\hat{A}_0$ is a unital associative algebra under the convolution operation. Indeed for $\forall a \in A$,

$$(\varepsilon \otimes \iota)\Delta(a) = (\iota \otimes \varepsilon)\Delta(a) = a,$$

which implies that $\hat{A}_0$ has a unit ε. And for all $a \in A$,

$$((\omega_1 \star \omega_2) \star \omega_3 \otimes \iota)\Delta(a) = (\omega_3 \otimes \iota)\Delta((\omega_2 \otimes \iota)\Delta((\omega_1 \otimes \iota)\Delta(a))) = (\omega_1 \star (\omega_2 \star \omega_3) \otimes \iota)\Delta(a),$$

$$(\iota \otimes (\omega_1 \star \omega_2) \star \omega_3))\Delta(a) = (\iota \otimes \omega_1)\Delta((\iota \otimes \omega_2)\Delta((\iota \otimes \omega_3)\Delta(a))) = (\iota \otimes \omega_1 \star (\omega_2 \star \omega_3))\Delta(a).$$

So $(\omega_1 \star \omega_2) \star \omega_3 = \omega_1 \star (\omega_2 \star \omega_3)$. Henceforth, $\hat{A}_0$ is an associative algebra with identity.

(3) As we have known, $A^{'}$ is also an associative algebra. Then $\forall f \in M(A)$, $\forall \omega \in \hat{A}_0$, $\forall b\varphi, c\varphi \in \hat{A}$, the associativity of the (convolution) product in $A^{'}$ leads to the relations

$$(b\varphi \star f) \star \omega = b\varphi \star (f \star \omega),$$
$$\omega \star (f \star c\varphi) = (\omega \star f) \star c\varphi,$$

which implies

$$(\hat{A} \star M(\hat{A})) \star \hat{A}_0 = \hat{A} \star (M(\hat{A}) \star \hat{A}_0),$$
$$\hat{A}_0 \star (M(\hat{A}) \star \hat{A}) = (\hat{A}_0 \star M(\hat{A})) \star \hat{A}.$$

Since $M(\hat{A})$ is the multiplier algebra of $\hat{A}$, i.e., $\hat{A} = \hat{A} \star M(\hat{A}) = M(\hat{A}) \star \hat{A}$,

$$\hat{A} \star \hat{A}_0 = \hat{A} \star (M(\hat{A}) \star \hat{A}_0),$$

$$\hat{A}_0 \star \hat{A} = (\hat{A}_0 \star M(\hat{A})) \star \hat{A}.$$

From the non-degeneracy of the (convolution) product, one can get

$$\hat{A}_0 = M(\hat{A}) \star \hat{A}_0 = \hat{A}_0 \star M(\hat{A}),$$

which shows that $\hat{A}_0$ is a two-sided ideal of $M(\hat{A})$. Again $\hat{A}_0$ is unital, therefore $\hat{A}_0 = M(\hat{A})$. ■

Remark 2.4 One can prove that $\hat{A}$ is a two-sided ideal of A^* and A^* is a subalgebra of $M(\hat{A})$ (see [6]). Then

$$\hat{A} \subset A^* \subset \hat{A}_0 = M(\hat{A}).$$

Corresponding to integrals, one can get the notion of cointegrals. A left cointegral in a regular multiplier Hopf algebra is a non-zero element $h \in A$ such that $ah = \varepsilon(a)h$ for all $a \in A$. Similarly, a non-zero element $k \in A$ satisfying $ka = \varepsilon(a)k$ is called a right cointegral. They do not always exist and need not coincide as if they exist. However, they are unique (up to a scalar) if they exist. They are faithful if

$$(\omega \otimes \iota)\Delta(h) = 0 \text{ implies } \omega = 0,$$
$$(\iota \otimes \omega)\Delta(h) = 0 \text{ implies } \omega = 0.$$

Using the cointegral, we have the following definition.
Definition 2.5[8] *Let (A,Δ) be an algebraic quantum group. We call (A,Δ) of compact type if A has an identity, i.e., A is a Hopf algebra. We call (A,Δ) of discrete type if A has a left (resp. right) cointegral.*

With the help of Lemma 2.3 and Remark 2.4, we have the following precise results for a special class of algebraic quantum group—discrete quantum group.
Proposition 2.6 *Let (A,Δ) be a discrete quantum group and $A^{'}$, A^* as defined above. Then $\forall f\in A^{'}$, $f\in A^*$ if and only if for all $a\in A$,*

$$(\iota\otimes f)\Delta(a)\in A \text{ and } (f\otimes\iota)\Delta(a)\in A.$$

Proof It suffices to prove the sufficiency. As (A,Δ) is a discrete quantum group, it is natural of discrete type. By Proposition 5.3 in [7], the dual $\hat{A}$ of A is of compact type. So $\hat{A}$ has an identity and hence $M(\hat{A})=\hat{A}$. Using Remark 2.4, $\hat{A}_0=A^*$ and therefore $f\in A^*$. ■
Proposition 2.7 *Let (A,Δ) be a discrete quantum group and $\widetilde{A}$ as described above. Then $A=\widetilde{A}$.*
Proof Firstly, we show that $A_1=A_2$. Indeed, for any $a,b\in A$,

$$\begin{aligned}((\omega\otimes\iota)\Delta(a))^*b &= (b^*(\omega\otimes\iota)\Delta(a))^*\\ &= ((\omega\otimes\iota)((1\otimes b^*)\Delta(a)))^*\\ &= ((\omega\otimes\iota)\sum_{(a)}a_{(1)}\otimes b^*a_{(2)})^*\\ &= \sum_{(a)}\overline{\omega(a_{(1)})}a^*_{(2)}b\\ &= \sum_{(a)}\omega^*(S(a^*_{(1)}))a^*_{(2)}b\\ &= S^{-1}((\iota\otimes\omega^*)\Delta(S(a^*)))b.\end{aligned}$$

So

$$((\omega\otimes\iota)\Delta(a))^*=S^{-1}((\iota\otimes\omega^*)\Delta(S(a^*))),$$

and similarly

$$((\iota\otimes\omega)\Delta(a))^*=S^{-1}((\omega^*\otimes\iota)\Delta(S(a^*))),$$

which means that $(\omega\otimes\iota)(\Delta(a)$ and $(\iota\otimes\omega)(\Delta(a)$ can be represented each other.

Secondly, we shows that $A\subseteq\widetilde{A}$. Using Proposition 3.1 in [4], A has a left cointegral h satisfying $h^2=h^*=h$. Set

$$I=\{(\omega\otimes\iota)\Delta(h)|\omega\in A_0^{'}\},$$

where $A_0^{'}$ is the set of linear functionals on A which are supported on finitely many components of A. Here $A_0^{'}=A^*$. In fact, if $f\in A_0^{'}$, then for all $x\in A$, by Proposition 3.1 in [8] there exists an element e (call it a local unit) such that $xe=ex$. Hence

$$f(x)=f(xe)=f(ex)=(efe)(x),$$

which implies $f\in A^*$. If $f=af^{'}b\in A^*$, $a,b\in A, f^{'}\in A^{'}$. Then $\forall x\in A$, $f(x)=f^{'}(bxa)$. Since A is a direct sum of matrix algebra, bxa is in finitely many simple summands of A and hence $f^{'}(bxa)$ is non-zero on finitely many components of A. So $f \in A_0^{'}$. Moreover, I is a two-sided ideal of A. Indeed, for any $a \in A$,

$$\Delta(h)(1 \otimes a) = \Delta(h^2)(1 \otimes a) = \Delta(h)\Delta(h)(1 \otimes a) \in \Delta(h)(A \otimes 1),$$

and therefore

$$((\omega \otimes \iota)\Delta(h))\, a = (\omega \otimes \iota)\,(\Delta(h)(1 \otimes a)) \in I.$$

Then $Ia \subseteq I$. Similarly, $aI \subseteq I$. Pick an element $a \neq 0$ such that $Ia = 0$. Then for all $\omega \in A^*$, $(\omega \otimes \iota)\,(\Delta(h)(1 \otimes a)) = 0$. So $\Delta(h)(1 \otimes a) = 0$, which implies $a = 0$ and leads to a contradiction. Thus $I = A$. Obviously $I \subseteq \widetilde{A}$ and hence $A \subseteq \widetilde{A}$. ■

With the help of Lemma 2.3, we propose a characterization for a discrete quantum group as follows.

Proposition 2.8 (A, Δ) *is a discrete quantum group if and only if* (A, Δ) *is an algebraic quantum group with a standard *-operation and a non-zero element* $z \in A$ *such that* $\forall a \in A$,

$$(1 \otimes a)\Delta(z) = \Delta(z)(a \otimes 1).$$

Proof We just need to prove the sufficiency. As $\widetilde{A} \subseteq A$ holds for any algebraic quantum group (A, Δ), in particular we have $(\omega \otimes \iota)\Delta(z) \in A$. Define a map $\Gamma : \hat{A}_0 \longrightarrow A$ by $\Gamma(\omega) = (\omega \otimes \iota)\Delta(z)$, where $\hat{A}_0$ is defined as in Lemma 2.3. It is obvious that Γ is well defined. Furthermore, one can prove that Γ is an injective A-module homomorphism. Indeed, if $(\omega \otimes \iota)\Delta(z) = 0$, then $\forall a \in A$

$$\begin{aligned} 0 &= a(\omega \otimes \iota)\Delta(z) \\ &= (\omega \otimes \iota)((1 \otimes a)\Delta(z)) \\ &= (\omega \otimes \iota)(\Delta(z)(a \otimes 1)) \\ &= \sum_{(a)} \omega(z_{(1)}a)z_{(2)}. \end{aligned}$$

Applying Δ and S to this formula, one can get

$$\sum_{(a)} \omega(z_{(3)}a)z_{(1)} \otimes S(z_{(2)}) = 0.$$

And replacing a by $S(z_{(2)})a$, one can obtain

$$\sum_{(a)} \omega(z_{(3)}S(z_{(2)})a)z_{(1)} = 0$$

and hence also $\omega(a)z = 0$ $(\forall a \in A)$, which implies $\omega = 0$ considering the fact $z \neq 0$.

For any $a \in A$,

$$\begin{aligned} a\Gamma(\omega) &= (\omega \otimes \iota)((1 \otimes a)\Delta(z)) \\ &= (\omega \otimes \iota)(\Delta(z)(a \otimes 1)) \\ &= (a\omega \otimes \iota)\Delta(z) \\ &= \Gamma(a\omega), \end{aligned}$$

which shows that Γ is A-module homomorphism.

Take $h = \Gamma(\varepsilon)$. Then $\forall a \in A$,

$$ah = a\Gamma(\varepsilon) = \Gamma(a\varepsilon) = \varepsilon(a)\Gamma(\varepsilon) = \varepsilon(a)h.$$

Thus h is a left cointegral of (A, Δ) and (A, Δ) is of discrete type. From [5, Theorem 3.1], A has local units. Denote them by $\{e_\alpha\}_{\alpha\in I}$. Then

$$A = \bigoplus_{\alpha\in I} Ae_\alpha.$$

Combining with the fact that (A, Δ) is an algebraic quantum group with a standard *-operation, A can be written as a direct sum of full matrix algebras. Namely, A is a discrete quantum group and this completes the proof. ■

Example 2.9 Let us look closer at $K(G)$, the space of all complex functions with finite support on G. Considering the fact that

$$(f\delta_e)(p) = (\delta_e f)(p) = f(p)\delta_e(p) = \varepsilon(f)\delta_e(p), \quad (\forall f \in K(G), p \in G)$$

where δ_e is the function taking value 1 on the unit e of G and 0 elsewhere, the element δ_e, denoted by h, is a cointegral on $K(G)$. Therefore $K(G)$ is an algebraic quantum group of discrete type. Again $(K(G), \Delta)$ has a standard *-operation, $K(G)$ is a discrete quantum group. It is clear that $\widetilde{K(G)} \subseteq K(G)$. On the other hand, suppose that φ is the left integral on $(K(G), \Delta)$. Then $\forall a \in K(G)$,

$$\begin{aligned} a &= (\varphi \otimes \iota)(h \otimes a) \\ &= (\varphi \otimes \iota)(\Delta(a)(h \otimes 1)) \\ &= (h\varphi \otimes \iota)\Delta(a). \end{aligned}$$

Here we use the relation

$$(1 \otimes a)\Delta(h) = \Delta(h)(a \otimes 1) \ (\forall a \in K(G)).$$

Therefore $a = (h\varphi \otimes \iota)\Delta(a) \in \widetilde{K(G)}$ and $K(G) = \widetilde{K(G)}$.

Assume that (A, Δ) is an algebraic quantum group, φ and ψ are the left and right Haar measures on (A, Δ), respectively. Then there exists an invertible multiplier $\delta \in M(A)$ (call it a modular function) (see [7]) so that $\forall a \in A$,

$$(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta, \quad (\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}, \quad \psi(a) = \varphi(a\delta)$$

and that

$$\Delta(\delta) = \delta \otimes \delta, \quad \varepsilon(\delta) = 1, \quad S(\delta) = \delta^{-1}.$$

In general case, $\delta \in M(A) - A$. In the following, we give a necessary and sufficient condition for $\delta \in A$.

Proposition 2.10 *Let (A, Δ) be an algebraic quantum group and δ as described above. Then $\delta \in A$ if and only if A is unital.*

Proof $\Leftarrow$) If A has a unit e, then $\varphi = e\varphi \in \hat{A} \subset \hat{A}_0$. So $\varphi(a)\delta = (\varphi\otimes\iota)\Delta(a) \in A$ and hence $\delta \in A$.

$\Rightarrow$) If $\delta \in A$, then $(\varphi\otimes\iota)\Delta(a) = \varphi(a)\delta \in A$ (resp. $(\iota\otimes\psi)\Delta(a) = \psi(a)\delta^{-1} \in A$). So $\varphi \in \hat{A}_0$ (resp. $\psi \in \hat{A}_0$) and thus $(\iota \otimes \varphi)\Delta(a) \in A$ (resp. $(\psi \otimes \iota)\Delta(a) \in A$). Again $(\iota\otimes\varphi)\Delta(a) = \varphi(a)1$ (resp. $(\psi\otimes\iota)\Delta(a) = \psi(a)1$), $\forall a \in A$. Then $\varphi(a)1 \in A$ (resp. $\psi(a)1 \in A$), i.e., $1 \in A$. ■

From Proposition 2.10, one can get the following conclusion at once.

Corollary 2.11 *Let (A, Δ) be a discrete quantum group and δ as described above. Then $\delta \in A$ if and only if A is of finite dimension.*

Remark 2.12 From Proposition 2.7 one can see that if A is a discrete quantum group, then $A = \widetilde{A}$, where

$$\widetilde{A} = \{(\omega \otimes \iota)\Delta(a)|a \in A, \omega \in A^*\},$$

and A^* is the space of all reduced functionals on A. On the other hand, if A is an algebraic quantum group with the property of $A = \widetilde{A}$, is A a discrete quantum group? The question is under consideration now.

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Ming Liu and Lining Jiang
Department of Mathematics, Beijing Institute of Technology,
Beijing, 100081, P.R.China
E-mail: jiangjl@yahoo.com.cn

Guosheng Zhang
Department of Mathematics, Shijiazhuang College,
Shijiazhuang, 050801, P.R.China