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On the index complex of a maximal subgroup and the group-theoretic properties of a finite group *

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ABSTRACT: Let G be a finite group, $S^p(G), \Phi'(G)$ and $\Phi_1(G)$ be generalizations of the Frattini subgroup of G. Based on these characteristic subgroups and using Deskins index complex, this paper gets some necessary and sufficient conditions for G to be a p-solvable, π-solvable, solvable, super-solvable and nilpotent group.

Key Words:: index complex; solvable groups; super-solvable groups; nilpotent groups.

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1. Intruduction

The relationship between the properties of maximal subgroups of a finite group and its structure has been studied extensively. The concept of index complex(see [1]) associated with a maximal subgroup plays an important role in the study of group theory.

Suppose that G is a finite group, and M is a maximal subgroup of G. A subgroup C of G is said to be a completion for M in G if C is not contained in M while every proper subgroup of C which is normal in G is contained in M. The set of all completions of M, denote it by $I(M)$, is called the index complex of M in G. Clearly $I(M)$ contains a normal subgroup, and is a nonempty partially ordered set by set inclusion relation. If $C \in I(M)$ and C is the maximal element of $I(M)$, C is said to be a maximal completion for M. If moreover $C \lhd G$, C then is said to be a normal completion for M. Clearly every normal completion of M

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is a maximal completion of M. Furthermore, by $k(C)$ we denote the product of all normal subgroups of G which are also proper subgroups of C, $k(C)$ is a proper normal subgroup of C.

In [2], Deskins studied the group-theoretic properties of the completions and its influences on the solvability of a finite group. He also raised a conjecture concerning super-solvability of a finite group in the same paper. Deskins's conjecture and other investigations were continued by many successive works [3-5]. This paper will study the structure of a finite group G. Using the concept of index complex and applying Frattini-Like subgroups such as $S^p(G), \Phi'(G)$ and $\Phi_1(G)$, the paper improves main results of [3-5] and obtains some necessary and sufficient conditions for the G to be a p-solvable, π-solvable, solvable, super-solvable and nilpotent group.

Throughout this paper, G denotes a finite group. The terminologies and notations agree with standard usage as in [6]. The notation $M \lessdot G$ means M is a maximal subgroup of G, and $N \lhd G$ means that N is a normal subgroup of G. If p is a prime, then $p^{'}$ denotes the complementary sets of primes and $|G:M|_p$ the p-part of $|G:M|$.

2. Preliminaries

For convenience, we give some notations and definitions firstly. Suppose that p is a prime, put

$$\begin{array}{rl} F^c = & \{M : M \lessdot G \text{ and } |G:M| \text{ is composite}\}; \\ F^p = & \{M : M \lessdot G \text{ and } M \geq N_G(P) \text{ for a } P \in Syl_p(G)\}; \\ F^{pc} = & F^p \bigcap F^c; \\ F_G = & \bigcup_{p\in\pi(G)} F^p \\ F'_G = & F_G \bigcap F^c \end{array} .$$

Using subgroups above, one can define Frattini-Like subgroups of G as follows.

Definition 2.1

$$S^p(G) = \bigcap\{M : M \in F^{pc}\} \ if \ F^{pc} \ is \ nonempty, \ otherwise \ S^p(G) = G;$$

$$\Phi_1(G) = \bigcap\{M : M \in F_G\} \ if \ F_G \ is \ nonempty, \ otherwise \ \Phi_1(G) = G;$$

$$\Phi'(G) = \bigcap\{M : M \in F'_G\} \ if \ F^{'}_G \ is \ nonempty, \ otherwise \ \Phi^{'}(G) = G.$$

We begin with a preliminary result which will be used frequently in connection with induction arguments in the next section.

Lemma 2.1 Let M be a maximal subgroup of a group G and N a normal subgroup of G. If $C \in I(M)$ and $N \leq k(C)$, then $C/N \in I(M/N)$ and $k(C/N) = k(C)/N$.

Proof. Since $C \in I(M)$, $C \nleq M$. Also $C/N \nleq M/N$. And if $A/N < C/N$, $A/N \lhd G/N$, then $A < C$ and $A \lhd G$. Since $A \leq M$, $A/N \leq M/N$, and $C/N \in$

$I(M/N)$. Also $C \not\leq M$ means $k(C) \neq C$. Then $k(C/N) \leq C/N$ and moreover $k(C)/N \leq M/N$. So $k(C/N) \leq k(C)/N$.

On the other hand, let $k(C/N) = H/N$, then $H \lhd G$ and $H/N < C/N$. Thus, $H < C$ and $k(C/N) = H/N \leq k(C)/N$. Therefore, $k(C/N) = k(C)/N$. □

Lemma 2.2[2] Let C and D be normal completions of a maximal subgroup M of G. Then $C/k(C) \cong D/k(D)$.

The order of $C/k(C)$, where C is a normal completion of M, is called the normal index of M in G, denoted by $\eta(G : M)$.

Lemma 2.3[7] $\Phi_1(G)$ is a nilpotent group; $\Phi'(G)$ is a Sylow tower group.

Lemma 2.4 If G is a group with a maximal core-free subgroup, the followings are equivalent:

(1) There exists a nontrivial solvable normal subgroup of G.

(2) There exists a unique minimal normal subgroup N of G and the index of all maximal subgroups of G in F_G with core-free are powers of a unique prime.

Proof. Using Ref.[7], it suffices to prove that (2) implies (1). Indeed for every $L \in F_G$ with core-free, let p be the unique prime divisor of $|G : L|$. Since $N \not\leq L$, $G = LN$. Moreover $|G : L| \Big| |N|$, thus $p \Big| |N|$. Let $P \in Syl_p(N)$. If $P \ntriangleleft G$, by the Frattini argument we have $G = N \cdot N_G(P)$. Suppose that $N_G(P) \leq M \lessdot G$, there exists $G_p \in Syl_p(G)$ satisfying $N_G(P) \geq N_G(G_p)$. This means $M \geq N_G(G_p)$ and therefore $M \in F_G$. But $N \not\leq M$, by the uniqueness of N we get that M is core-free. By the hypothesis, $p \Big| |G : M|$. Since $M \geq N_G(G_p)$, $p \not\Big| |G : M|$. This leads to a contradiction. Thus $P \lhd G$ and $P = N$ is a nontrivial solvable normal subgroup of G. □

3. Main Results

The following is the main result of the paper which gives a description of p-solvable group.

Theorem 3.1 Let p be the largest prime divisor of the order of G. The G is p-solvable if and only if for each non-nilpotent maximal subgroup M of G in F^{pc}, there exists a normal completion C in $I(M)$ such that $C/k(C)$ is a p'-group.

Proof. It suffices to prove the sufficient condition. Suppose that the result is false and let G be a counterexample of minimal order, now we can claim that:

i) F^{pc} is not empty. Indeed if F^{pc} is empty, then $S^p(G) = G$. Using [9, Lemma 2.2], $S^p(G)$ is p-closed. So $P \in Syl_p(G) \lhd G$ and G is p-solvable. This leads to a contradiction.

ii) Every maximal subgroup M of G in F^{pc} must be non-nilpotent. Indeed if there exists a maximal subgroup M in F^{pc} which is also nilpotent, then $|G : M|_p =$

1 and G is p-solvable. It is a contradiction.

iii) G has a unique minimal normal subgroup N such that G/N is p-solvable. Indeed if G is simple, then for every M of G in F^{pc}, G is the only normal completion in $I(M)$ with $k(G) = 1$. By hypothesis, $G = G/k(G)$ is a p'-group. This contradicts with the fact that p is the largest prime dividing $|G|$, hence G is not simple. Let N be a minimal normal subgroup of G, we will according to cases of $N \leq k(C)$ or $N \not\leq k(C)$ prove that G/N satisfies the hypothesis of the theorem.

If $N \leq k(C)$, then $N \leq C$ and C/N is a normal completion for M/N in G/N. By Lemma 2.1, $C/N\Big/k(C/N) = C/N\Big/k(C)/N \cong C/k(C)$. Again $C/k(C)$ is a p'-group, so $C/N\Big/k(C/N)$ is a p'-group.

If $N \not\leq k(C)$, then $N \not\leq C$. For otherwise, either $N = C$ or $N < C$, so either $G = MC = MN = M$ or $N < k(C)$. Each of which is a contradiction. Since N is a minimal normal subgroup of G, we have either $C \bigcap N = N$ or $C \bigcap N = 1$. If $C \bigcap N = N$, then $N \leq C$. It is also a contradiction. So $C \bigcap N = 1$. Then CN/N is a normal completion for M/N in G/N. We are to show that $C/N\Big/k(C/N)$ is a p'-group. Since $k(C) < C$ and $C \bigcap N = 1$, it follows that $k(C)N < CN$, and hence $k(C)N/N < CN/N$. Also $k(C)N/N \lhd G/N$, so we have $k(C)N/N \leq k(CN/N)$. We define a map ϕ: $C/k(C) \to CN/N\Big/k(CN/N)$, by

$$\phi(xk(C)) = xNk(CN/N)$$

for all $xk(C) \in C/k(C)$. Now $xk(C) = yk(C)$ implies that $x^{-1}y \in k(C)$, so $(xN)^{-1}(yN) = (x^{-1}y)N \in k(C)N/N \leq k(CN/N)$ and

$$(xN)k(CN/N) = (yN)k(CN/N).$$

That is to say, $\phi(xk(C)) = \phi(yk(C))$. Hence the map is well defined. It can be verified that ϕ is an epimorphism and $CN/N\Big/k(CN/N)$ is an epimorphic image of a p'-group. Thus G/N satisfies the hypothesis of the theorem. By the minimality of G, G/N is p-solvable.

Similarly, it can be shown that G/N_1 is p-solvable if N is another minimal normal subgroup N_1 of G. Thus $G = G/N \bigcap N_1$, which is isomorphic a subgroup of the p-solvable group $G/N \times G/N_1$, is p-solvable. So in the following suppose that N is the unique minimal normal subgroup of G.

If $p \not\Big| \ |N|$ or N is a p-group, then N is p-solvable and so G is p-solvable. It is a contradiction. Hence, $|N|_p \neq 1$ and $N \neq N_p \in Syl_p(N)$. Let M be a maximal subgroup of G such that $N_G(N_p) \leq M$. By the Frattini argument, we obtain that $G = N \cdot N_G(N_p)$. Using [7, lemma 5], there exists a $G_p \in Syl_p(G)$ with $N_G(N_p) \geq N_G(G_p)$, so $M \in F^p$ and $|G : M|_p = 1$. If $|G : M| = q$ be a prime less than p, then $|G|$ divides $q!$. This leads to another contradiction. Thus $|G : M|$ is composite and $M \in F^{pc}$. By ii) and hypothesis, there exists a normal completion C in $I(M)$ such that $C/k(C)$ is a p'-group. Obviously N is a normal completion of M. Combining with Lemma 2.2, we have $C/k(C) \cong N/k(N) = N$. Thus N is a p'-group, which leads to the final contradiction. This completes the proof. $\square$

As we have known in [3], a group G is π-solvable if and only if for every maximal subgroup M of G there exists a normal completion C in $I(M)$ such that $C/k(C)$is π-solvable. We now extend this result by considering a smaller class of maximal subgroups.

Theorem 3.2 Let G be a finite group. G is π-solvable if and only if for every maximal subgroup M of G in F'_G there exists a normal completion C in $I(M)$ such that $C/k(C)$is π-solvable.

Proof. $\Leftarrow$) Let G be a group satisfying the hypothesis of the theorem. If F'_G is empty then $\Phi'(G) = G$, and G is solvable. Thus assume that F'_G is not empty. If G is simple, then for every M in F'_G, G is the only normal completion in $I(M)$ with $k(G) = 1$ and thus $G = G/k(G)$ is π-solvable. So suppose that G is not simple. Let N be a minimal normal subgroup of G. Without loss of generality, one can suppose that $F'_{G/N}$ is not empty. We will use induction on the order of G. For each M/N in $F'_{G/N}$, by [7, Lemma 3], it follows that $M \in F'_G$. So by hypothesis there exists a normal completion C in $I(M)$ such that $C/k(C)$ is π-solvable.

Similar to the proof in Theorem 3.1, $CN/N \big/ k(CN/N)$ is π-solvable. Thus G/N satisfies the hypothesis of the theorem. Using the induction we obtain that G/N is π-solvable. Furthermore, we can assume that N is the unique minimal normal subgroup of G. By the same way, G/N is still a π-solvable group.

Now if $N \leq \Phi'(G)$, then from Lemma 2.3 $\Phi'(G)$ is solvable. Thus, N is π-solvable, and furthermore G is π-solvable. If $N \not\leq \Phi'(G)$, there exists a maximal subgroup $M_0 \in F'_G$ with $N \not\leq M_0$. Then $Core_G M_0 = 1$ and $G = NM_0$. So N is a normal completion in $I(M_0)$. By hypothesis there exists a normal completion C in $I(M_0)$ such that $C/k(C)$ is π-solvable. By Lemma 2.2, $N/k(N) = N \cong C/k(C)$. Again $C/k(C)$ is π-solvable, therefore N is π-solvable and moreover, G is π-solvable.

$\Rightarrow$) The converse is obvious. $\square$

The following theorem can be proved similarly as Theorem 3.2, and we omit it here.

Theorem 3.3 Let G be a finite group. G is solvable if and only if for every maximal subgroup M of G in F'_G there exists a normal completion C in $I(M)$ such that $C/k(C)$is solvable.

As we have known [4], if G is S_4-free, then G is super-solvable if and only if for each maximal subgroup M of G, there exists a maximal completion C in $I(M)$ such that $G = CM$ and $C/k(C)$ is cyclic. The following theorem extends this result.

Theorem 3.4 Suppose that G is S_4-free. G is super-solvable if and only if for each maximal subgroup M of G in F'_G, there exists a maximal completion C in $I(M)$ such that $G = CM$ and $C/k(C)$ is cyclic.

Proof. Let G be a super-solvable group. Then every chief factor of G is a cyclic group of prime order. $\forall M \in F'_G$, it is clear that the set $S = \{T \lhd G | T \not\leq M\}$ is not empty. Choose an H to be the minimal element in S. Clearly, $H \in I(M)$ and $H/k(H)$is a chief factor of G, hence $H/k(H)$ is cyclic.

Let G be a group satisfying the hypothesis of the Theorem. If F'_G is empty then $G = \Phi'(G)$ and G is super-solvable [9]. We now assume that F'_G is not empty and then G is solvable. In the remainder of the proof we will drop the maximality imposed on the completion C in $I(M)$ in the hypothesis. For each maximal subgroup M in F'_G, there exists a completion C in $I(M)$ such that $G = CM$ and $C/k(C)$ is cyclic. From [5, Lemma 2], we can get a normal completion A in $I(M)$ such that $A/k(A)$ is either cyclic or elementary abelian of order 2^2.

First suppose that there exists an M in F'_G which has a normal completion A such that $A/k(A)$ is elementary abelian of order 2^2. Let $\overline{G} = G/core_G(M)$ and $\overline{C}$, $\overline{M}$, $\overline{A}$ be the images of C, M and A in $\overline{G}$ respectively. Then $\overline{G} = \overline{C} \cdot \overline{M} = \overline{A} \cdot \overline{M}$. It is easy to verify that $k(A) = A \bigcap core_G M$, so $A/k(A) \cong A\ core_G M/core_G M = \overline{A}$. Since $core_{\overline{G}} \overline{M} = 1$, $k(\overline{A}) = 1$, $\overline{A}$ is a minimal normal subgroup of $\overline{G}$. $\overline{A}$ is an elementary abelian of order 2^2 and $\overline{M} \bigcap \overline{A} = 1$. Considering the permutation representation of $\overline{G}$ on 4 cosets of $\overline{M}$, G is isomorphic to a subgroup of S_4. Again S_4 and A_4 are the only non-super-solvable subgroups of S_4, A_4 doesn't satisfy the hypothesis of the theorem, and G is S_4-free, so G is super-solvable.

Now assume that for each maximal subgroup M in F'_G, M has a normal completion A so that $A/k(A)$ is cyclic. Let N be a minimal normal subgroup of G. Obviously, that G is S_4-free is quotient-closed. By [4, Lemma 3] and [7, Lemma 3], we can assume that the hypothesis holds for G/N. Using induction, we obtain that G/N is super-solvable. Similar to Theorem 3.1, we can suppose that N is the unique minimal normal subgroup of G. If $N \leq \Phi'(G)$, then G is super-solvable. If $N \not\leq \Phi'(G)$, there exists a maximal subgroup M in F'_G so that $G = NM$ and $core_G(M) = 1$. Obviously N is a normal completion in $I(M)$. By hypothesis, there exists a normal completion A so that $A/k(A)$ is cyclic. By Lemma 2.2, $A/k(A) \cong N/k(N) = N$. Thus N is cyclic and G is super-solvable.

Remark Let G be a solvable group. To obtain the conclusion in Theorem 3.4, the condition of maximality imposed on the completion C is nonsignificant. So we have the following result: If G is S_4-free and solvable, G is super-solvable if and only if for each maximal subgroup M of G in F'_G, there exists a completion C in $I(M)$ so that $G = CM$ and $C/k(C)$ is cyclic.

Theorem 3.5 Let G be a group and M be an arbitrary maximal subgroup of G in F_G. Then G is nilpotent if and only if for each normal completion C of M,

$$|C/k(C)| = |G : M|.$$

Proof. $\Leftarrow$) Let G be a group satisfying the hypothesis of the theorem. If F_G is empty then $G/N = \Phi_1(G/N)$. Using [9, Lemma 2.3], G/N is nilpotent. If G is simple, then for every M in F_G, G is the only normal completion in $I(M)$ with $k(G) = 1$. By hypothesis $|G/k(G)| = G = |G : M|$, $M = 1$, hence G is a cyclic group of prime order. So assume that G is not simple. Let N be a minimal normal subgroup of G. Without loss of generality, suppose that $F_{G/N}$ is not empty. For any maximal subgroup M/N in $F_{G/N}$, suppose that C/N is an arbitrary normal completion in $I(M/N)$. From [7, Lemma 3] we have M in F_G. Obviously C is a normal completion in $I(M)$ and $|C/k(C)| = |G : M|$. Using Lemma 2.1,

$$|C/N\Big/k(C/N)| = |C/N\Big/k(C)/N| = |C/k(C)| = |G : M| = |G/N\Big/M/N|.$$

Thus G/N satisfies the hypothesis of the theorem. Applying induction one can see G/N is nilpotent. Similar to the proof in Theorem 3.1, we may assume N is the unique minimal subgroup of G.

If $N \leq \Phi_1(G)$, by [5, Lemma 2.3] G is nilpotent. If $N \not\leq \Phi_1(G)$, there exists an M in F_G so that $G = NM$. Clearly, N is a normal completion in $I(M)$.By hypothesis $|N/k(N)| = |N| = |G : M|$. For any L in F_G with $core_G(L) = 1$, obviously $N \not\leq L$ and $G = NL$. N is also a normal completion in $I(M)$, so $|N/k(N)| = |N| = |G : L|$. By Lemma 2.4 G has a nontrivial solvable subgroup K, so $N \leq K$ and N is solvable. Since G/N is nilpotent, G is solvable. Thus N is an elementary abelian p-group. If G is not a p-group, we assume that $|G|$ has a prime factor q different from p. If the subgroup $Q = \langle a | a \in G \text{ and } |a| = q\rangle \leq M$, this contradicts with the fact that $core_G M = 1$. So there exists an of order q element a in $G - M$. This implies that $G = \langle M, \langle a\rangle\rangle$. However, $|N| = |G : M|$ is a power of p. This leads to another contradiction. So G must be a p-group and then is a nilpotent group.

$\Rightarrow$) The converse holds obviously. □

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