Bol. Soc. Paran. Mat.
(3s.) **v. 23** 1-2 (2005): 73–84.
©SPM –ISNN-00378712

On δ-Semiopen Sets And A Generalization Of Functions

Erdal Ekici

ABSTRACT: In this paper, we introduce and investigate a weaker form of R-maps and δ-continuous functions which is called almost δ-semicontinuity. We obtain its characterizations, its basic properties and their relationships with other types of functions between topological spaces.

Key Words: δ-semicontinuity, R-map, δ-continuity, almost δ-semicontinuity.

Contents

1. Introduction and preliminaries

By using various forms of open sets many authors introduced and studied various types of continuity. In 1973, Carnahan introduced the notion of R-maps. In 1980, Noiri studied the notion of δ-continuous functions. The aim of this paper is to introduce the notion of almost δ-semicontinuous functions which generalize R-maps and δ-continuous functions. Various characterizations and properties of such functions are obtained. Throughout the present paper, spaces mean topological spaces and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let S be a subset of a space X. The closure and the interior of S are denoted by $cl(S)$ and $int(S)$, respectively.

Definition 1 *A subset S of a space X is said to be*

(1) regular open [22] if $S = int(cl(S))$,

(2) δ-open [23] if for each $x \in S$, there exists a regular open set W such that $x \in W \subset S$,

(3) α-open [14] if $S \subset int(cl(int(S)))$,

(4) semi-open [9] if $S \subset cl(int(S))$,

(5) preopen [11] if $S \subset int(cl(S))$,

(6) γ-open [7] if $S \subset int(cl(S)) \cup cl(int(S))$,

(7) β-open [1] or semi-preopen [2] if $S \subset cl(int(cl(S)))$.

2000 *Mathematics Subject Classification:* 54C08, 54C10
Date submission 06-Apr-2005.

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$style.
© Soc. Paran. Mat.

The complement of a regular open set is said to be regular closed [22].

The complement of a semiopen set is said to be semiclosed [6]. The intersection of all semiclosed sets containing a subset A of X is called the semi-closure [6] of A and is denoted by $s\text{-}cl(A)$. The union of all semiopen sets contained in a subset A of X is called the semi-interior of A and is denoted by $s\text{-}int(A)$.

A point $x \in X$ is called a δ-cluster (resp. θ-cluster) point of A [23] if $A \cap int(cl(U)) \neq \varnothing$ (resp. $A \cap cl(U) \neq \varnothing$) for each open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of A is called the δ-closure (resp. θ-closure) of A and is denoted by $\delta\text{-}cl(A)$ (resp. $\theta\text{-}cl(A)$). If $\delta\text{-}cl(A) = A$ (resp. $\theta\text{-}cl(A) = A$), then A is said to be δ-closed (resp. θ-closed). The complement of a δ-closed (resp. θ-closed) set is said to be δ-open (resp. θ-open).

A subset S of a topological space X is said to be δ-semiopen [20] iff $S \subset cl(\delta\text{-}int(S))$. The complement of a δ-semiopen set is called a δ-semiclosed set [20]. The union (resp. intersection) of all δ-semiopen (resp. δ-semiclosed) sets, each contained in (resp. containing) a set S in a topological space X is called the δ-semiinterior (resp. δ-semiclosure) of S and it is denoted by $\delta\text{-}sint(S)$ (resp. $\delta\text{-}scl(S)$) [20].

The family of all δ-semiopen (resp.regular open, preopen, β-open. α-open, semi-open, δ-open) sets of a space X will be denoted by $\delta SO(X)$ (resp. $RO(X)$, $PO(X)$, $\beta O(X)$, $\alpha O(X)$, $SO(X)$, $\delta O(X)$). The family of all δ-semiclosed (resp. regular closed, δ-closed) sets in a space X is denoted by $\delta SC(X)$ (resp. $RC(X)$, $\delta C(X)$). The family of all δ-semiopen (resp.regular open, δ-open) sets containing a point $x \in X$ will be denoted by $\delta SO(X, x)$ (resp. $RO(X, x)$, $\delta O(X, x)$).

Lemma 2 *Let (X, τ) be a topological space. Intersection of arbitrary of δ-closed sets in X is δ-closed.*

Lemma 3 *Let A be a subset of a topological space (X, τ). Then $\delta\text{-}cl(A) = \cap\{F \in \delta C(X) : A \subset F\}$.*

Corollary 4 *$\delta\text{-}cl(A)$ is δ-closed for a subset A in a topological space (X, τ).*

Proof. It is obvious from the above lemmas. ■

Definition 5 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*
(1) R-map [5] if $f^{-1}(V) \in RO(X)$ for every $V \in RO(Y)$,
(2) almost semi-continuous [12] if $f^{-1}(V) \in SO(X)$ for every $V \in RO(Y)$,
(3) δ-continuous [15] if $f^{-1}(V)$ is δ-open in X for every $V \in RO(Y)$.

Lemma 6 *(Park et. al. [20]) Let A be a subset of a space X. Then*
(1) $\delta\text{-}scl(X \backslash A) = X \backslash \delta\text{-}sint(A)$,
(2) $x \in \delta\text{-}scl(A)$ if and only if $A \cap U \neq \varnothing$ for each $U \in \delta SO(X, x)$,
(3) A is δ-semiclosed in X if and only if $A = \delta\text{-}scl(A)$,
(4) $\delta\text{-}scl(A)$ is δ-semiclosed in X.

Lemma 7 *(Noiri [17]) For a subset of a space Y, the following hold:*
(1) $\alpha\text{-}cl(V) = cl(V)$ for every $V \in \beta O(Y)$.
(2) $p\text{-}cl(F) = cl(V)$ for every $V \in SO(Y)$.

Lemma 8 *(Noiri [18]) s-$cl(V) = int(cl(V))$ for every preopen set V of a space X.*

Definition 9 *A space (X, τ) is said to be*
(1) submaximal [3] if every dense subset of X is open in X,
(2) extremally disconnected [3, 16] if $cl(U) \in \tau$ for every $U \in \tau$.

2. Almost δ-semicontinuous functions

In this section, we obtain several characterizations of almost δ-semicontinuous functions.

Definition 10 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost δ-semicontinuous if for each $x \in X$ and each $V \in RO(Y)$ containing $f(x)$, there exists $U \in \delta SO(X)$ containing x such that $f(U) \subset V$.*

Theorem 11 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*
(1) f is almost δ-semicontinuous;
(2) for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \delta SO(X)$ containing x such that $f(U) \subset int(cl(V))$;
(3) $f^{-1}(F) \in \delta SC(X)$ for every $F \in RC(Y)$;
(4) $f^{-1}(V) \in \delta SO(X)$ for every $V \in RO(Y)$.
(5) $f(\delta\text{-}scl(A)) \subset \delta\text{-}cl(f(A))$ for every subset A of X ;
(6) $\delta\text{-}scl(f^{-1}(B)) \subset f^{-1}(\delta\text{-}cl(B))$ for every subset B of Y;
(7) $f^{-1}(F) \in \delta SC(X)$ for every δ-closed set F of (Y, σ);
(8) $f^{-1}(V) \in \delta SO(X)$ for every δ-open set V of (Y, σ);
(9) $\delta\text{-}scl(f^{-1}(cl(int(cl(B))))) \subset f^{-1}(cl(B))$ for every subset B of Y;
(10) $\delta\text{-}scl(f^{-1}(cl(int(F)))) \subset f^{-1}(F)$ for every closed set F of Y;
(11) $\delta\text{-}scl(f^{-1}(cl(V))) \subset f^{-1}(cl(V))$ for every open set V of Y;
(12) $f^{-1}(V) \subset \delta\text{-}sint(f^{-1}(s\text{-}cl(V)))$ for every open set V of Y;
(13) $f^{-1}(V) \subset cl(\delta\text{-}int(f^{-1}(s\text{-}cl(V))))$ for every open set V of Y;
(14) $f^{-1}(V) \subset \delta\text{-}sint(f^{-1}(int(cl(V))))$ for every open set V of Y;
(15) $f^{-1}(V) \subset cl(\delta\text{-}int(f^{-1}(int(cl(V)))))$ for every open set V of Y;
(16) $\delta\text{-}scl(f^{-1}(V)) \subset f^{-1}(cl(V))$ for each $V \in \beta O(Y)$;
(17) $\delta\text{-}scl(f^{-1}(V)) \subset f^{-1}(cl(V))$ for each $V \in SO(Y)$;
(18) $f^{-1}(V) \subset \delta\text{-}sint(f^{-1}(int(cl(V))))$ for each $V \in PO(Y)$;
(19) $\delta\text{-}scl(f^{-1}(V)) \subset f^{-1}(\alpha\text{-}cl(V))$ for each $V \in \beta O(Y)$;
(20) $\delta\text{-}scl(f^{-1}(V)) \subset f^{-1}(p\text{-}cl(V))$ for each $V \in SO(Y)$;
(21) $f^{-1}(V) \subset \delta\text{-}sint(f^{-1}(s\text{-}cl(V)))$ for each $V \in PO(Y)$.

Proof. (1)⇒(2). Let $x \in X$ and $V \in \sigma$ containing $f(x)$. We have $int(cl(V)) \in RO(Y)$. Since f is almost δ-semicontinuous, then there exists $U \in \delta SO(X, x)$ such that $f(U) \subset int(cl(V))$.

(2)⇒(1). Obvious.

(3)⇔(4). Obvious.

(1)⇒(4). Let $x \in X$ and $V \in RO(Y, f(x))$. Since f is almost δ-semicontinuous, then there exists $U_x \in \delta SO(X, x)$ such that $f(U_x) \subset V$. We have $U_x \subset f^{-1}(V)$. Thus, $f^{-1}(V) = \cup U_x \in \delta SO(X)$.

(4)⇒(1). Obvious.

(1)⇒(5). Let A be a subset of X. Since δ-$cl(f(A))$ is δ-closed in Y, it is denoted by $\cap\{F_i : F_i \in RC(Y),\ i \in I\}$, where I is an index set. By (1)⇔(3), we have

$$A \subset f^{-1}(\delta - cl(f(A))) = \cap\{f^{-1}(F_i) : i \in I\} \in \delta SC(X)$$

and hence δ-$scl(A) \subset f^{-1}(\delta$-$cl(f(A)))$. Therefore, we obtain $f(\delta$-$scl(A)) \subset \delta$-$cl(f(A))$.

(5)⇒(6). Let B be a subset of Y. We have $f(\delta$-$scl(f^{-1}(B))) \subset \delta$-$cl(f(f^{-1}(B))) \subset \delta$-$cl(B)$ and hence δ-$scl(f^{-1}(B)) \subset f^{-1}(\delta$-$cl(B))$.

(6)⇒(7). Let F be any δ-closed set of (Y,σ). We have δ-$scl(f^{-1}(F)) \subset f^{-1}(\delta$-$cl(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is δ-semiclosed in (X,τ).

(7)⇒(8). Let V be any δ-open set of (Y,σ). We have $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V) \in \delta SC(X)$ and hence $f^{-1}(V) \in \delta SO(X)$.

(8)⇒(1). Let V be any regular open set of (Y,σ). Since V is δ-open in (Y,σ), $f^{-1}(V) \in \delta SO(X)$ and hence, by (1)⇔(4), f is almost δ-semicontinuous.

(1)⇒(9). Let B be any subset of Y. Assume that $x \in X\backslash f^{-1}(cl(B))$. Then $f(x) \in Y\backslash cl(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \varnothing$; hence $int(cl(V)) \cap cl(int(cl(B))) = \varnothing$. Since f is almost δ-semicontinuous, there exists $U \in \delta SO(X,x)$ such that $f(U) \subset int(cl(V))$. Therefore, we have $U \cap f^{-1}(cl(int(cl(B)))) = \varnothing$ and hence $x \in X\backslash\delta$-$scl(f^{-1}(cl(int(cl(B)))))$. Thus we obtain

$$\delta\text{-}scl(f^{-1}(cl(int(cl(B))))) \subset f^{-1}(cl(B)).$$

(9)⇒(10). Let F be any closed set of Y. Then we have

$$\begin{aligned}\delta\text{-}scl(f^{-1}(cl(int(F))) = {}& \delta\text{-}scl(f^{-1}(cl(int(cl(F))))) \\ & \subset f^{-1}(cl(F)) = f^{-1}(F).\end{aligned}$$

(10)⇒(11). For any open set V of Y, $cl(V)$ is regular closed in Y and we have

$$\delta\text{-}scl(f^{-1}(cl(V)) = \delta\text{-}scl(f^{-1}(cl(int(cl(V))))) \subset f^{-1}(cl(V)).$$

(11)⇒(12). Let V be any open set of Y. Then $Y\backslash cl(V)$ is open in Y and by using Lemma 8 we have

$$\begin{aligned}& X\backslash\delta\text{-}sint(f^{-1}(s\text{-}cl(V))) \\ = {}& \delta\text{-}scl(f^{-1}(Y\backslash int(cl(V)))) \subset f^{-1}(cl(Y\backslash cl(V))) \subset X\backslash f^{-1}(V).\end{aligned}$$

Therefore, we obtain $f^{-1}(V) \subset \delta$-$sint(f^{-1}(s$-$cl(V)))$.

(12)⇒(13). Let V be any open set of Y. We obtain

$$f^{-1}(V) \subset \delta\text{-}sint(f^{-1}(s\text{-}cl(V))) \subset cl(\delta\text{-}int(f^{-1}(s\text{-}cl(V)))).$$

(13)⇒(1). Let x be any point of X and V any open set of Y containing $f(x)$. Then $x \in f^{-1}(int(cl(V))) \subset cl(\delta$-$int(f^{-1}(s$-$cl(int(cl(V)))))) = cl(\delta$-$int(f^{-1}(int(cl(V)))))$. Thus, $f^{-1}(int(cl(V))) \in \delta SO(X)$. Take $U = f^{-1}(int(cl(V)))$. We obtain $x \in U$ and $f(U) \subset int(cl(V))$. Therefore, f is almost δ-semicontinuous.

(12)$\Leftrightarrow$(14) and (13)$\Leftrightarrow$(15). Obvious.

(1)$\Rightarrow$(16). Let V be any β-open set of Y. It follows from [2, Theorem 2.4] that $cl(V)$ is regular closed in Y. Since f is almost δ-semicontinuous, by (1)$\Leftrightarrow$(3), $f^{-1}(cl(V))$ is δ-semiclosed in X. Therefore, we obtain $\delta\text{-}scl(f^{-1}(V)) \subset f^{-1}(cl(V))$.

(16)$\Rightarrow$(17). This is obvious since $SO(Y) \subset \beta O(Y)$.

(17)$\Rightarrow$(1). Let F be any regular closed set of Y. Then F is semi-open in Y and hence $\delta\text{-}scl(f^{-1}(F)) \subset f^{-1}(cl(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is δ-semiclosed. Therefore, by (1)$\Leftrightarrow$(3), f is almost δ-semicontinuous.

(1)$\Rightarrow$(18). Let V be any preopen set of Y. Then $V \subset int(cl(V))$ and $int(cl(V))$ is regular open in Y. Since f is almost δ-semicontinuous, by (1)$\Leftrightarrow$(4), $f^{-1}(int(cl(V)))$ is δ-semiopen in X and hence we obtain that $f^{-1}(V) \subset f^{-1}(int(cl(V))) \subset \delta\text{-}sint(f^{-1}(int(cl(V))))$.

(18)$\Rightarrow$(1). Let V be any regular open set of Y. Then V is preopen and $f^{-1}(V) \subset \delta\text{-}sint(f^{-1}(int(cl(V)))) = \delta\text{-}sint(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is δ-semiopen in X and hence, by (1)$\Leftrightarrow$(4), f is almost δ-semicontinuous.

(16)$\Leftrightarrow$(19), (17)$\Leftrightarrow$(20), (18)$\Leftrightarrow$(21). Obvious. ∎

3. Relationships

In this section, the relationships of almost δ-semicontinuity are investigated.

almost semi-continuous $\Leftarrow$ almost δ-semicontinuous $\Leftarrow$ δ-continuous $\Leftarrow$ R-map

However, the converses are not true in general as shown by the following examples:

Example 12 *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f : X \to X$ be a function defined by $f(a) = a$, $f(b) = d$, $f(c) = c$, $f(d) = d$. Then, f is almost semi-continuous but not almost δ-semicontinuous.*

Example 13 *Let $X = \{a, b, c\}$ and $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$. Let $f : X \to X$ be a function defined by $f(a) = b$, $f(b) = a$, $f(c) = a$. Then, f is almost δ-semicontinuous but not δ-continuous.*

The other example for the last implication can be seen in [15].

Definition 14 *Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ_s. It is called the semiregularization. In case when $\tau = \tau_s$, the space (X, τ) is called semi-regular [22].*

Theorem 15 *Let (X, τ) be a semi-regular space. Then a function $f : (X, \tau) \to (Y, \sigma)$ is almost semi-continuous if and only if it is almost δ-semicontinuous.*

Definition 16 *A function $f : X \to Y$ is said to be*

(1) weakly δ-semicontinuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset cl(V)$.

(2) δ-semicontinuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset V$,

(3) δ-semiirresolute [4] if for each $x \in X$ and each δ-semiopen set V of Y containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset V$.

The following example shows that the composition of two δ-semicontinuous functions is not δ-semicontinuous.

Example 17 *Let $X = Y = Z = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : X \to Y$ and $g : Y \to Z$ be functions defined by $f(a) = b$, $f(b) = b$, $f(c) = c$, $f(d) = d$ and $g(a) = a$, $g(b) = c$, $g(c) = a$, $g(d) = d$, respectively. Then, f and g are δ-semicontinuous but $g \circ f$ is not δ-semicontinuous.*

Theorem 18 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the following hold:*

(1) If f is almost δ-semicontinuous and g is an R-map, then the composition $g \circ f : X \to Z$ is almost δ-semicontinuous,

(2) If f is δ-semiirresolute and g is almost δ-semicontinuous, the composition $g \circ f : X \to Z$ is almost δ-semicontinuous.

Theorem 19 *The following properties are equivalent for a function $f : X \to Y$*

(1) f is δ-semicontinuous,

(2) $f^{-1}(F)$ is δ-semiclosed in X for every closed set F in Y.

Definition 20 *A function $f : X \to Y$ is said to be faintly δ-semicontinuous if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset V$.*

Theorem 21 *The following properties are equivalent for a function $f : X \to Y$*

(1) f is faintly δ-semicontinuous,

(2) $f^{-1}(F)$ is δ-semiclosed in X for θ-closed set F in Y.

Theorem 22 *Let $f : X \to Y$ be a function. Suppose that Y is regular. Then, the following properties are equivalent:*

(1) f is δ-semicontinuous,

(2) $f^{-1}(\delta\text{-}cl(B))$ is δ-semiclosed in X for every subset B of Y,

(3) f is almost δ-semicontinuous,

(4) f is weakly δ-semicontinuous,

(5) f is faintly δ-semicontinuous.

Proof. (1)⇒(2). Since $\delta\text{-}cl(B)$ is closed in Y for every subset B of Y, $f^{-1}(\delta\text{-}cl(B))$ is δ-semiclosed in X.

(2)⇒(3). For any subset B of Y, $f^{-1}(\delta\text{-}cl(B))$ is δ-semiclosed in X and hence we have $\delta\text{-}scl(f^{-1}(B)) \subset \delta\text{-}scl(f^{-1}(\delta\text{-}cl(B))) = f^{-1}(\delta\text{-}cl(B))$. It follows that f is almost δ-semicontinuous

(3)⇒(4). This is obvious.

(4)⇒(5). Let A be any subset of X. Let $x \in \delta\text{-}scl(A)$ and V be any open set of Y containing $f(x)$. There exists $U \in \delta SO(X, x)$ such that $f(U) \subset cl(V)$. Since $x \in \delta\text{-}scl(A)$, we have $U \cap A \neq \varnothing$ and hence $\varnothing \neq f(U) \cap f(A) \subset cl(V) \cap f(A)$. Therefore, we have $f(x) \in \theta\text{-}cl(f(A))$ and hence $f(\delta\text{-}scl(A)) \subset \theta\text{-}cl(f(A))$.

Let B be any subset of Y. We have $f(\delta\text{-}scl(f^{-1}(B))) \subset \theta\text{-}cl(B)$ and $\delta\text{-}scl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}cl(B))$.

Let F be any θ-closed set of Y. It follows that δ-$scl(f^{-1}(F)) \subset f^{-1}(\theta$-$cl(F)) = f^{-1}(F)$. Therefore $f^{-1}(F)$ is δ-semiclosed in X and hence f is faintly δ-semicontinuous.

(5)⇒(1). Let V be any open set of Y. Since Y is regular, V is θ-open in Y. By the faint δ-semicontinuity of f, $f^{-1}(V)$ is δ-semiopen in X. Therefore, f is δ-semicontinuous. ■

Definition 23 *A function $f : X \to Y$ is said to be faintly continuous [10] (resp. faintly semi-continuous [19], faintly precontinuous [19], faintly β-continuous [13, 19], faintly α-continuous [13]) if $f^{-1}(V)$ is open (resp. semi-open, preopen, β-open, α-open) in X for each θ-open set V of of Y.*

Theorem 24 *If (X, τ) is submaximal extremally disconnected semi-regular and (Y, σ) is regular, then the following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:*

(1) f is faintly α-continuous,

(2) f is faintly semi-continuous,

(3) f is faintly precontinuous,

(4) f is faintly γ-continuous,

(5) f is faintly β-continuous,

(6) f is faintly continuous,

(7) f is faintly δ-semicontinuous,

(8) f is δ-semicontinuous,

(9) f is almost δ-semicontinuous,

(10) f is weakly δ-semicontinuous.

Definition 25 *A function $f : X \to Y$ is said to be almost δ-semiopen if $f(U) \subset int(cl(f(U)))$ for every δ-semiopen set U of X.*

Theorem 26 *If $f : X \to Y$ is an almost δ-semiopen and weakly δ-semicontinuous function, then f is almost δ-semicontinuous*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since f is weakly δ-semicontinuous, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset cl(V)$. Since f is almost δ-semiopen, $f(U) \subset int(cl(f(U))) \subset int(cl(V))$ and hence f is almost δ-semicontinuous. ■

Definition 27 *A space X is said to be*

(1) almost regular [21] if for any regular closed set F of X and any point $x \in X \backslash F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$,

(2) semi-regular if for any open set U of X and each point $x \in U$ there exists a regular open set V of X such that $x \in V \subset U$.

Theorem 28 *If $f : X \to Y$ is a weakly δ-semicontinuous function and Y is almost regular, then f is almost δ-semicontinuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the almost regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset cl(G) \subset int(cl(V))$ [21, Theorem 2.2]. Since f is weakly δ-semicontinuous, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset cl(G) \subset int(cl(V))$. Therefore, f is almost δ-semicontinuous. ■

Theorem 29 *If $f : X \to Y$ is an almost δ-semicontinuous function and Y is semi-regular, then f is δ-semicontinuous.*

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. By the semi-regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost δ-semicontinuous, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset int(cl(G)) = G \subset V$ and hence f is δ-semicontinuous. ■

4. Properties

Theorem 30 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then g is almost δ-semicontinuous if and only if f is almost δ-semicontinuous.*

Proof. Necessity. Let $x \in X$ and $V \in RO(Y)$ containing $f(x)$. Then, we have $g(x) = (x, f(x)) \in X \times V \in RO(X \times Y)$. Since g is almost δ-semicontinuous, there exists a δ-semiopen set U of X containing x such that $g(U) \subset X \times V$. Therefore, we obtain $f(U) \subset V$ and hence f is almost δ-semicontinuous.

Sufficiency. Let $x \in X$ and W be a regular open set of $X \times Y$ containing $g(x)$. There exist $U_1 \in RO(X)$ and $V \in RO(Y)$ such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is almost δ-semicontinuous, there exists $U_2 \in \delta SO(X)$ such that $x \in U_2$ and $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then we obtain $x \in U \in \delta SO(X)$ and $g(U) \subset U_1 \times V \subset W$. This shows that g is almost δ-semicontinuous. ■

Let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be any two families of spaces with the same index set I. For each $i \in I$, let $f_i : X_i \to Y_i$ be a function. The product space $\prod_{i \in I} X_i$ will be denoted by $\prod X_i$ and the product function $\prod f_i : \prod X_i \to \prod Y_i$ is simply denoted by $f : \prod X_i \to \prod Y_i$.

Theorem 31 *If a function $f : X \to \prod Y_i$ is almost δ-semicontinuous, then $p_i \circ f : X \to Y_i$ is almost δ-semicontinuous for each $i \in I$, where p_i is the projection of $\prod Y_i$ onto Y_i.*

Proof. Let V_i be any regular open set of Y_i. Since p_i is continuous open, it is an R-map and hence $p_i^{-1}(V_i) \in RO(\prod Y_i)$. By Theorem 11, $f^{-1}(p_i^{-1}(V_i)) = (p_i \circ f)^{-1}(V_i) \in \delta SO(X)$. This shows that $p_i \circ f$ is almost δ-semicontinuous for each $i \in I$. ■

Theorem 32 *The product function $f : \prod X_i \to \prod Y_i$ is almost δ-semicontinuous if and only if $f_i : X_i \to Y_i$ is almost δ-semicontinuous for each $i \in I$.*

Proof. Necessity. Let k be an arbitrarily fixed index and V_k any regular open set of Y_k. Then $\prod Y_j \times V_k$ is regular open in $\prod Y_i$, where $j \in I$ and $j \neq k$, and hence $f^{-1}(\prod Y_j \times V_k) = \prod Y_j \times f_k^{-1}(V_k)$ is δ-semiopen in $\prod X_i$. Thus, $f_k^{-1}(V_k)$ is δ-semiopen in X_k and hence f_k is almost δ-semicontinuous.

Sufficiency. Let $\{x_i\}$ be any point of $\prod X_i$ and W any regular open set of $\prod Y_i$ containing $f(\{x_i\})$. There exists a finite subset I_0 of I such that $V_k \in RO(Y_k)$ for each $k \in I_0$ and $\{f_i(x_i)\} \in \prod\{V_k : k \in I_0\} \times \prod\{Y_j : j \in I \backslash I_0\} \subset W$. For each $k \in I_0$, there exists $U_k \in \delta SO(X_k)$ containing x_k such that $f_k(U_k) \subset V_k$. Thus, $U = \prod\{U_k : k \in I_0\} \times \prod\{X_j : j \in I \backslash I_0\}$ is a δ-semiopen set of $\prod X_i$ containing $\{x_i\}$ and $f(U) \subset W$. This shows that f is almost δ-semicontinuous. ■

Lemma 33 *A set S in X is δ-semiopen if and only if $S \cap G \in \delta SO(X)$ for every δ-open set G of X.*

Lemma 34 *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta SO(X)$ and $X_0 \in \delta O(X)$, then $A \cap X_0 \in \delta SO(X_0)$ [8].*

Theorem 35 *If $f : (X, \tau) \to (Y, \sigma)$ is almost δ-semicontinuous and A is δ-open in (X, τ), then the restriction $f\mid_A : (A, \tau_A) \to (Y, \sigma)$ is almost δ-semicontinuous.*

Proof. Let V be any regular open set of Y. By Theorem 11, we have $f^{-1}(V) \in \delta SO(X)$ and hence $(f\mid_A)^{-1}(V) = f^{-1}(V) \cap A \in \delta SO(A)$ by Lemma 34. Thus, it follows that $f\mid_A$ is almost δ-semicontinuous. ■

Lemma 36 *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta SO(X_0)$ and $X_0 \in \delta O(X)$, then $A \in \delta SO(X)$ [8].*

Theorem 37 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $\{U_i : i \in I\}$ a cover of X by δ-open sets of (X, τ). If $f\mid_{U_i} : (U_i, \tau_{U_i}) \to (Y, \sigma)$ is almost δ-semicontinuous for each $i \in I$, then f is almost δ-semicontinuous.*

Proof. Let V be any regular open set of (Y, σ). Then, we have

$$f^{-1}(V) = X \cap f^{-1}(V) = \cup\{U_i \cap f^{-1}(V) : i \in I\} = \cup\{(f\mid_{U_i})^{-1}(V) : i \in I\}.$$

Since $f\mid_{U_i}$ is almost δ-semicontinuous, $(f\mid_{U_i})^{-1}(V) \in \delta SO(U_i)$ for each $i \in I$. By Lemma 36, for each $i \in I$, $(f\mid_{U_i})^{-1}(V)$ is δ-semiopen in X and hence $f^{-1}(V)$ is δ-semiopen in X. Therefore, f is almost δ-semicontinuous. ■

Definition 38 *The δ-semifrontier of a subset A of X, denoted by δ-$sfr(A)$, is defined by δ-$sfr(A) = \delta$-$scl(A) \cap \delta$-$scl(X \backslash A) = \delta$-$scl(A) \backslash \delta$-$sint(A)$ [8].*

Theorem 39 *The set of all points x of X at which a function $f : X \to Y$ is not almost δ-semicontinuous is identical with the union of the δ-semifrontiers of the inverse images of regular open sets containing $f(x)$.*

Proof. Let x be a point of X at which f is not almost δ-semicontinuous Then, there exists a regular open set V of Y containing $f(x)$ such that $U \cap (X \backslash f^{-1}(V)) \neq \varnothing$ for every $U \in \delta SO(X, x)$. Therefore, we have $x \in \delta\text{-}scl(X \backslash f^{-1}(V)) = X \backslash \delta\text{-}sint(f^{-1}(V))$ and $x \in f^{-1}(V)$. Thus, we obtain $x \in \delta\text{-}sfr(f^{-1}(V))$.

Conversely, suppose that f is almost δ-semicontinuous at $x \in X$ and let V be a regular open set containing $f(x)$. Then there exists $U \in \delta SO(X, x)$ such that $U \subset f^{-1}(V)$; hence $x \in \delta\text{-}sint(f^{-1}(V))$. Therefore, it follows that $x \in X \backslash \delta\text{-}sfr(f^{-1}(V))$. This completes the proof. ∎

Theorem 40 *If $f : X \to Y$ is almost δ-semicontinuous, $g : X \to Y$ is δ-continuous and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is δ-semiclosed in X.*

Proof. Let $A = \{x \in X : f(x) = g(x)\}$ and $x \in X \backslash A$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V and W of Y such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \varnothing$; hence $int(cl(V)) \cap int(cl(W)) = \varnothing$. Since f is almost δ-semicontinuous, there exists $G \in \delta SO(X, x)$ such that $f(G) \subset int(cl(V))$. Since g is δ-continuous, there exists an δ-open set H of X containing x such that $g(H) \subset int(cl(W))$. Now, put $U = G \cap H$, then $U \in \delta SO(X, x)$ and $f(U) \cap g(U) \subset int(cl(V)) \cap int(cl(W)) = \varnothing$. Therefore, we obtain $U \cap A = \varnothing$ and hence $x \in X \backslash \delta\text{-}scl(A)$. This shows that A is δ-semiclosed in X. ∎

Theorem 41 *If $f_1 : X_1 \to Y$ is weakly δ-semicontinuous, $f_2 : X_2 \to Y$ is almost δ-semicontinuous and Y is Hausdorff, then the set $\{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = f(x_2)\}$ is δ-semiclosed in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = f(x_2)\}$ and $(x_1, x_2) \in (X_1 \times X_2) \backslash A$. Then $f(x_1) \neq f(x_2)$ and there exist open sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \varnothing$; hence $cl(V_1) \cap int(cl(V_2)) = \varnothing$. Since f_1 (resp, f_2) is weakly δ-semicontinuous (resp. almost δ-semicontinuous), there exists $U_1 \in \delta SO(X_1, x_1)$ such that $f_1(U_1) \subset cl(V_1)$ (resp. $U_2 \in \delta SO(X_2, x_2)$ such that $f_2(U_2) \subset int(cl(V_2))$). Therefore, we obtain $(x_1, x_2) \in U_1 \times U_2 \subset (X_1 \times X_2) \backslash A$ and $U_1 \times U_2 \in \delta SO(X_1 \times X_2)$. This shows that A is δ-semiclosed in $X_1 \times X_2$. ∎

Definition 42 *A space X is said to be δ-semi-T_2 [4] if for any distinct points x, y of X, there exist disjoint δ-semiopen sets U, V of X such that $x \in U$ and $y \in V$.*

Theorem 43 *If for each pair of distinct points x_1 and x_2 in a space X, there exists a function f of X into a Hausdorff space Y such that*
(1) $f(x_1) \neq f(x_2)$,
(2) f is weakly δ-semicontinuous at x_1 and
(3) almost δ-semicontinuous at x_2,
then X is δ-semi-T_2.

Proof. Since Y is Hausdorff, there exist open sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \varnothing$; hence $cl(V_1) \cap int(cl(V_2)) = \varnothing$. Since f

is weakly δ-semicontinuous at x_1, there exists $U_1 \in \delta SO(X, x_1)$ such that $f(U_1) \subset cl(V_1)$. Since f is almost δ-semicontinuous at x_2, there exists $U_2 \in \delta SO(X, x_2)$ such that $f(U_2) \subset int(cl(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \varnothing$. This shows that X is δ-semi-T_2. ■

Definition 44 *A space X is said to be δ-semi-compact if every δ-semiopen cover of X has a finite subcover.*

Let $f : X \to Y$ be a function. The subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 45 *A function $f : X \to Y$ has a (δ_s, r)-graph if for each $(x, y) \in X \times Y \backslash G(f)$, there exist $U \in \delta SO(X, x)$ and a regular open set V of Y containing y such that $(U \times V) \cap G(f) = \varnothing$.*

Lemma 46 *A function $f : X \to Y$ has a (δ_s, r)-graph if and only if for each $(x, y) \in X \times Y$ such that $y \neq f(x)$, there exist a δ-semiopen set U and a regular open set V containing x and y, respectively, such that $f(U) \cap V = \varnothing$.*

Theorem 47 *If $f : X \to Y$ is an almost δ-semicontinuous function and Y is Hausdorff, then f has a (δ_s, r)-graph.*

Proof. Let $(x, y) \in X \times Y$ such that $y \neq f(x)$. Then there exist open sets V and W such that $y \in V$, $f(x) \in W$ and $V \cap W = \varnothing$; hence $int(cl(V)) \cap int(cl(W)) = \varnothing$. Since f is almost δ-semicontinuous, there exists $U \in \delta SO(X, x)$ such that $f(U) \subset int(cl(W))$. This implies that $f(U) \cap int(cl(V)) = \varnothing$. Therefore, f has a (δ_s, r)-graph. ■

Theorem 48 *If $f : (X, \tau) \to (Y, \sigma)$ has a (δ_s, r)-graph, then $f(K)$ is δ-closed in (Y, σ) for each subset K which is δ-semi-compact relative to (X, τ).*

Proof. Suppose that $y \notin f(K)$. Then $(x, y) \notin G(f)$ for each $x \in K$. Since $G(f)$ is (δ_s, r)-graph, there exist $U_x \in \delta SO(X)$ containing x and a regular open set V_x of Y containing y such that $f(U_x) \cap V_x = \varnothing$. The family $\{U_x : x \in K\}$ is a cover of K by δ-semiopen sets. Since K is δ-semi-compact relative to (X, τ), there exists a finite subset K_0 of K such that $K \subset \cup\{U_x : x \in K_0\}$. Set $V = \cap\{V_x : x \in K_0\}$. Then V is a regular open set in Y containing y. Therefore, we have

$$f(K) \cap V \subset [\bigcup_{x \in K_0} f(U_x)] \cap V \subset \bigcup_{x \in K_0} [f(U_x) \cap V] = \varnothing.$$

It follows that $y \notin \delta$-$cl(f(K))$. Therefore, $f(K)$ is δ-closed in (Y, σ). ■

Corollary 49 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost δ-semicontinuous function and Y is Hausdorff, then $f(K)$ is δ-closed in (Y, σ) for each subset K which is δ-semi-compact relative to (X, τ).*

References

1. M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings. Bull Fac. Sci. Assiut Univ., 12 (1983), 77-90.
2. D. Andrijević, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.
3. N. Bourbaki, General Topology, Part I, addison Wesley, reading, Mass., 1996.
4. M. Caldas, D. N. Georgiou, S. Jafari and T. Noiri, More on δ-semiopen sets, Note di Matematica, 22 (2) (2003/2004), 113-126.
5. D. Carnahan, Some Properties Related to Compactness in Topological Spaces, Ph. D. Thesis (Univ. Arkansas, 1973).
6. S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas J. Sci., 22 (1971) 99-112.
7. A. A. El-Atik, A study of some types of mappings on topological spaces, MSc Thesis, Tanta Univ., Egypt, 1997.
8. B. Y. Lee, M. J. Son and J. H. Park, δ-semiopen sets and its applications, Far East J. Math. Sci., 3 (5) (2001), 745-759.
9. N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
10. P. E. Long and L. L. Herrington, The T_θ-topology and faintly continuous functions, Kyungpook Math. J., 22 (1982), 7-14.
11. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.
12. B. M. Munshi and D. S. Bassan, Almost semi-continuous mappings, Math. Student, 49 (1981). 239-248.
13. A. A. Nasef, Another weak forms of faint continuity, Chaos, Solitons and Fractals, 12 (2001), 2219-2225.
14. O. Njåstad, On some classes of nearly open sets, Pacific J. Math., 15 (1965), 96J-970.
15. T. Noiri, On δ-continuous functions, J. Korean Math. Soc., 16 (1980), 161-166.
16. T. Noiri, Characterizations of extremally disconnected spaces, Indian J. Pure Appl. Math., 19 (1988), 325-329.
17. T. Noiri, On almost continuous functions, Indian J. Pure Appl. Math., 20 (1989), 571-576.
18. T. Noiri, Almost quasi-continuous functions, Bull. Inst. Math. Acad. Sinica, 18 (1990), 321-332.
19. T. Noiri and V. Popa, Weak forms of faint continuity, Bull. Math. Soc. Sci. Math. Roumanie, 34 (82) (1990), 270-363.
20. J. H. Park, B. Y. Lee and M. J. Son, On δ-semiopen sets in topological space, J. Indian Acad. Math., 19 (1) (1997), 59-67.
21. M. K. Signal and S. P. Arya, On almost-regular spaces, Glasnik Mat. III, 4(24) (1969), 89-99.
22. M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41 (1937) 375-381.
23. N. V. Veličko, H-closed topological spaces, Amer. Math. Soc. Transl. (2), 78 (1968), 103-118.

Erdal Ekici
Department of Mathematics,
Canakkale Onsekiz Mart University,
Terzioglu Campus,
17020 Canakkale - TURKEY
E-mail: eekici@comu.edu.tr