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Independence Number, Neighborhood Intersection and Hamiltonian Properties *

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ABSTRACT: Let G be a 2-connected simple graph of order n with the independence number α. We show here that $\forall u,\ v \in V(G)$ and any $z \in \{u,v\}$, $w \in V(G)\backslash\{u,v\}$ with $d(w,z) = 2$, if $|N(u) \cap N(w)| \geq \alpha - 1$ or $|N(v) \cap N(w)| \geq \alpha - 1$, then G is Hamiltonian, unless G belongs to a kind of special graphs.

Key words: Independence number, Neighborhood, Cycle.

Contents

1. Introduction

Hamiltonian graph is a very useful graph class in graph theory and many applications [1,2,3]. The research on sufficient conditions of Hamiltonian graphs is very active. Here we establish a new sufficient condition for Hamiltonian graphs using only independence number and neighborhood intersection properties. The result is very useful for the research of Hamiltonian graphs. By the new condition, it does not need to check all pairs of nonadjacent vertices in G. The following is our main result.

Theorem 1 *Let G be a 2-connected simple graph of order n with the independence number α. For any three vertices $u, v, w \in V(G)$ with $d(u,w) = d(v,w) = 2$, if $|N(u) \cap N(w)| \geqslant \alpha - 1$ or $|N(v) \cap N(w)| \geq \alpha - 1$, then G is Hamiltonian, except $G \cong G'(\alpha - 1, \alpha)$*

The outline of the paper is as follows. We propose our main result in the current section. The proof of the main result is given in the next section. For the proof, we shall prove the six useful lemmas. With several claims and these lemmas, we complete our demonstration.

For the simplicity, we shall use following terms and notations throughout this paper. $G = (V,E)$ denotes an undirected connected simple graph of order $n(\geq 3)$ with the independence number $\alpha(G) = \alpha$. Let $C \subseteq V(G), B \subseteq G$ and x be any vertex in G. Define $N_C(x) = \{v | v \in C \text{ and } xv \in E(G)\}$,$N_C(B) = \bigcup_{x \in B} N_C(x)$.

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Specially, if $C = V(G)$, we simply write it as $N(x)$ and $N(B)$. If no ambiguity can rise,we sometimes write B instead of $V(B)$.

Let $G'(r,t)$ be a kind of special graph, V_1, V_2 is a pair of sets of vertices with $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Here $|V_1| = r, G[V_1]$ is any simple graph; $V_2 = C_1 \cup C_2 \cup \cdots \cup C_t$, $C_i \cap C_j = \emptyset$ $(i \neq j)$ for any j with $1 \leq j \leq t$, $G[C_j]$ is complete and $C_j = C_{j_1} \cup C_{j_2}$, $C_{j1} \cap C_{j2} = \emptyset$, every vertex in C_{j_1} is adjacent to each vertices of V_1, as well as for any vertex of C_{j_2} is not adjacent to each vertices of V_1. For any $u \in C_i$, $v \in C_j$ with $i \neq j$, satisfying $uv \notin E(G)$

2. The proof of Theorem 1

Proof: The theorem is true for $\alpha = 1$ because G is complete. Now we assume that $\alpha \geqslant 2$. With the conditions of the theorem, we shall show that if G is not Hamiltonian, then $G \cong G'(\alpha-1, \alpha)$. Let C be a cycle of maximum length in G. It is clear that $|V(C)| < n$. Let B be any component of $G \backslash V(C)$. Denote $\overrightarrow{C}$ as the cycle with a given orientation. $u\overrightarrow{C}v$ means the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$ for $u, v \in V(C)$. The same vertices, in reverse order are given by $v\overleftarrow{C}u$. We here consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. uBv stands for the path from u via B to v. We use u^+ and u^- to denote the successor and predecessor respectively of u on $\overrightarrow{C}$. We write u^{++} instead of $(u^+)^+$ and u^{--} instead of $(u^-)^-$. Put $N_C(B) = \{v_1, v_2, \cdots v_m\}$, where v_i occurs on $\overrightarrow{C}$ in the order of their indices. Clearly, $m \geq 2$ and $N^+ = \{v_1^+, v_2^+, \cdots v_m^+, \}$, $N^- = \{v_1^-, v_2^-, \cdots v_m^-, \}$. For any $j(1 \leq j \leq m)$, x_j is a vertex in B adjacent to v_j. It is possible that $x_i = x_j$ for $i \neq j$. Then the following claims are obvious from results in [2,3].

Claim 1: *For any $j(1 \leq j \leq m)$, $x_j v_j^- \notin E(G)$ and $x_j v_j^+ \notin E(G)$.*

By Claim 1, for any $j(1 \leq j \leq m)$

$$d(x_j, v_j^-) = d(x_j, v_j^+) = 2 \tag{1}$$

Claim 2: *Let x be any vertex in B, then $N^+ \cup \{x\}$ and $N^- \cup \{x\}$ are independent sets.*

Claim 3: $N(x) \cap N(v_j^-) \subseteq N_C(B)$ *and* $N(x) \cup N(v_j^+) \subseteq N_C(B)$.

Hence,

$$|N(x) \cap N(v_j^+)| \leq m \ \ and \ \ |N(x) \cap N(v_j^-)| \leq m \tag{2}$$

By (1) and the conditions of theorem, we have:

$$\alpha - 1 \leq |N(x) \cap N(v_j^-)| \ \ or \ \ \alpha - 1 \leq |N(x) \cap N(v_j^+)| \tag{3}$$

It follows from claims that there are the following two cases:

Case 1: By (2) and (3),$\alpha - 1 \leq m$

and

Case 2: By Claim 2,$|N^+ \cup \{x\}| \leq \alpha \Longrightarrow m + 1 \leq \alpha \Longrightarrow m \leq \alpha - 1$.

Combining Cases 1 and 2, we have that

$$m = \alpha - 1 \tag{4}$$

Claim 4 *For any j, $1 \leq j \leq m$, $N(x_j) \cap N(v_j^-) = N_C(B)$ or $N(x_j) \cap N(v_j^+) = N_C(B)$.*

Based on claims above, we shall prove the following 6 lemmas to complete the proof of the theorem.

Lemma 1 For any u, $v \in N^+$

1. $uw \notin E(G)$, or $vw^- \notin E(G)$, when $w \in u^+\overrightarrow{C}v^-$
2. $uw \notin E(G)$ or $vw^+ \notin E(G)$ when $w \in v^+\overrightarrow{C}u^-$

Proof of Lemma 1: Suppose that $u = v_i^+$, $v = v_j^+$, $(i \neq j)$. We get

1. $uw \in E$ and $vw^- \in E$ when $w \in u^+\overrightarrow{C}v^-$
2. $uw \in E$ and $vw^+ \in E$ when $w \in v^+\overrightarrow{C}u^-$.

Then,

1. the cycle $v_i^+w\overrightarrow{C}v_jBv_i\overleftarrow{C}v_j^+w^-\overleftarrow{C}v_i^+$ is longer than C;
2. the cycle $v_i^+w\overleftarrow{C}v_j^+w^+\overrightarrow{C}v_iBv_j\overleftarrow{C}v_i^+$ is also longer than C.

both cases lead to a contradiction. □

Remark: Similarly, Lemma 1 holds as well for N^+ when N^+ is substituted by N^- in Lemma 1.

Lemma 2 For any $v_i^- \in N^-$ and $v_j^+ \in N^+$ with $i \neq j+1$, $v_i^-v_j^+ \notin E(G)$.
Proof of Lemma 2: Assume that there exist vertices $v_i^- \in N^-$ and $v_j^+ \in N^+$ with $i \neq j+1$, and $v_i^-v_j^+ \in E(G)$. Without loss of generality, Claim 4 implies that if $v_iv_{j+1}^+ \in E(G)$, then the cycle $v_i^-v_j^+\overrightarrow{C}v_{j+1}Bv_j\overleftarrow{C}v_iv_{j+1}^+\overrightarrow{C}v_i^-$ is longer than C which is a contradiction. □

For any $v_j^+ \in N^+$, $1 \leq j \leq m$, Let $C_j = \{u | u \in v_j^+\overrightarrow{C}v_{j+1}^-\}$ and $C_0 = V(B)$. Then we can prove the following lemma.

Lemma 3 For any j with $0 \leq j \leq m$, $G[C_j]$ is complete graph.
Proof of Lemma 3: It follows immediately from Claim 2 and 4 that $G[C_0]$ is complete. For any $j \neq 0$, while $|C_j| = 1, 2$, Lemma 3 holds. We here consider only $|C_j| \geq 3$. Suppose that $v_{j+1}^-v_j^+ \notin E(G)$. By Claim 1 and Lemma 2, $N^+ \cup \{x_j, v_{j+1}^-\}$ is an independent set of cardinality $m+2$ which contradicts $\alpha = m+1$. Thus $v_{j+1}^-v_j^+ \in E(G)$. Moreover, by Lemma 1 with $w = v_{j+1}^-$, we have for any $v_k^+ \in N^+$ with $k \neq j$, $v_k^+v_{j+1}^{--} \notin E(G)$. If $v_j^+v_{j+1}^{--} \notin E(G)$. then $N^+ \cup \{x_j, v_{j+1}^{--}\}$ is also an independent set in G. Note that$|N^+ \cup \{x_j, v_{j+1}^{--}\}| = m+2 = \alpha+1$ leads a contradiction. Hence $v_j^+v_{j+1}^{--} \in E(G)$. Similarly, we have that v_j^+ is adjacent to each vertex of C_j, by symmetry, v_{j+1}^- is adjacent to each vertex of C_j. Up to now, if $G[C_j]$ is not complete yet, we take vertex s and t from C_j,

such that $st \notin E(G)$ and the $s\overrightarrow{C}t$ as long as possibly. By the choice of s, t, $s,\ t \in v_j^{++}\overrightarrow{C}v_{j+1}^{--}$, and t is adjacent each vertex in $v_j^+\overrightarrow{C}s^-$, and s is adjacent each vertex in $t^+\overrightarrow{C}v_{j+1}^-$. So $ts^- \in E(G)$ and $t^+v_j^+ \in E(G)$ imply that for any $v_k^+ \in (N^+ \setminus \{v_j^+\})$, $sv_k^+ \notin E(G)$ and $tv_k^+ \notin E(G)$. If it is not true, assume $sv_k^+ \in E(G)$, then the cycle $t^+v_j^+\overrightarrow{C}s^-t\overleftarrow{C}sv_k^+\overrightarrow{C}v_jBv_k\overleftarrow{C}t^+$ is longer than C which is a contradiction. If $tv_k^+ \in E(G)$, then the cycle $t^+v_j^+\overrightarrow{C}tv_k^+\overrightarrow{C}v_jBv_k\overleftarrow{C}t^+$ is longer than C. This is a contradiction as well. Therefore, $(N^+ \setminus \{v_j^+\}) \cup \{s, t, x_j\}$ is an independent set of cardinality $m + 2$ which contradicts to. $\alpha = m + 1$. □

Lemma 4 For any $u \in C_i$, $v \in C_j$ with $i \neq j$, $uv \notin E(G)$.

Proof of Lemma 4: If there exists a vertex $u \in C_i$ and $v \in C_j$, (we may let $i < j$), then $uv \in E(G)$ It follows from Claim 2 and Lemma 2, without loss of generality, that $u \in v_i^{++}\overrightarrow{C}v_{i+1}^{--}$ and $v \in v_j^+\overrightarrow{C}v_{j+1}^-$. Note that Lemma 3 and Claim 4 imply $v_{j+1}^+v_j \in E(G)$. Hence, the cycle

$$v_i\overrightarrow{C}u^-v_{i+1}^-\overleftarrow{C}uv\overleftarrow{C}v_j^+v^+\overrightarrow{C}v_{j+1}Bv_{i+1}\overrightarrow{C}v_jv_{j+1}^+\overrightarrow{C}v_i$$

is longer than C which is a contradiction. □

Lemma 5 $V(G) = V(C) \cup V(B)$.

Proof of Lemma 5: Assume that the lemma is not true. Suppose that B_1 is another component of $G\setminus V(C)$. Then $froallx \in B, \forall y \in B_1$, there is $yx \notin E(G)$. It follows from Claim 2 and and $\alpha = m + 1$ that there exists vertices $v_k^+ \in N^+$ such that $yv_k^+ \in E(G)$. By Lemma 4, $N(y)\cap C_j = \phi$ for any j, $1 \le j \le m$, $j \neq k$. $N(y)\cap C_k \subseteq \{v_k^+, v_{k+1}^-\}$, because C is a longest cycle in G. By $|C_j| \ge 3$, there exists at least a vertex $u \in v_k^{++}\overrightarrow{C}v_{k+1}^{--}$, such that $yu \notin E(G)$. Then $(N^- \setminus \{v_{k+1}^-\}) \cup \{x, y, u\}$ is an independent set of cardinality $m + 2$ which is a contradiction. □

In terms of the definition of C_j, there is $V(C) = N_C(B) \cup C_1 \cup \cdots \cup C_m$ where $C_i \cap C_j = \emptyset$ for $i \neq j$. Lemma 5 tells us that

$$V(G) = N_C(B) \cup C_1 \cup \cdots \cup C_m \cup V(B).$$

Since $C_0 = V(B)$, define $V_1 = N_C(B)$ which leads

$$|V_1| = m = \alpha - 1. \tag{5}$$

Therefore, $V(G) = V_1 \cup C_0 \cup C_1 \cup \cdots \cup C_m$ where $C_0 \cap C_j = \emptyset$ for all j. Set $V_2 = C_0 \cup C_1 \cup \cdots \cup C_m$ and

$$V(G) = V_1 \cup V_2 \quad \text{and} \quad V_1 \cap V_2 = \emptyset. \tag{6}$$

Lemma 6 For any $x \in C_0$, if there exists $v_j \in V_1$ such that $xv_j \in E(G)$ then $xv_k \in E(G)$ for any $k(1 \le k \le m)$.

Proof of Lemma 6: Suppose that $v_j \in V_1$ and $xv_j \in E(G)$. From the definition of $N_C(B)$ and x_j $(1 \leq j \leq m)$, there is $x = x_j$. By Claim 4, $N_C(B) \subseteq N(x)$. Hence, $xv_k \in E(G)$, for any k $(1 \leq j \leq m)$ □

By symmetry of C_0 and C_j, by replacement of C_j $(1 \leq j \leq m)$ instead of C_0 in Lemma 6, Lemma 6 is true for C_j.

Set $C_{j1} = \{x \in C_j \ \mid v_k x \in E(G), \ \ \forall v_k \in V_1\}$ and $C_{j2} = C_j$ $setminus C_{j1}$. Then $\forall j$ $(0 \leq j \leq m)$,

$$C_j = C_{j1} \cup C_{j2} \quad \text{and} \quad C_{j1} \cap C_{j2} = \emptyset. \tag{7}$$

Finally, $G \cong G'(\alpha - 1, \alpha)$ follows from (5)-(7) and Lemmas 3, 4, and 6. The proof of the theorem is complete. □

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