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The Homotopy Type of Seiberg-Witten Configuration Space

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ABSTRACT: Let X be a closed smooth 4-manifold. In the Theory of the Seiberg-Witten Equations, the configuration space is $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$, where $\mathcal{A}_\alpha$ is defined as the space of $\mathfrak{u}_1$-connections on a complex line bundle over X, $\Gamma(S^+_\alpha)$ is the space of sections of the positive complex spinor bundle over X and $\mathcal{G}_\alpha$ is the gauge group. It is shown that $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ has the same homotopic type of the Jacobian Torus

$$T^{b_1(X)} = \frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})},$$

where $b_1(X) = dim_{\mathbb{R}} H^1(X,\mathbb{R})$.

Key words: connections, Gauge fields, 4-manifolds

Contents

1. Introduction

Although the physical meaning of the Seiberg-Witten equations ($\mathcal{SW}_\alpha$-eq.) is yet to be discovered, the mathematical meaning is rather deep and efficient to understand one of the most basic phenomenon of differential topology in four dimension, namely, the existence of non-equivalent differential smooth structures on the same underlying topological manifold. The Seiberg-Witten equations arose through the ideas of duality described in Witten [12]. It is conjectured that the Seiberg-Witten equations are dual to Yang-Mills equations; the duality being at the quantum level. A necessary condition is the equality of the expectation values for the dual theories. In topology, this means that fixed a 4-manifold its Seiberg-Witten invariants are equal to Donaldson invariants. A basic reference for $\mathcal{SW}_\alpha$-eq. is [2].

Let (X,g) represent a fixed riemannian structure on X. Originally, the $\mathcal{SW}_\alpha$-equations were 1^{st}-order differential equations and their solutions (A,ϕ) satisfying

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$\phi \neq 0$ were called $\mathcal{SW}_\alpha$-monopoles. These equations were not obtained by a variational principle. In [12], Witten used some special identities to obtain an integral useful to prove that the moduli spaces of sw-monopoles were empty, but a finite number of them. This integral defines the $\mathcal{SW}_\alpha$-functional on the configuration space $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ and satisfies the Palais-Smale condition, as proved by Jost-Peng-Wang in [7].

The nature of the $\mathcal{SW}_\alpha$-monopoles is rather subtle than the anti-self-dual connections considered in the Donaldson theory. It is known that the scalar curvature k_g plays a important role to the non-existence of $\mathcal{SW}_\alpha$-monopoles on X, e.g.: on S^4 endowed with the round metric, the only solution is the trivial one $(0, 0)$. It is a open question to find necessary and sufficient conditions for the existence of a $\mathcal{SW}_\alpha$-monopoles in $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$. In [11], Taubes shows that if X admits a symplectic structure then the *spin*^{c}-structure α defined by the canonical class admits a $\mathcal{SW}_\alpha$-monopole. In [5], Fintushel-Stern proved that whenever there is a class $\alpha \in Spin^c(X)$ which $\mathcal{SW}_\alpha$-invariant is non-zero, then we can construct many closed 4-manifolds Y, all non diffeomorphic to X, admitting a $\mathcal{SW}_\beta$-monopole for some $\beta \in Spin^c(Y)$.

Once the Seiberg-Witen theory can be formulated in a variational framework, and the functional satisfies the Palais-Smale condition, it is natural to search for a Morse Theory framework. As a first step, our attempt is to prove that the homotopy type of the configuration space is completely determined by the classical Hodge theory . This fact contrast with Donaldson theory, where there are an abundace of instantons and the Atiyah-Jones conjecture shows the interplay among the homotopy type of the moduli space of connections and the homotopy type of the moduli space of instantons.

By considering the embedding of the Jacobian Torus

$$i : T^{b_1(X)} = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})} \hookrightarrow \mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha), \quad b_1(X) = dim_{\mathbb{R}} H^1(X, \mathbb{R})$$

in the configuration space, the variational formulation of the $\mathcal{SW}_\alpha$-equations give us a interpretation to the topology of $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$.

Theorem 1.1. *Let X be a closed smooth 4-manifold endowed with a riemannian metric g which scalar curvature is k_g. Let*

1. *If $k_g \geq 0$, then the gradient flow of the $\mathcal{SW}_\alpha$-functional defines an homotopy equivalence among $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ and $i(T^{b_1(X)})$.*
2. *If $k_g < 0$, then $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ has the same homotopy type of $T^{b_1(X)}$.*

2. Basic Set Up

From a duality principle applicable to SUSY theories in Quantum Field Theory, Seiberg-Witten discovered a nice coupling of the self-dual(SD) equation, of a U_1 Yang-Mills Theory, to the Dirac^c equation. The coupling is performed by a particular isomorphism relating the space $\Omega^2_+(X)$, of *self-dual* 2-forms, and the bundle $End^0({\mathcal{S}_\alpha}^+)$.

By considering the projection $p_1 : H^2(X,\mathbb{Z}) \oplus H^1(X;\mathbb{Z}_2) \to H^2(X,\mathbb{Z})$, the space of
$Spin^c$-strutures on X is given by

$$Spin^c(X) = \{\alpha \in H^2(X,\mathbb{Z}) \oplus H^1(X;\mathbb{Z}_2) \mid w_2(X) = p_1(\alpha)\ mod\ 2\}.$$

For each $\alpha \in Spin^c(X)$, there is a representation $\rho_\alpha : SO_4 \to \mathbb{C}l_4$, induced by a $Spin^c$ representation, and consequently, a pair of vector bundles $({\mathcal{S}_\alpha}^+, \mathcal{L}_\alpha)$ over X (see [8]), where

- $\mathcal{S}_\alpha = P_{SO_4} \times_{\rho_\alpha} V = {\mathcal{S}_\alpha}^+ \oplus {\mathcal{S}_\alpha}^-$.
  The bundle ${\mathcal{S}_\alpha}^+$ is the positive complex spinors bundle (fibers are $Spin_4^c-$ *modules* isomorphic to $\mathbb{C}^2$).
- $\mathcal{L}_\alpha = P_{SO_4} \times_{det(\alpha)} \mathbb{C}$.
  It is called the *determinant line bundle* associated to the $Spin^c$-struture α. ($c_1(\mathcal{L}_\alpha) = \alpha$).

Thus, for a given $\alpha \in Spin^c(X)$ we associate a pair of bundles:

$$\alpha \in Spin^c(X) \quad \rightsquigarrow \quad (\mathcal{L}_\alpha, {\mathcal{S}_\alpha}^+).$$

From now on, we consider

- a Riemannian metric g over X,
- a Hermitian structure h on $\mathcal{S}_\alpha$.

Remark 2.1. Let $E \to X$ be a vector bundle over X;

1. The space of sections of E (usually denoted by $\Gamma(E)$) is denoted by $\Omega^0(E)$.
2. The space of p-forms ($1 \leq p \leq 4$) with values in E is denoted by $\Omega^p(E)$.
3. For each fixed covariant derivative ∇ [1] on E, there is a 1^{st}-order differential operator $d^\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$.

For each class $\alpha \in Spin^c(X)$ corresponds a U_1-principal bundle over X, denoted P_α, with $c_1(P_\alpha) = \alpha$. Also, we consider the adjoint bundles

$$Ad(U_1) = P_{U_1} \times_{Ad} U_1 \quad ad(\mathfrak{u}_1) = P_{U_1} \times_{ad} \mathfrak{u}_1.$$

$Ad(U_1)$ is a fiber bundle with fiber U_1, and $ad(\mathfrak{u}_1)$ is a vector bundle with fiber isomorphic to the Lie Algebra $\mathfrak{u}_1$. Once a covariant derivative is considered on $ad(\mathfrak{u}_1)$, it induces the sequence

$$\begin{aligned} \Omega^0(ad(\mathfrak{u}_1)) \xrightarrow{d^\nabla} \Omega^1(ad(\mathfrak{u}_1)) \xrightarrow{d^\nabla} \Omega^2(ad(\mathfrak{u}_1)) \xrightarrow{d^\nabla} \\ \xrightarrow{d^\nabla} \Omega^3(ad(\mathfrak{u}_1)) \xrightarrow{d^\nabla} \Omega^4(ad(\mathfrak{u}_1)).. \end{aligned} \tag{2.1}$$

[1] on E, connection 1-form A $\leftrightarrow$ ∇^A covariant derivative

The 2-form of curvature F_∇, induced by the connection ∇, is the operator

$$F_\nabla = d^\nabla \circ d^\nabla : \Omega^0(ad(\mathfrak{u}_1)) \to \Omega^2(ad(\mathfrak{u}_1)).$$

Since $Ad(U_1) \sim X \times U_1$ and $ad(\mathfrak{u}_1) \sim X \times \mathfrak{u}_1$, the spaces $\Omega^0(ad(\mathfrak{u}_1))$ and $\Gamma(Ad(U_1))$ are identified, respectively, to the spaces $\Omega^0(X, i\mathbb{R})$ and $Map(X, U_1)$. It is well known from the theory (see in [3]) that a $\mathfrak{u}_1$-connection defined on $\mathcal{L}_\alpha$ can be identified with a section of the vector bundle $\Omega^1(ad(\mathfrak{u}_1))$, and a Gauge transformation with a section of the bundle $Ad(U_1)$.

On a complex vector bundle E over (X,g), endowed with a hermitian metric and a covariant derivative ∇, we consider the Sobolev Norm of a section $\phi \in \Omega^0(E)$ as

$$|| \phi ||_{L^{k,p}} = \sum_{|i|=0}^{k} (\int_X | \nabla^i \phi |^p)^{\frac{1}{p}}$$

and the Sobolev Spaces of sections of E as

$$L^{k,p}(E) = \{\phi \in \Omega^0(X, E) \mid \quad || \phi ||_{L^{k,p}} < \infty\}$$

Now, consider the spaces

- $\mathcal{A}_\alpha = L^{1,2}(\Omega^0(ad(\mathfrak{u}_1)))$,
- $\Gamma({\mathcal{S}_\alpha}^+) = L^{1,2}(\Omega^0(X, {\mathcal{S}_\alpha}^+)$,
- $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma({\mathcal{S}_\alpha}^+)$,
- $\mathcal{G}_\alpha = L^{2,2}(X, U_1) = L^{2,2}(Map(X, U_1))$.

The space $\mathcal{G}_\alpha$ is the Gauge Group acting on $\mathcal{C}_\alpha$ by the action

$$\mathcal{G}_\alpha \times \mathcal{C}_\alpha \to \mathcal{C}_\alpha; \quad (g, (A, \phi)) \to (g^{-1}dg + A, g^{-1}\phi). \tag{2.2}$$

Since we are in dimension 4, the vector bundle $\Omega^2(ad(\mathfrak{u}_1))$ splits as

$$\Omega^2_+(ad(\mathfrak{u}_1)) \oplus \Omega^2_-(ad(\mathfrak{u}_1)), \tag{2.3}$$

where $(+)$ is the seld-dual component and (-) the anti-self-dual. The 1^{st}-order (original) *Seiberg-Witten* equations are defined over the configuration space $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma({\mathcal{S}_\alpha}^+)$ as

$$\begin{cases} D^+_A(\phi) = 0, \\ F^+_A = \sigma(\phi), \end{cases} \tag{2.4}$$

where

- D^+_A is the $Spinc^c$-Dirac operator defined on $\Gamma(\mathcal{S}^+_\alpha)$,

- The quadratic form $\sigma : \Gamma(\mathcal{S}_\alpha^+) \to End^0(S_\alpha^+)$ given by

$$\sigma(\phi) = \phi \otimes \phi^* - \frac{\mid \phi \mid^2}{2}.I \tag{2.5}$$

performs the coupling of the ASD-equation with the $Dirac^c$ operator. Locally, for $\phi = (\phi_1, \phi_2)$ the quadratic form takes the value

$$\sigma(\phi) = \begin{pmatrix} \frac{|\phi_1|^2 - |\phi_2|^2}{2} & \phi_1.\bar{\phi_2} \\ \phi_2.\bar{\phi_1} & \frac{|\phi_2|^2 - |\phi_1|^2}{2} \end{pmatrix}.$$

The $\mathcal{SW}_\alpha$-monopoles form the set of solutions of equations (2.4), this space can be described as the inverse image $\mathcal{F}^{-1}(0)$ by a map $\mathcal{F}_\alpha : \mathcal{C}_\alpha \to \Omega_+^2(X) \oplus \Gamma(\mathcal{S}_\alpha{}^-)$, defined as

$$\mathcal{F}_\alpha(A, \phi) = (F_A^+ - \sigma(\phi), D_A^+(\phi)).$$

The $\mathcal{SW}_\alpha$-equations are $\mathcal{G}_\alpha$-invariant.

3. The W-Homotopy Type of $\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S_\alpha^+)$

The space $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S_\alpha^+)$ isn't a manifold since the action isn't free. We observe that the isotropic group $G_{(A,o)}$ of a element $(A, 0) \in \mathcal{C}_\alpha$ is formed by the constant maps $g : X \to U_1$, since

$$g.(A, 0) = (A, 0) \Leftrightarrow g^{-1}dg = 0 \Leftrightarrow \quad g \quad \text{is constant.}$$

Therefore, $G_{(A,0)} \overset{\text{iso}}{\simeq} U_1$ and we consider the Gauge Group

$$\widehat{\mathcal{G}}_\alpha = \frac{\mathcal{G}_\alpha}{\{g : X \to U_1 \mid \text{ g=constant}\}} \overset{\text{iso}}{\simeq} \mathcal{G}_\alpha / U_1.$$

From now on, instead of the $\mathcal{G}_\alpha$-action, we consider on $\mathcal{C}_\alpha$ the $\widehat{\mathcal{G}_\alpha}$-action; consequently, the quotient space $\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S_\alpha^+)$ is a manifold. Nevertheless, the spaces $\mathcal{A}_\alpha/\mathcal{G}_\alpha$ and $\mathcal{A}_\alpha/\widehat{\mathcal{G}}_\alpha$ are diffeomorphic because all elements $A \in \mathcal{A}_\alpha$ have the same isotropic group.

In this section, the hypothesis of the theorem A.4 are checked to the space $\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S_\alpha^+)$, and the study of its weak homotopy type is performed.

We begin with the following remarks;

1. The quotiente spaces $\mathcal{B}_\alpha = \mathcal{A}_\alpha / \mathcal{G}_\alpha$ and $\Gamma(\mathcal{S}_\alpha^+)/\mathcal{G}_\alpha$ are Hausdorff spaces ([6]).
2. the $\mathcal{G}_\alpha$-action on $\mathcal{A}_\alpha$ is not free since the action of the subgroup of constant maps $g : M \to U_1$, $g(x) = g, \forall x \in M$, acts trivially on $\mathcal{A}_\alpha$.

As mentioned before, instead of the $\mathcal{G}_\alpha$-action, we consider on $\mathcal{C}_\alpha$ the $\widehat{\mathcal{G}_\alpha}$-action. On $\mathcal{A}_\alpha$, the $\widehat{\mathcal{G}_\alpha}$-action is free, and so, the space $\widehat{\mathcal{B}}_\alpha = \mathcal{A}_\alpha/\widehat{\mathcal{G}}_\alpha$ is a manifold.

The $\widehat{\mathcal{G}}_\alpha$-action on $\Gamma(\mathcal{S}^+_\alpha)$ is free except on the 0-section, where the isotropic group is the full group $\widehat{\mathcal{G}}_\alpha$. The action also preserves the spheres in $\Gamma(\mathcal{S}^+_\alpha)$, consequently, the quotient space is a cone over the quotient of a sphere by the $\widehat{\mathcal{G}}_\alpha$-action. Therefore, the quotient space is contractible.

It follows from the Corollary of A.3 that there exists the fibration

$$\Gamma(\mathcal{S}^+_\alpha) \to \mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(\mathcal{S}^+_\alpha) \to \widehat{\mathcal{B}}_\alpha.$$

By the contractibility of $\Gamma(\mathcal{S}^+_\alpha)$, it follows that

$$\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(\mathcal{S}^+_\alpha) \overset{\text{htpy}}{\sim} \widehat{\mathcal{B}}_\alpha.$$

In [1], they studied the homotopy type of the space $\mathcal{B}^* = \mathcal{A}/\mathcal{G}^*$, where $\mathcal{A}$ is the space of connections defined on a G-Principal Bundle P and

$$\mathcal{G}^* = \{g \in \mathcal{G} \mid g(x_0) = I\},$$

is a subgroup of the Gauge Group $\mathcal{G} = \Gamma(Ad(P))$. They observed that $\mathcal{G}^*$ acts freely on $\mathcal{A}$, and so, the quotiente space $\mathcal{B}^*$ is a manifold. We need to compare the $\widehat{\mathcal{G}}_\alpha$ and $\mathcal{G}^*_\alpha$ actions on $\mathcal{A}_\alpha$, neverthless, they turn out to be equal.

Proposition 3.1. *The gauge groups $\widehat{\mathcal{G}}_\alpha$ and $\mathcal{G}^*_\alpha$ are diffeomorphic and perform the same action on $\mathcal{A}_\alpha$.*

Proof. The projection ρ in the exact sequence

$$1 \to U_1 \to \mathcal{G}_\alpha \xrightarrow{\rho} \mathcal{G}^*_\alpha \to 1, \quad \rho(g) = g(x_0)^{-1}.g,$$

induces the diffeomorphims. Let $g^* = \rho(\widehat{g}), \quad \widehat{g} \in \widehat{\mathcal{G}}_\alpha$, so

$$g^*.A = A + (g^*)^{-1}dg^* = A + \widehat{g}^{-1}.\widehat{g}(x_0)^{-1}.\widehat{g}(x_0).d\widehat{g} = A + \widehat{g}^{-1}d\widehat{g} = \widehat{g}.A.$$

The same computation implies that $[g] = [h] \in \widehat{\mathcal{G}}_\alpha \Rightarrow g.A = h.A \quad \forall A \in \mathcal{A}_\alpha$. □

Consequently, $\mathcal{A}_\alpha/\widehat{\mathcal{G}}_\alpha = \mathcal{A}_\alpha/\mathcal{G}^*_\alpha$.

In this way, the results of [1] can be applied to the understanding of the topology of the space $\mathcal{A}_\alpha/\widehat{\mathcal{G}}_\alpha$.

The weak homotopy type of $\mathcal{B}^*_\alpha$ has been studied in [1] and [3] where they proved the following;

Theorem 3.2. *Let $\mathcal{L}_\alpha$ be a complex line with $c_1(\mathcal{L}_\alpha) = \alpha$, $\mathcal{E}U_1$ be the Universal bundle associated to U_1 and*

$$Map^0_\alpha(X, \mathbb{C}P^\infty) = \{f : X \to \mathbb{C}P^\infty \mid f^*(\mathcal{E}U_1) \overset{iso}{\sim} \mathcal{L}_\alpha, f(x_0) = y_0\}.$$

Then,

$$\mathcal{B}^*_\alpha \overset{w\text{-}htpy}{\sim} Map^0_\alpha(X, \mathbb{C}P^\infty).$$

Corollary 3.3. *The space* $\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S^+_\alpha)$ *is path-connected and*

$$\pi_n(\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S^+_\alpha)) = \pi_n(Map^0_\alpha(X, \mathbb{C}P^\infty)), \quad n \in \mathbb{N}.$$

The set of path-connected components of $Map^0(X, \mathbb{C}P^\infty)$ is equal to the space of homotopic classes $f : X \to \mathbb{C}P^\infty$, denoted by $[X, \mathbb{C}P^\infty]$. From Algebraic Topology, we know that

1. There is a 1-1 correspondence
$$\{\mathcal{L} \mid \mathcal{L} \quad \text{is a complex line bundle over X}\} \leftrightarrow Map^0(X, \mathbb{C}P^\infty),$$
2. The space of isomorphic classes of complex line bundles is 1-1 with $[X, \mathbb{C}P^\infty]$, i.e., if $\mathcal{L}$ is isomorphic to $\mathcal{L}_\alpha$ then $f \in Map^0_\alpha(X, \mathbb{C}P^\infty)$.
3. $[X, \mathbb{C}P^\infty] = H^2(X, \mathbb{Z})$.

In other words,

$$\pi_0(Map^0(X, \mathbb{C}P^\infty)) = H^2(X, \mathbb{Z}). \tag{3.1}$$

Theorem 3.4. *Let* $\alpha \in Spin^c(X)$. *For each* $n \in \mathbb{N}$, *the homotopy group* $\pi_n(Map^0_\alpha(X, \mathbb{C}P^\infty))$ *is isomorphic to*

$$\mathcal{H} = H^1(S^n, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}),$$

Consequently,

$$\pi_n(Map^0_\alpha(X, \mathbb{C}P^\infty)) = \begin{cases} 0, & \textit{if } n \neq 1, \\ H^1(X, \mathbb{Z}), & \textit{if } n{=}1. \end{cases} \tag{3.2}$$

Proof. Since

$$\pi_n(Map^0(X, \mathbb{C}P^\infty)) \simeq H^2(S^n \times X, \mathbb{Z}),$$

we can perform the computation of $\pi_n(Map^0_\alpha(X, \mathbb{C}P^\infty))$ by fixing a class of $[X, \mathbb{C}P^\infty]$.

For a class $\alpha \in H^2(X, \mathbb{Z})$, we fix a map $f : X \to \mathbb{C}P^\infty$ representing α, $x_0 \in X$ and $a \in S^n$. Thus,

$$\pi_n(Map^0_\alpha(X, \mathbb{C}P^\infty)) = [(S^n \times X, \{a\} \times X \cup S^n \times \{x_0\}), (\mathbb{C}P^\infty, f(x_0))] =$$
$$= [(S^n \times X, \{a\} \times X \cup S^n \times \{x_0\}), \mathbb{C}P^\infty].$$

However,

$$[(S^n \times X, \{a\} \times X), \mathbb{C}P^\infty] = H^2(S^n \times X, \mathbb{Z})/H^2(\{a\} \times X \cup S^n \times \{x_0\}, \mathbb{Z}).$$

Let $\mathcal{H} = H^2(S^n \times X, \mathbb{Z})/H^2(\{a\} \times X \cup S^n \times \{x_0\}, \mathbb{Z})$. By Kuneth's formula,

$$H^2(S^n \times X, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus \{H^1(S^n, \mathbb{Z}) \otimes H^1(X, \mathbb{Z})\} \oplus H^2(S^n, \mathbb{Z}).$$

and

$$H^2(\{a\} \times X \cup S^n \times \{x_0\}, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus H^2(S^n, \mathbb{Z}).$$

Hence,

$$\mathcal{H} = H^1(S^n, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}).$$

□

4. $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ **has same homotopy type as** $T^{b_1(X)} \subset \mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$**.**

Let us to consider the functional

$$SW(A,\phi) = \frac{1}{2}\int_X \{| F^+_A - \sigma(\phi) |^2 + | D^+_A(\phi) |^2\} dv_g. \qquad (4.1)$$

Proposition 4.1. *For each $\alpha \in Spin^c(X)$, let $\mathcal{L}_\alpha$ be the determinant line bundle associated to α and (A,ϕ) $\in \mathcal{C}_\alpha$. Also, assume that k_g=scalar curvature of (X,g). Then,*

1. $< F^+_A, \sigma(\phi) >= \frac{1}{2} < F^+_A.\phi, \phi >$.
2. $< \sigma(\phi), \sigma(\phi) >= \frac{1}{4} | \phi |^4$.
3. *Weitzenböck formula*
$$D^2\phi = \nabla^*\nabla\phi + \frac{k_g}{4}\phi + \frac{F_A}{2}.\phi.$$
4. $\sigma(\phi)\phi = \frac{|\phi|^2}{2}\phi$.
5. *The intersection form of X $Q_X : H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to \mathbb{R}$ is given by*
$$Q(\omega,\eta) = \int_X \omega \wedge \eta.$$
6. $\int_X | F^+_A |^2 dv_g = \int_X \frac{1}{2} | F_A |^2 dv_g + 2\pi^2\alpha^2$.

The identities above are applied to the functional (4.1) and, as consequence, a new functional turns up into the scenario. The new functional is defined as follows;

Definition 4.1. *For each $\alpha \in Spin^c(X)$, the Seiberg-Witten Functional is the functional $\mathcal{SW}_\alpha : \mathcal{C}_\alpha \to \mathbb{R}$ given by*

$$\mathcal{SW}_\alpha(A,\phi) = \int_X \{\frac{1}{4} | F_A |^2 + | \nabla^A\phi |^2 + \frac{1}{8} | \phi |^4 + \frac{1}{4}k_g | \phi |^2\} dv_g + \pi^2\alpha^2, \quad (4.2)$$

where k_g= scalar curvature of (X,g).

Let $k_{g,X} = min_{x\in X} k_g$ and

$$k^-_{g,X} = min\{0, -k^{\frac{1}{2}}_{g,X}\}. \qquad (4.3)$$

Remark 4.2.

1. Since X is compact and $|| \phi ||_{L^4} < || \phi ||_{L^{1,2}}$, the functional is well defined on $\mathcal{C}_\alpha$,
2. Once the $\mathcal{SW}_\alpha$-functional *(4.2)* is Gauge invariant, it induces a functional $\mathcal{SW}_\alpha : \mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha) \to \mathbb{R}$.

3. The $\mathcal{SW}_\alpha$-functional is bounded below by 0, and it is equal to 0 if and only if either there exists a $\mathcal{SW}_\alpha$-monopole or an self-dual U_1-connection.

Proposition 4.2. *The Euler-Lagrange equations of the $\mathcal{SW}_\alpha$-functional (4.2) are*

$$\Delta_A\phi + \frac{|\phi|^2}{4}\phi + \frac{k_g}{4}\phi = 0, \tag{4.4}$$

$$d^*F_A + 4\Phi^*(\nabla^A\phi) = 0, \tag{4.5}$$

where $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1({\mathcal{S}_\alpha}^+)$.

Remark 4.3. Locally, in a orthonormal basis $\{\eta^i\}_{1\leq i\leq 4}$ of T^*X, the operator Φ^* can be written as

$$\Phi^*(\nabla^A\phi) = \sum_{i=1}^{4} < \nabla_i^A\phi, \phi > \eta^i, \quad \textit{where} \quad \nabla_i^A = \nabla_{X_i}^A \quad (\eta_i(X_j) = \delta_{ij}).$$

The regularity of the solutions of (4.4) and (4.5) was studied by Jost-Peng-Wang in [7]. They observed that the L^∞ estimate of ϕ, already known to be satisfied by the $\mathcal{SW}_\alpha$-monopoles, is also obeyed by the solutions of (4.4) e (4.5). The estimate is the following;

Proposition 4.3. *If $(A,\phi) \in \mathcal{C}_\alpha$ is a solution of (4.4) and (4.5), then*

$$\| \phi \|_\infty \leq k_{g,X}^-, \tag{4.6}$$

where $k_{g,X}^- = max_{x\in X}\{0, -k_g^{\frac{1}{2}}(X)\}$.

In [7], Jost-Peng-Wang studied the analytical properties of the $\mathcal{SW}_\alpha$-functional. They proved that the Palais-Smale Condition, up to gauge equivalence, is satisfied. Therefore, by the *Minimax Principle* the $\mathcal{SW}_\alpha$-functional always attains its minimum in $\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S_\alpha^+)$ and, consequently, on $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S_\alpha^+)$. In this way, it is left the following question: Under which conditions the minimum in $\mathcal{A}_\alpha \times_{\widehat{\mathcal{G}}_\alpha} \Gamma(S_\alpha^+)$ is a $\mathcal{SW}_\alpha$-monopole ?

As a consequence of the estimate (4.6), if the Riemannian metric g on X has non-negative scalar curvature then the only solutions are $(A, 0)$, where

$$d^*F_A = 0.$$

The following result is well known [3];

Proposition 4.4. *Let X be a closed, smooth 4-manifold. The solutions of $d^*F_A = 0$, module the $\mathcal{G}_\alpha$-action, define the Jacobian Torus*

$$T^{b_1(X)} = \frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})}, \quad b_1(X) = dim_{\mathbb{R}}H^1(X,\mathbb{Z}).$$

Proof. Let's recall that $\alpha = \frac{i}{2\pi}\int F_A$. The equation $d^*F_A = 0$ implies that F_A is an harmonic 2-form, and by Hodge theory, it is the only one. Let A and B be solutions and consider $B = A + b$, so,

$$d^*F_B + d^*F_A + d^*db = 0 \quad \Rightarrow db = 0,$$

from where we can associate $B \rightsquigarrow b \in H^1(X,\mathbb{R})$ (and $F_B = F_A$).

If a connection B_1 is gauge equivalent to B_2, then there exists $g \in \mathcal{G}_\alpha$ such that $B = A + g^{-1}dg$ and $F_B = F_A$. However, the 1-form $g^{-1}dg \in H^1(X,\mathbb{Z})$. Consequently, if b_1, b_2 are the respectives elements in $H^1(X,\mathbb{R})$, then $b_2 = b_1$ in $\frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})}$. □

The fact that $T^{b_1(X)}$ has the same the homotopy type as $\mathcal{A}_\alpha \times_{\hat{\mathcal{G}}_\alpha} \Gamma(S^+_\alpha)$ (3.4) leads to the main result;

Theorem 4.5. *The space of solutions of $d^*F_A = 0$ and $\mathcal{A}_\alpha \times_{\hat{\mathcal{G}}_\alpha} \Gamma(S^+_\alpha)$ have the same homotopy type. Beyond, if $k_g \geq 0$ then there exists a homotopy equivalence among $\mathcal{A}_\alpha \times_{\hat{\mathcal{G}}_\alpha} \Gamma(S^+_\alpha)$ and $T^{b_1(X)}$.*

$$\begin{cases} 0, & \text{if } n \neq 1, \\ H^1(X,\mathbb{Z}), & \text{if } n=1. \end{cases} \tag{4.7}$$

Proof. The computaions of the homotopy groups has been performed in 3.2. Whenever
$k_g \geq 0$, the *Deformation Lemma* of Morse Theory implies that the gradient flow of $\mathcal{SW}_\alpha$ defines a homotopy equivalence among $\mathcal{A}_\alpha \times_{\hat{\mathcal{G}}_\alpha} \Gamma(S^+_\alpha)$ and $T^{b_1(X)}$, since

$$k_g \geq 0 \quad \Rightarrow \quad \mathcal{SW}_\alpha(A,0) < \mathcal{SW}_\alpha(A,\phi), \quad \forall \phi \neq 0 \in \Gamma(\mathcal{S}^+_\alpha).$$

□

Whenever k_g is non-positive, may be a $\mathcal{SW}_\alpha$-monopole is present and the homotopy equivalence can't be performed by the gradient flow.

If $(A,0)$ is a solution of the 1^{st}-order $\mathcal{SW}_\alpha$-equation (minimum for $\mathcal{SW}_\alpha$), then $F^+_A = 0$. It is known ([3]) that if $b^+_2 > 1$, then such solutions do not exists for a dense set of the space of metrics on X. Therefore, these facts can be packed in the following proposition;

Proposition 4.6. *Suppose that $b^+_2(X) > 1$. There exists a dense set of metrics on X such that;*

1. *For a finite number of classes $\alpha \in Spin^c(X)$ there exists a $\mathcal{SW}_\alpha$-monopole attaining the minimum,*
2. *If $\alpha \in Spin^c(X)$ is none of the classes considered in the previous item, then*

$$\inf_{(A,\phi)\in\mathcal{A}_\alpha\times_{\mathcal{G}_\alpha}\Gamma(S^+_\alpha)} \mathcal{SW}_\alpha(A,\phi) > 0.$$

Final Remark: The main result implies that the nature of the $\mathcal{SW}_\alpha$-monopoles are rather subtle than we may expect. The following questions were not reached by the methods applied to prove the main result;

1. Is $\mathcal{SW}_\alpha$ a Morse function for a generic subset of metrics on X ? If the answer is positive, is it possible to transform $\mathcal{SW}_\alpha$ into a perfect Morse function ?
2. Under which condition there exists a $\mathcal{SW}_\alpha$-monopole in $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S_\alpha^+)$?
3. Are there unstable critical points of the $\mathcal{SW}_\alpha$-functional?

A. The Diagonal Action and its Quotient Space

Let M, N be smooth manifolds endowed with G-actions α_M, α_N (respec.). About the G-actions, we will assume that;

1. There exists a subgroup H of G such that for all $m \in M$ G_m is conjugate to H.
2. The quotient spaces $M \diagup G$ and $N \diagup G$ are Hausdorff spaces.

The product action of $G \times G$ on the manifold $M \times N$, is defined by

$$\alpha_M \times \alpha_N : G \times G \times (M \times N) \to M \times N,$$

$$\alpha_M \times \alpha_N(g_1, g_2, m, n) = (\alpha_M(g_1, m), \alpha_N(g_2, n)),$$

or equivalently,

$$(g_1, g_2).(m, n) = (g_1.m, g_2.n).$$

Definition A.1. *The diagonal action* $\alpha_{\mathcal{D}} : G \times (M \times N) \to M \times N$ *is defined as*

$$\alpha_{\mathcal{D}}(g, (m, n)) = (\alpha_M(g.m), \alpha_N(g, n)),$$

and denoted as g.(m,n)=(g.m,g.n). The quotiente space is denoted by $M \times_G N$*.*

Definition A.2. *Let* $m \in M$ *and* $n \in N$*. The corresponding orbits are defined as follows;*

1. *For the action* α_M *on M, let* $\mathcal{O}_m^M = \{g.m \mid g \in G\}$*.*
2. *For the action* α_N *on N, let* $\mathcal{O}_n^N = \{g.n \mid g \in G\}$*.*
3. *For the product action (P-action)* $\alpha_M \times \alpha_N$ *on* $M \times N$*, let*
$$\mathcal{O}_{(m,n)}^P = \{(g_1.m, g_2.n) \mid g_1, g_2 \in G\}.$$
4. *For the diagonal action $\mathcal{D}$-action)* $\alpha_{\mathcal{D}}$ *on* $M \times N$*, let*
$$\mathcal{O}_{(m,n)}^{\mathcal{D}} = \{(g.m, g.n) \mid g \in G\}.$$

The orbit of (m,n), by the product action, is easily described by the orbits in M and N as

$$\mathcal{O}^P_{(m,n)} = \mathcal{O}^M_m \times \mathcal{O}^N_n.$$

Consequently,

$$(M \times N)/(G \times G) = (M/G) \times (N/G),$$

which induces the fibration

$$N/G \longrightarrow (M \times N)/(G \times G) \longrightarrow M/G.$$

In order to describe the topology of the space $M \times_G N$, we consider the commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{p_1} & M \\ {\scriptstyle \pi^{M\times N}}\big\downarrow & & {\scriptstyle \pi^M}\big\downarrow \\ M \times_G N & \xrightarrow{\mathfrak{p}} & M/G \end{array}, \qquad \text{(A.1)}$$

where

1. $p_1 : M \times N \to M$ is the projection on the 1^{st} factor;
2. $\pi^{M\times N} : M \times N \to M \times_G N$ is the projection induced by the quotient;
3. $\mathfrak{p} : M \times_G N \to M/G$ is the natural map induced by the projection $\mathcal{O}^{\mathcal{D}}_{(m,n)} \to \mathcal{O}^M_m$.

From now on, we fix $[m_0] \in M/G$ in order to describe $\mathfrak{p}^{-1}([m_0])$.
From the diagram, we get that

1. $(\pi^M)^{-1}([m_0]) = \mathcal{O}^M_{m_0}$
2. $(\pi^M \circ p_1)^{-1}([m_0]) = \mathcal{O}^M_{m_0} \times N$,
3. $(\pi^{M\times N})^{-1}([m_0, n_0]) = \mathcal{O}^{\mathcal{D}}_{[(m_0,n_0)]}$

Proposition A.1. *The subspace $\mathcal{O}^M_{m_0} \times N$ is a G-space with respect to the $\mathcal{D}$-action.*

Proof. The proof is splited into two easy claims;

1. If $(m,n) \in \mathcal{O}^M_{m_0} \times N$, then $\mathcal{O}^{\mathcal{D}}_{(m,n)} \subset \mathcal{O}^M_{m_0} \times N$.
   Let $m = g.m_0$;

$$g'.(m,n) = (g'g.m_0, g'.n) \in \mathcal{O}^M_{m_0} \times N.$$

2. If $(m,n) \in \mathcal{O}^M_{m_0} \times N$, then there exits $g \in G$ and $n^{,} \in N$ such that $(m,n) \in \mathcal{O}^{\mathcal{D}}_{(m_0,n^{,})}$
   Let $m = g.m_0$ and $n^{,} = g^{-1}.n$;
$$(m,n) = g.(m_0, g^{-1}.n) \quad \Rightarrow (m,n) \in \mathcal{O}^{\mathcal{D}}_{(m_0,n^{,})}.$$
□

Consequently,
$$\mathfrak{p}^1([m_0]) = \mathcal{O}^M_{m_0} \times_G N.$$

Proposition A.2.
$$\mathcal{O}^{\mathcal{D}}_{(m,n)} \cap p_1^{-1}(m_0) = \{g.n \mid g \in G_{m_0}\}$$

Proof. Let $m = g.m_0$; so $(m,n) = g.(m_0, g^{-1}.n) \Rightarrow \mathcal{O}^{\mathcal{D}}_{(m,n)} = \mathcal{O}^{\mathcal{D}}_{(m_0,g^{-1}.n)}$. Nevertheless,
$$g.(m_0,n) \in p_1^{-1}(m_0) \Leftrightarrow \exists g \in G \quad \text{such that} \quad g.(m,n) = (m_0, n^{,}),$$
this implies that $g \in G_{m_0}$ and $n^{,} = g.n$ □

Therefore, every $\mathcal{D}$-orbit meets the set $p_1^{-1}(m_0)$, and, beyond that, the intersection of the $\mathcal{D}$-orbits with $p_1^{-1}(m_0)$ defines a G_{m_0}-action on N;
$g.(m_0,n) = (m_0, g.n)$. For each $n \in N$, the isotropic group is given by $G_{m_0} \cap G_n$, where G_n stands for the isotropic group relative to the G-action on N. Consequently, the orbit space of the $\mathcal{D}$-action on $\mathcal{O}_{m_0} \times N$ can be identified with N/G_{m_0}.

Proposition A.3.
$$M \times_G N = \bigcup_{[m]\in M/G} N/G_m.$$

Proof. Since it was concluded that $\mathfrak{p}^{-1}([m_0]) = N/G_{m_0}$, the claim follows from the discussion above. □

Corollary A.4. 1. *If the G-action on M is free, then there is a fibration*
$$N \longrightarrow M \times_G N \longrightarrow M/G. \tag{A.2}$$

2. *Suppose that there exists a subgroup H of G such that for all $m \in M$ the isotropic group G_m is conjugate to H. So, there is a fibration*
$$N/H \longrightarrow M \times_G N \longrightarrow M/G, \tag{A.3}$$
   whose fiber N/H may be is a singular space.

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