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Periodic Solutions of a Neutral Difference System

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ABSTRACT: Sufficient conditions in terms of the matrix measure for the periodic solutions of a neutral type delay difference system

$$\Delta\left[x(n)+cx(n-\tau)\right]=A(n,x(n))\,x(n)+f(n,x(n-\sigma))$$

are given.

Key words: Kransnolselskii fixed point theorem, periodic solution, neutral system

Contents

1. Introduction

There are many studies related to periodic solutions of difference equations such as

$$\Delta x(n)=A(n,x(n))x(n)+f(n,x(n-\sigma)),\ \ n\in Z,$$

see e.g. [1,2,3,4,5]. One basic assumption behind such an equation is that the change $x(n+1)-x(n)$ is, aside from a perturbation, 'proportional' to $x(n)$. Yet there are cases when the effect of the change $x(n-\tau+1)-x(n-\tau)$ is also important. In this paper, we consider difference systems of the form

$$\Delta\left[x(n)+cx(n-\tau)\right]=A(n,x(n))\,x(n)+f(n,x(n-\sigma))\,,\ n\in Z, \tag{1}$$

where $Z=\{0,\pm1,\pm2,...\}$, τ and σ are integers, $c\in R$ and $|c|<1$, $A:Z\times R^s\to R^{s\times s}$ and $f:Z\times R^s\to R^s$ are continuous functions such that for some positive integer ω, $A(n+\omega,x)=A(n,x)$ and $f(n+\omega,x)=f(n,x)$ for $(n,x)\in Z\times R^s$.

A solution of (1) is a real vector sequence of the form $x=\{x(n)\}_{n\in Z}$ which renders (1) into an identity after substitution. As in the previous studies, we are concerned with the existence of solutions which are ω-periodic, that is, solutions that satisfy $x(n+\omega)=x(n)$ for $n\in Z$.

We will invoke the Krasnolselskii fixed point theorem for finding ω-periodic solutions of (1): Suppose B is a Banach space and G is a bounded, convex and closed subset of B. Let $S,P:X\to B$ satisfy the following conditions: (i) $Sx+Py\in G$, for any $x,y\in G$, (ii) S is a contraction mapping, and (iii) P is completely continuous. Then $S+P$ has a fixed point in G.

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2. Preliminaries

First of all, for any real (scalar) sequence $\{u_n\}_{n\in Z}$, we define a nonstandard summation operation:

$$\bigoplus_{n=\alpha}^{\beta} u_n = \begin{cases} \sum_{n=\alpha}^{\beta} u_n, & \alpha \le \beta \\ 0, & \beta = \alpha - 1 \\ -\sum_{n=\beta+1}^{\alpha-1} u_n, & \beta < \alpha - 1 \end{cases}.$$

Next, we recall the matrix norms and matrix measures. Let C be the set of complex numbers. Let $|\cdot|_p$ be the standard p norm for the linear space C^s. For each matrix $A \in C^{s\times s}$, the quantity $\|A\|_p$ defined by

$$\|A\|_p = \sup_{|x|_p \neq 0} \frac{|Ax|_p}{|x|_P} \tag{2}$$

is called the induced (matrix) norm of A corresponding to the vector norm $|\cdot|_p$. The matrix measure corresponding to $\|\cdot\|_p$ is the function $\mu_p : C^{s\times s} \to R$ defined by

$$\mu_p(A) = \lim_{k\to+\infty} k\left(\left\|I + \frac{1}{k}A\right\|_p - 1\right). \tag{3}$$

It is known (see e.g. [6]) that μ_p has the following properties:
(i) For each $A \in C^{s\times s}$, the limit indicated in (3) exists and is well defined;
(ii) $-\|A\|_p \le -\mu_p(-A) \le \mu_p(A) \le \|A\|_p$ for $A \in C^{s\times s}$,
(iii) $\mu_p(\alpha A) = \alpha\mu_p(A)$ for $\alpha \geqslant 0$ and $A \in C^{s\times s}$,
(iv) for $A, B \in C^{s\times s}$,

$$\max\{\mu_p(A) - \mu_p(-B), -\mu_p(-A) + \mu_p(B)\} \le \mu_p(A+B) \le \mu_p(A) + \mu_p(B),$$

(v) μ_p is convex, that is, for $\alpha \in [0,1]$ and $A, B \in C^{s\times s}$,

$$\mu_p\{\alpha A + (1-\alpha)B\} \le \alpha\mu_p(A) + (1-\alpha)\mu_p(B),$$

(iv) $-\mu_p(-A) \le \mathrm{Re}\lambda \le \mu_p(A)$ whenever λ is an eigenvalue of A.

As examples (see e.g. [6]), let $x = (x_1, ..., x_s)^T$, $A = (a_{ij})_{s\times s} \in C^{s\times s}$, then $|x|_\infty = \max_{0\le i\le s}|x_i|$, $\|A\|_\infty = \max_{0\le i\le s}\sum_j |a_{ij}|$, $\mu_\infty(A) = \max_{0\le i\le s}\{a_{ii} + \sum_{j\neq i}|a_{ij}|\}$, $|x|_1 = \sum_i |x_i|$, $\|A\|_1 = \max_{0\le j\le s}\sum_i |a_{ij}|$, $\mu_1(A) = \max_{0\le j\le s}\left\{a_{jj} + \sum_{i\neq j}|a_{ij}|\right\}$.

LEMMA 1. Let $A = (a_{ij})_{s\times s} \in R^{s\times s}$ and $|a_{ii}| \le 1$ for $i = 1, 2, ..., s$. Then for all positive integer k,

$$\|I + A\|_p \le k\left\|I + \frac{1}{k}A\right\|_p - (k-1),\ p = 1, \infty. \tag{4}$$

Proof: By definition, for each positive integer k, there is an integer $i_0 \in \{1, 2, ..., s\}$ such that

$$\begin{aligned} \left\| \frac{1}{k} I + \frac{1}{k} A \right\|_\infty &= \frac{a_{i_0 i_0}}{k} + \frac{1}{k} + \frac{1}{k} \sum\nolimits_{j \neq i_0} |a_{ij}| \\ &= 1 + \frac{a_{i_0 i_0}}{k} + \frac{1}{k} \sum\nolimits_{j \neq i_0} |a_{ij}| - \frac{k-1}{k} \\ &\leq \left| 1 + \frac{a_{i_0 i_0}}{k} \right| + \frac{1}{k} \sum\nolimits_{j \neq i_0} |a_{ij}| - \frac{k-1}{k} \\ &\leq \left\| I + \frac{1}{k} A \right\|_\infty - \frac{k-1}{k}. \end{aligned} \tag{5}$$

It follows that

$$\|I + A\|_p = k \left\| \frac{1}{k} I + \frac{1}{k} A \right\|_\infty \leq k \left\| I + \frac{1}{k} A \right\|_\infty - (k-1). \tag{6}$$

The other case where $p = 1$ may similarly be proved.

Next we recall some basic facts about linear periodic difference systems. Consider the system

$$\Delta x(n) = A(n) x(n), \ n \in Z, \tag{7}$$

where $A(n) = (a_{ij}(n))_{s \times s} \in R^{s \times s}$, $I + A(n)$ is nonsingular and $A(n + \omega) = A(n)$ for $n \in Z$. Let $\Phi(n, n_0)$ be the fundamental matrix of (7) which satisfies $\Phi(n_0, n_0) = I$. Recall that

$$\Phi(n, n_0) = \prod_{i=n_0}^{n-1} (I + A(i)), \ n > n_0$$

and

$$\Phi(n, n_0) = \prod_{i=n}^{n_0-1} (I + A(i))^{-1}, \ n < n_0,$$

and any solution of (7) is of the form $x(n) = \Phi(n, n_0) x(n_0)$, and for $n, \delta, t \in Z$,

$$\Phi(n, \delta) \Phi(\delta, t) = \Phi(n, t), \tag{8}$$

and

$$\Phi(n + 1, \delta) - \Phi(n, \delta) = A(n) \Phi(n, \delta), \tag{9}$$

As a consequence, if $\{x(n)\}_{n \in Z}$ is any one nontrivial ω-periodic solution of (7), then $x(0) \neq 0$ and

$$(I - \Phi(\omega, 0)) x(0) = (\Phi(0, 0) - \Phi(\omega, 0)) x(0) = 0,$$

so that

$$\det(I - \Phi(\omega, 0)) = 0.$$

Conversely, if $\det(I - \Phi(\omega, 0)) = 0$, then there is some $x_0 \neq 0$ such that $Ix_0 = \Phi(\omega, 0)x_0$. Let $x = \{x(n)\}_{n\in Z}$ be the unique solution of (7) which satisfies $x(0) = x_0$. Since $x(\omega) = \Phi(\omega, 0)x_0 = x(0)$, x is a nontrivial ω-periodic solution of (7).

LEMMA 2. Let $\{x(n)\}_{n\in Z}$ be a solution of (7). If $A(n) = (a_{ij}(n))_{s\times s} \in R^{s\times s}$ and $|a_{ii}(n)| < 1$ for $1 \le i \le s$ and $n, m \in Z, n \geqslant m$, then

$$|x(n)|_\infty \le |x(m)|_\infty \exp\left\{\bigoplus\nolimits_{i=m}^{n-1} \mu_\infty (A(i))\right\}. \tag{10}$$

Proof: In view of (7), we have

$$x(i+1) = (I + A(i))\, x(i), \ i \geqslant m. \tag{11}$$

By (11) and Lemma 1, we see that

$$\begin{aligned} |x(i+1)|_\infty &\le \|I + A(i)\|_\infty |x(i)|_\infty \le \left(k\left\|I + \frac{1}{k}A(i)\right\|_\infty - (k-1)\right)|x(i)|_\infty \\ &\le \exp\left\{\left(k\left\|I + \frac{1}{k}A\right\|_\infty - k\right)\right\}|x(i)|_\infty \end{aligned}$$

Taking limits on both sides as $k \to +\infty$, we see that

$$|x(i+1)|_\infty \le \exp(\mu_\infty(A(i)))\, |x(i)|_\infty, \ i \geqslant m, \tag{12}$$

which implies (10). The proof is complete.

As an immediate consequence, the fundamental matrix of (7) satisfies

$$\|\Phi(n, m)\|_\infty \le \exp\left\{\bigoplus\nolimits_{i=m}^{n-1} \mu_\infty (A(i))\right\}, \ n \geqslant m. \tag{13}$$

Let us seek a solution $x = \{x(n)\}_{n\in Z}$ of the following nonhomogeneous system associated with (7):

$$\Delta x(n) = A(n)x(n) + F(n), \ n \in Z, \tag{14}$$

where $F : Z \to R^s$ satisfies $F(n + \omega) = F(n)$ for $n \in Z$. By the method of undetermined coefficients, we assume

$$x(n) = \Phi(n, n_0)y(n), \ n \in Z, \tag{15}$$

where $\Phi(n, n_0)$ is the fundamental matrix of (7) satisfying $\Phi(n_0, n_0) = I$ but $y = \{y(n)\}_{n\in Z}$ is to be sought. Since

$$\Phi(n+1, n_0)\, y(n+1) = (I + A(n))\, \Phi(n, n_0)\, y(n) + F(n), \tag{16}$$

and

$$\Phi(n+1, n_0) = (I + A(n))\, \Phi(n, n_0), \tag{17}$$

we have

$$\Phi(n+1, n_0)\, \Delta y(n) = F(n). \tag{18}$$

Thus

$$\Delta y(n) = \Phi(n+1, n_0)^{-1} F(n) = \Phi(n_0, n+1) F(n), \tag{19}$$

so that

$$y(n) = y(n_0) + \bigoplus\nolimits_{i=n_0}^{n-1} \Phi(n_0, i+1) F(i), \ n \in Z, \tag{20}$$

We have thus found a solution $\{x(n)\}_{n\in Z}$ of (14) defined by

$$\begin{aligned} x(n) &= \Phi(n, n_0) x(n_0) + \Phi(n, n_0) \bigoplus\nolimits_{i=n_0}^{n-1} \Phi(n_0, i+1) F(i) \\ &= \Phi(n, n_0) x(n_0) + \bigoplus\nolimits_{i=n_0}^{n-1} \Phi(n, i+1) F(i) \end{aligned} \tag{21}$$

for $n \in Z$.

THEOREM 1. Suppose (7) does not have any nontrivial ω-periodic solutions.

$$\exp\left\{\bigoplus\nolimits_{i=0}^{\omega-1} \mu_\infty(A(i))\right\} < 1. \tag{22}$$

If the nonhomogeneous system (14) has an ω-periodic solution $\{x(n)\}_{n\in Z}$, then $\{x(n)\}_{n\in Z}$ is an ω-periodic solution of the system

$$x(n) = (I - \Phi(n+\omega, n))^{-1} \bigoplus\nolimits_{i=n}^{n+\omega-1} \Phi(n+\omega, i+1) F(i), \ n \in Z. \tag{23}$$

Conversely, if $\{x(n)\}_{n\in Z}$ is an ω-periodic solution of (23), then it is also an ω-periodic solution of (14).

Indeed, recall that (7) does not have any nontrivial ω-periodic solutions if, and only if, $\det(I - \Phi(\omega, 0)) \neq 0$. Let $\{x(n)\}_{n\in Z}$ be an ω-periodic solution of (14). Then in view of (21),

$$x(n_0) = (I - \Phi(\omega, 0))^{-1} \bigoplus\nolimits_{i=n_0}^{n_0+\omega-1} \Phi(n_0+\omega, i+1) F(i). \tag{24}$$

By (21) again and relations (13) and (22),

$$x(n) = (I - \Phi(n+\omega, n))^{-1} \bigoplus\nolimits_{i=n}^{n+\omega-1} \Phi(n+\omega, i+1) F(i), \ n \in Z. \tag{25}$$

The converse is easily seen by reversing the arguments above. The proof is complete.

For the sake of simplicity, let the norm $|\cdot|_\infty$, induced norm $\|\cdot\|_\infty$ and the corresponding matrix measure $\mu_\infty(\cdot)$ be denoted by $|\cdot|, \|A\|$ and $\mu(A)$ respectively. Let l^ω be the Banach space of all real vector ω-periodic sequences of the form $x = \{x(n)\}_{n\in Z}$ (where $x(n) \in R^s$) endowed with the usual linear structure as well as the norm $\|x\|_2 = \|x\|^0 + \|x\|^1$ where $\|x\|^0 = \max_{0\le i\le \omega-1} |x(i)|$ and $\|x\|^1 = \max_{0\le i\le\omega-1} |\Delta x(i)|$.

LEMMA 3. A subset D of l^ω is relatively compact if and only if D is bounded.

Proof: It is easy to see that if D is relatively compact in l^ω, then D is bounded. Conversely, if the subset D of l^ω is bounded, then there is a subset

$$\Gamma := \{x \in l^\omega | \ \ \|x\|^0 \le H, \|x\|^1 \le H\},$$

where H is a positive constant, such that $D \subset \Gamma$. It suffices to show that Γ is relatively compact in l^ω. To see this, note that for each $\varepsilon > 0$, we may choose numbers $y_0 < y_1 < ... < y_m$ such that $y_0 = -H$, $y_m = H$ and $y_{i+1} - y_i < \varepsilon/4$, for $i = 0, ..., m-1$. Then the set Γ_1 of all real ω-periodic vector sequence of the form

$$\left\{(v_1(n), v_2(n), ..., v_s(n))^T\right\}_{n\in Z}$$

that satisfies $v_j(i) \in \{y_0, y_1, ..., y_{m-1}\}$ for $j = 1, 2, ..., s$ and $i = 0, ..., \omega - 1$ is a finite ε-net of Γ. Indeed, it is easy to see that Γ_1 is a finite subset of l^ω, furthermore, for any $x = \{x(n)\}_{n\in Z} \in \Gamma$, we can let $\nu = \{v(n)\}_{n\in Z} \in \Gamma_1$ such that $|x_j(n) - v_j(n)| < \varepsilon/4$ for $j = 1, 2, ..., s$ and $n = 0, ..., \omega - 1$. Then $|x(n) - v(n)| \le \varepsilon/4$ and

$$|\Delta x(n) - \Delta v(n)| \le |x(n+1) - v(n+1)| + |x(n) - v(n)| \le \varepsilon/2,$$

for $n = 0, ..., \omega - 1$, so that

$$\|x - \nu\|_2 = \|x - \nu\|^0 + \|x - \nu\|^1 \le \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

The proof is complete.

3. Main Results

We first recall the conditions imposed on (1): $|c| < 1$ and $A : Z \times R^s \to R^{s\times s}$ and $f : Z \times R^s \to R^s$ are continuous functions such that for some positive ω, $A(n+\omega, x) = A(n, x)$ and $f(n+\omega, x) = f(n, x)$ for $(n, x) \in Z \times R^s$. Let $A(n, x) = (a_{ij}(n, x))_{s\times s}$.

THEOREM 2. Suppose there is a nontrivial ω-periodic sequence $\{\alpha(n)\}_{n\in Z}$ such that

$$\beta = \exp\left(\bigoplus_{i=0}^{\omega-1} \alpha(i)\right) < 1$$

and $|a_{ij}(n, x)| < 1$ for $1 \le i, j \le s$ and $(n, x) \in Z \times R^s$ and

$$\mu(A(n, x)) \le \alpha(n), \ n \in Z. \tag{26}$$

Suppose further that there is $M > 0$ such that

$$\bigoplus_{n=0}^{\omega-1} \sup_{|x|\le M} |f(n, x)| < \frac{(1-\beta) M (1 - 2|c|)}{M_0} - \frac{ML + b_0}{(1 - |c|)} |c| \omega \tag{27}$$

where

$$L = \sup_{|x|<M, 0\le n\le\omega} \|A(n, x)\|, \tag{28}$$

$$b_0 = \sup_{0\le n\le\omega-1, |x|\le M} |f(n, x)|$$

and

$$M_0 = \sup_{0\le s\le t\le\omega-1} \exp\left(\bigoplus_{i=s}^{t} \alpha(i)\right).$$

Then (1) has an ω-periodic solution.

Proof: For each $u = \{u(n)\}_{n\in Z} \in l^{\omega}$, consider the periodic system of the form

$$\Delta x(n) = A(n, u(n))\, x(n), \ n \in Z, \tag{29}$$

and

$$\Delta x(n) = A(n, u(n))\, x(n) + f(n, u(n-\sigma)) - c\Delta u(n-\tau), \ n \in Z. \tag{30}$$

Since $|a_{ij}(n,x)| < 1$ for $1 \le i, j \le s$ and $(n,x) \in Z \times R^s$, $I + A(n, u(n))$ is nonsingular for each $n \in Z$. Let $\Phi_u(n, n_0)$ be the fundamental matrix of (29) which satisfies $\Phi_u(n_0, n_0) = I$. By (13) and our assumption, we have

$$\|\Phi_u(\omega, 0)\| \le \exp\left\{\bigoplus_{i=0}^{\omega-1} \mu(A(i, u(i)))\right\} \le \exp\left(\bigoplus_{i=0}^{\omega-1} \alpha(i)\right) < 1, \tag{31}$$

thus $(I - \Phi_u(\omega, 0))^{-1}$ exists, which shows that (29) has no nontrivial ω-periodic solutions.

Define the mappings $S : l^{\omega} \to l^{\omega}$ and $P : l^{\omega} \to l^{\omega}$ by

$$(Su)(n) = -cu(n-\tau), \tag{32}$$

and

$$\begin{aligned}(Pu)(n) \; &= \; cu(n-\tau) + (I - \Phi_u(n+\omega, n))^{-1} \times \\ & \bigoplus_{i=n}^{n+\omega-1} \{\Phi_u(n+\omega, i+1)[f(i, u(i-\sigma)) - c\Delta u(i-\tau)]\}\end{aligned} \tag{33}$$

for $n \in Z$. Then

$$\begin{aligned}(Su + Pu)(n) \; &= \; (I - \Phi_u(n+\omega, n))^{-1} \times \\ & \bigoplus_{i=n}^{n+\omega-1} \{\Phi_u(n+\omega, i+1)[f(i, u(i-\sigma)) - c\Delta u(i-\tau)]\}\end{aligned}$$

for $n \in Z$. Thus if u is a fixed point of the operator $S + P$, then by Theorem 1, it is also an ω-periodic solution of (30).

We now show that the asusmptions in the Krasnoselskii's Theorem are satisfied, so that a fixed point of $S + P$ can indeed be found. Let

$$N = \frac{ML + b_0}{1 - |c|}. \tag{34}$$

Define

$$G = \left\{x \in l^{\omega} : \|x\|^0 \le M, \|x\|^1 \le N\right\}, \tag{35}$$

it is easy to see that G is a bounded, closed and convex subset of l^{ω}.

It is easily seen that the condition $|c| < 1$ implies S is a contraction mapping. Next we assert that for any $u, v \in G$, that satisfy $\|Su + Pv\|^0 \le M$. Indeed, since

$$\begin{aligned}\|\Phi_u(n+\omega, s)\| \; &\le \; \exp\left\{\bigoplus_{i=s}^{n+\omega-1} \mu(A(i, u(i)))\right\} \\ &\le \; \exp\left(\bigoplus_{i=s}^{n+\omega-1} \alpha(i)\right) < M_0, \ n \le s \le n+\omega-1,\end{aligned} \tag{36}$$

and by using (13) get

$$\begin{aligned}\left\|\left(I-\Phi_u\left(n+\omega,n\right)\right)^{-1}\right\| &= \left\|\bigoplus\nolimits_{i=0}^{\infty}\left(\Phi_u\left(n+\omega,n\right)\right)^{(i)}\right\| \\ &\le \bigoplus\nolimits_{i=0}^{\infty}\left\|\left(\Phi_u\left(n+\omega,n\right)\right)^{(i)}\right\| \le \bigoplus\nolimits_{i=0}^{\infty}\beta^i = \frac{1}{1-\beta} \qquad (37)\end{aligned}$$

From (27), (32), (33), (34), (35), (36) and (37), we have

$$\begin{aligned} & \left|\left(Su\right)(n)+\left(Pv\right)(n)\right| \\ \le\ & \left|\left(Su\right)(n)\right|+\left|\left(Pv\right)(n)\right| \\ \le\ & 2\left|c\right|M+\left\|\left(I-\Phi_v\left(n+\omega,n\right)\right)^{-1}\right\|\bigoplus\nolimits_{i=n}^{n+\omega-1}\left\|\Phi_u\left(n+\omega,i+1\right)\right\|\left[\sup_{|x|\le M}\left|f\left(n,x\right)\right|+\left|c\right|N\right] \\ \le\ & 2\left|c\right|M+\frac{M_0}{1-\beta}\left[\bigoplus\nolimits_{i=n}^{n+\omega-1}\sup_{|x|\le M}\left|f\left(n,x\right)\right|+\left|c\right|N\omega\right] \\ \le\ & \frac{M_0}{1-\beta}\left\{\frac{1-\beta}{M_0}2\left|c\right|M+\left|c\right|N\omega+\bigoplus\nolimits_{i=n}^{n+\omega-1}\sup_{|x|\le M}\left|f\left(n,x\right)\right|\right\} \\ \le\ & \frac{M_0}{1-\beta}\left\{\frac{1-\beta}{M_0}2\left|c\right|M+\frac{ML+b_0}{1-|c|}\left|c\right|\omega+M\left[\frac{(1-\beta)}{M_0}\left(1-2\left|c\right|\right)-\frac{ML+b_0}{M\left(1-|c|\right)}\left|c\right|\omega\right]\right\} \\ =\ & M. \end{aligned} \tag{38}$$

Since

$$\Delta\left(\left(Su\right)(n)\right)=-c\Delta u\left(n-\tau\right). \tag{39}$$

and

$$\Delta\left(Pv\right)(n)=A\left(n,v\left(n\right)\right)\left\{\left(Pv\right)(n)+\left(Sv\right)(n)\right\}+f(n,v\left(n-\sigma\right)), \tag{40}$$

we have

$$\begin{aligned} & \left|\Delta\left(\left(Su\right)(n)+Pv\left(n\right)\right)\right| \\ \le\ & \left\|A\left(n,v\left(n\right)\right)\right\|\left\{\left|\left(Pv\right)(n)+\left(Sv\right)(n)\right|\right\}+\left|f(n,v\left(n-\sigma\right))\right|+\left|c\Delta u\left(n-\tau\right)\right| \\ =\ & LM+b_0+\left|c\right|N=N, \end{aligned}$$

so that $\|Su+Pv\|^1\le N$. We have now proved that for $u,v\in G$, $Su+Pv\in G$.

Next, we prove that P is a completely continuous operator from G into G. For $u,v\in G$, let $V=Pu-Pv$. By (40), we know that

$$\begin{aligned}\Delta\left(V\left(n\right)\right) &= A\left(n,u\left(n\right)\right)\left\{\left(Pu\right)(n)+\left(Su\right)(n)\right\}+f(n,u\left(n-\sigma\right)) \\ &\quad -A\left(n,v\left(n\right)\right)\left\{\left(Pv\right)(n)+\left(Sv\right)(n)\right\}-f(n,v\left(n-\sigma\right)) \\ &= A\left(n,u\left(n\right)\right)V\left(n\right)+\left[A\left(n,u\left(n\right)\right)-A\left(n,v\left(n\right)\right)\right]\left(Pv\right)(n) \\ &\quad +A\left(n,u\left(n\right)\right)\left[\left(Su\right)(n)-\left(Sv\right)(n)\right] \\ &\quad +\left[A\left(n,u\left(n\right)\right)-A\left(n,v\left(n\right)\right)\right]\left(Sv\right)(n) \\ &\quad +f(n,u\left(n-\sigma\right))-f(n,v\left(n-\sigma\right)). \end{aligned} \tag{41}$$

Let

$$\begin{aligned} w(t,u(n),v(n)) &= -A(n,u(n))\,c\,[u(n-\tau)-v(n-\tau)] \\ &\quad +[A(n,u(n))-A(n,v(n))]\,[(Pv)(n)+(Sv)(n)] \\ &\quad +f(n,u(n-\sigma))-f(n,v(n-\sigma)). \end{aligned} \tag{42}$$

Noting that $A(n,x)$ and $f(n,x)$ for $0 \le n \le \omega-1$ are continuous on G and $Pv+Sv$ is bouned, we see that when $\|u-v\|_2 \to 0$, $|w(t,u(n),v(n))| \to 0$ holds for $0 \le n \le \omega-1$. By (41), we have

$$\Delta(V(n)) = A(n,u(n))\,V(n) + w(t,u(n),v(n)), \tag{43}$$

that is, $V(n)$ is an ω-periodic solution of (43). By Thereom 1 we have

$$\begin{aligned} |V(n)| &\le \left\|(I-\Phi_u(n+\omega,n))^{-1}\right\| \bigoplus\nolimits_{i=n}^{n+\omega-1}\Phi_u(n+\omega,i+1)\,|w(t,u(i),v(i))| \\ &\le \frac{M_0}{1-\beta}\bigoplus\nolimits_{i=n}^{n+\omega-1}|w(t,u(i),v(i))|. \end{aligned} \tag{44}$$

Thus, we see that when $\|u-v\|^0 \to 0$, $\|Pu-Pv\|^0 = \|V\|^0 \to 0$. On the other hand, in view of (41), we see that $\|u-v\|^0 \to 0$ and $\|Pu-Pv\|^1 = \|V\|^1 = \|\Delta V\|^0 \to 0$. Hence if $\|u-v\|_2 \to 0$, then $\|u-v\|^0 \to 0$ and so $\|Pu-Pv\|_2 = \|Pu-Pv\|^0 + \|Pu-Pv\|^1 \to 0$, that is, P is a continuous mapping on G. On the other hand, note that $PG \subset G$ and G is bounded, from Lemma 3, we know that PG is relatively compact. Thus P is a completely continuous mapping from G into G. By means of the Krasnoselskii's thereom, we know that $P+S$ has a fixed point in G. By Theorem 1, (1) has an ω-periodic solution. The proof is complete.

COROLLARY 1. Suppose there is a nontrivial ω-periodic sequence$\{\alpha(n)\}_{n\in Z}$ such that

$$\beta = \exp\left(\bigoplus\nolimits_{i=0}^{\omega-1}\alpha(i)\right) < 1,$$

and $|a_{ii}(n,x)| < 1$ for $1 \le i,j \le s$ and $(n,x) \in Z \times R^s$ and

$$\mu(A(n,x)) \le \alpha(n) \le 0.$$

Suppose further that there is $M > 0$ such that

$$\bigoplus\nolimits_{i=0}^{\omega-1}\sup_{|x|\le M}|f(i,x)| < (1-\beta)\,M\,(1-2|c|) - \frac{ML+b_0}{(1-|c|)}|c|\,\omega,$$

where

$$L = \sup_{|x|<M,\,0\le n\le\omega}\|A(n,x)\|$$

and

$$b_0 = \sup_{0\le n\le\omega-1,\,|x|\le M}|f(n,x)|.$$

Then (1) has an ω-periodic solution.

As an example, consider the two dimensional nonlinear neutral difference system of the form

$$\Delta \left[x(n) - \frac{1}{16} x(n-\tau) \right] = A(n, x(n))\, x(n) + f(n, x(n-\sigma)), \quad n \in Z, \tag{45}$$

where τ and r are positive integers,

$$A(n,x) = \begin{pmatrix} \frac{-1}{4} & \frac{(-1)^n}{8} \exp\left(-x_1^2 - x_2^2\right) \\ \frac{(-1)^n}{8} \exp\left(-x_1^4 - x_2^4\right) & \frac{-1}{4} \end{pmatrix}, \quad n \in Z,$$

and

$$f(n,x) = \begin{pmatrix} \frac{(-1)^n}{4} \exp\left(-x_1^2 - x_2^2\right) \\ \frac{(-1)^{n+1}}{8} \exp\left(-x_1^8 - x_2^8\right) \end{pmatrix}, \quad n \in Z.$$

It is easy to see that $|a_{ii}(n,x)| = \frac{1}{4} < 1$ for $i = 1, 2$, $\mu_\infty(A(n,x)) \le -\frac{1}{8}$ and $\sup_{0 \le n \le 1, |x| \le M} |f(n,x)|_\infty \le \frac{1}{4}$. If we let $\alpha(n) = -\frac{1}{8}$ and $M = 16$, then $\beta = \exp\left(\bigoplus_{i=0}^{\omega-1} \alpha(i)\right) = e^{-\frac{1}{4}}$ and $L = \sup_{|x|<M, 0 \le n \le \omega} \|A(n,x)\|_\infty = \frac{3}{8}, b_0 = \sup_{0 \le n \le 1, |x| \le M} |f(n,x)| = \frac{1}{4}$, $\bigoplus_{i=0}^{\omega-1} \sup_{|x| \le M} |f(n,x)| \le \frac{1}{2}$. In view of these calculations, we may see that the conditions of Corollary 1 are satisfied. Hence (45) has a 2-periodic solution. This solution is also nontrivial, since $f(n,0) \neq 0$.

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