



Attractors and their structure for semilinear wave equations with nonlinear boundary dissipation

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ABSTRACT: Long time behavior of a semilinear wave equation with nonlinear boundary dissipation is considered. It is shown that weak solutions generated by the wave dynamics converge asymptotically to a finite dimensional attractor. It is known [CEL1] that the attractor consists of all full trajectories emanating from the set of stationary points. Under the additional assumption that the set of stationary points is finite it is proved that every solution converges to some stationary points at an exponential rate.

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1. Introduction

The aim of this paper is to present recent results on long time behaviour of solutions to semilinear wave equations with nonlinear boundary dissipation. The main questions asked are the following:

- (i) existence and structure of global attractors,
- (ii) regularity of attractors,
- (iii) fractal dimension of attractors,
- (iv) convergence of individual trajectories to equilibria points (an interesting case is when the equilibria points are multiple),

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Problems of long time behaviour of hyperbolic equations have attracted considerable attention in the literature [B-V,E-M-N,E-F-N-T,H,Ha,T] and references therein. However, majority of results in the field deal with *linear* and *internal* dissipation, as opposed to *nonlinear* and *boundary* dissipation that is considered in this manuscript. It has been recognized that nonlinearity of dissipation in hyperbolic structures leads to substantial mathematical difficulties. Rich body of results and techniques developed for parabolic like dynamics are no longer applicable. Hyperbolic flows with nonlinear dissipation are not C^1 -a feature that is fundamental to all treatments that are based on linearization of the flow [T,E-M-N,B-V]. Nonlinear wave equations are perturbations of Hamiltonian flows for which the long time behavior is not finite dimensional. Adding to this the fact that dissipation considered is localized at the boundary and the resulting semi-flow is not a group, makes the problem even more intricate. Indeed, propagation of dissipation from the boundary of a spatial domain to the entire region is a delicate issue even in the simplest case of linear dissipation with a single zero equilibrium. More complex structures of attractors that include multiple equilibria and other orbits put this problem into a different perspective. As recognized recently in [1] page 353 “globally dissipative dynamics of nonlinear wave equation or hyperbolic evolutionary equations with boundary damping remains an open problem”. The presence of boundary dissipation is no longer represented by a bounded (on the phase space) operator, in contrast to the interior (full or localized) damping [H-R,F1,F2,F3]. Thus, *nonlinearity of boundary dissipation in the context of structural properties of “hyperbolic” attractors* is a distinctive feature of the present work. Our main aim is to answer in affirmative the four questions raised above. This will be accomplished by combining recent observability estimates obtained in the context of control theory with some new developments in dynamical systems that pertain to Kolmogorov entropy and related fractal dimensions.

We begin by presenting PDE model to be considered. Let $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, be a bounded, connected set with a smooth boundary Γ . The exterior normal on Γ is denoted by ν . We consider the following equation

$$w'' - \Delta w + f(w) = 0 \text{ in } Q = [0, \infty) \times \Omega \quad (1.1)$$

subject to the boundary condition

$$\partial_\nu w + w = -g(w') \text{ in } \Sigma = [0, \infty) \times \Gamma \quad (1.2)$$

and the initial conditions

$$w(0) = w_0 \text{ and } w'(0) = w_1 . \quad (1.3)$$

Here f and g are nonlinear functions subject to the following assumption.

Assumption 1 (f-1) $f \in C^2(\mathbf{R})$ such that $|f''(s)| \leq c(1 + |s|^p)$ for all s and for some $c > 0$ where $0 < p < \infty$ for $n = 2$ and $0 < p \leq 1$ for $n = 3$.

(f-2)

$$\liminf_{|s| \rightarrow \infty} \frac{f(s)}{s} > -\lambda,$$

where λ is the best constant in the Poincaré type inequality

$$\int_{\Omega} |\nabla u|^2 + \int_{\Gamma} |u|^2 \geq \lambda \int_{\Omega} |u|^2.$$

(g-1) $g \in C^1(\mathbf{R})$ is an increasing function, $g(0) = 0$ and there exist two positive constants m_1 and m_2 such that $m_1 \leq g'(s) \leq m_2$ for all $|s| \geq 1$.

It is known [L-Ta] that the system described by (1.1)–(1.3) generates a semi-flow $S(t)$ in the usual finite energy space $\mathcal{H} \equiv H^1(\Omega) \times L_2(\Omega)$. Moreover, more regular initial conditions lead to corresponding solutions that display an additional regularity. The following wellposedness-regularity results are known.

Theorem 1.1 [CEL1]

Weak solutions Assume that the initial conditions satisfy $(w_0, w_1) \in \mathcal{H}$. Then there exists a unique generalized solution $(w, w') \in C([0, \infty); \mathcal{H})$ to (1.1)–(1.3).

Strong solutions Assume, in addition, that $w_0 \in H^2(\Omega)$, $w_1 \in H^1(\Omega)$ and w_0, w_1 satisfy the compatibility conditions on the boundary

$$\partial_\nu w_0 + w_0 + g(w_1) = 0 \text{ on } \Gamma.$$

Then a weak solution is "strong" and satisfies the regularity properties

$$w \in L_\infty(0, \infty; H^2(\Omega)), \quad w' \in L_\infty(0, \infty; H^1(\Omega)) \text{ and } w'' \in L_\infty(0, \infty; L_2(\Omega)).$$

The proof of this theorem is given in [CEL1, Section 2] and relies on the theory of monotone operators (see, e.g., [Ba, Br, S]). Regularity results of solutions stated above will be used (implicitly) throughout the manuscript. Indeed, many computations performed in the paper require higher regularity. To cope with this, we shall use the usual procedure of considering strong solutions and pass to the limit by using density theorems at the level of final estimates.

In what follows we shall adopt the following notation. By $L_p(\Omega)$ we denote the space of Lebesgue measurable functions whose p -th powers is integrable, and by $H^s(\Omega)$ we denote the L_2 based Sobolev space of order s . The scalar product in $L_2(\Omega)$ is $(u, v) = \int_{\Omega} uv$ and the scalar product in $L_2(\Gamma)$ is $\langle u, v \rangle = \int_{\Gamma} uv$. In the space $\mathcal{H} = H^1(\Omega) \times L_2(\Omega)$ we define the norm by the formula

$$\|U\|_{\mathcal{H}}^2 = \|\nabla u_0\|_{L_2(\Omega)}^2 + \|u_0\|_{L_2(\Gamma)}^2 + \|u_1\|_{L_2(\Omega)}^2, \quad U = (u_0, u_1) \in \mathcal{H}.$$

2. Formulation of the results

By Theorem 1.1 equations (1.1) and (1.2) generate a dynamical system $(S(t), \mathcal{H})$ with the phase space \mathcal{H} and the evolution operator $S(t)$ given by the formula

$$S(t)U = U = (u(t), u'(t)), \quad U = (u_0, u_1) \in \mathcal{H},$$

where $u(t)$ is a weak solution to (1.1)–(1.3). We recall the following definition (see, e.g. [B-V], or [C], or [T]).

Definition 2.1 A closed bounded set \mathcal{A} is said to be a global attractor for a dynamical system $(S(t), \mathcal{H})$ iff

- \mathcal{A} is a strictly invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$,
- \mathcal{A} uniformly attracts any other bounded set from the phase space, i.e.,

$$\lim_{t \rightarrow \infty} \sup_{U \in \mathcal{B}} \text{dist}_{\mathcal{H}}(S(t)U, \mathcal{A}) = 0$$

for any bounded set $B \subset \mathcal{H}$.

We begin with asserting the existence of global and compact attractors for a strongly continuous semi-flow $S(t), t \geq 0$.

Theorem 2.2 [*CEL1, Theorem 1.1*] *With reference to dynamics described by (1.1) and (1.2) we assume Assumption 1. Then there exists a global compact attractor $\mathcal{A} \subset \mathcal{H}$ for the dynamical system $(S(t), \mathcal{H})$.*

Theorem 2.2 was proved in [*CEL1*]. It should be noted that this result is also valid for critical exponents of nonlinear forcing terms $f(u)$. Indeed, we are allowed to take $p = 1$ when $n = 3$ in the Assumption 1. Once existence of attractors has been asserted, an interesting question is that of a structure of attractor. In fact, the result described below provides a rather precise characterization of the attractor. The attractor comprises of trajectories connecting equilibria. To state these results, we introduce the set of stationary points of $S(t)$ denoted by \mathcal{N} ,

$$\mathcal{N} = \{V \in \mathcal{H} : S(t)V = V \text{ for all } t \geq 0\}.$$

Every stationary point $W \in \mathcal{N}$ has the form $W = (w, 0)$, where $w = w(x)$ solves the problem

$$-\Delta w + f(w) = 0 \text{ in } \Omega \text{ and } \partial_\nu w + w = 0 \text{ in } \Gamma. \quad (2.1)$$

Let us define the *unstable manifold* $M^u(\mathcal{N})$ emanating from the set \mathcal{N} as a set of all $Y \in \mathcal{H}$ such that there exists a full trajectory $\gamma = \{W(t) : t \in \mathbf{R}\}$ with the properties

$$W(0) = Y \text{ and } \lim_{t \rightarrow -\infty} \text{dist}_{\mathcal{H}}(W(t), \mathcal{N}) = 0.$$

Our next result asserts that the attractor \mathcal{A} coincides with this unstable manifold.

Theorem 2.3 [*CEL1, Theorem 1.2*] *Under the assumptions of Theorem 2.2 we have*

- $\mathcal{A} = M^u(\mathcal{N})$,
- $\lim_{t \rightarrow +\infty} \text{dist}_{\mathcal{H}}(S(t)W, \mathcal{N}) = 0$ for any $W \in \mathcal{H}$.

From Theorem 2.3 we obtain the following corollaries.

Corollary 2.4 [*CEL1, Corollary 1.3*] *The global attractor \mathcal{A} consists of full trajectories $\gamma = \{W(t) : t \in \mathbf{R}\}$ such that*

$$\lim_{t \rightarrow -\infty} \text{dist}_{\mathcal{H}}(W(t), \mathcal{N}) = 0 \text{ and } \lim_{t \rightarrow +\infty} \text{dist}_{\mathcal{H}}(W(t), \mathcal{N}) = 0.$$

Corollary 2.5 [*CEL1, Corollary 1.4*] *Assume that problem (2.1) has a finite number of solutions.*

- (i) *The global attractor \mathcal{A} consists of full trajectories $\gamma = \{W(t) : t \in \mathbf{R}\}$ connecting pairs of stationary points, i.e. any $W \in \mathcal{A}$ belongs some full trajectory γ and for any $\gamma \subset \mathcal{A}$ there exists a pair $\{Z, Z^*\} \subset \mathcal{N}$ such that*

$$W(t) \rightarrow Z \text{ as } t \rightarrow -\infty \text{ and } W(t) \rightarrow Z^* \text{ as } t \rightarrow +\infty.$$

- (ii) *For any $V \in \mathcal{H}$ there exists a stationary point Z such that*

$$S(t)V \rightarrow Z \text{ as } t \rightarrow +\infty.$$

A question of particular interest is that of regularity of attractors. It is known that in the case of linear dissipation regularity of attractors is limited only by the regularity of the forcing terms [G-T]. However, in the case of nonlinear damping the situation is very different. Nonlinear damping poses a serious treat to propagation of regularity. The result stated below asserts that only certain amount of regularity (one derivative) can be gained in the case of boundary nonlinear damping.

Theorem 2.6 [*CEL1, Theorem 1.6*] *In addition to Assumption 1 we assume that $p < 1$ when $n = 3$ and $g'(0) > 0$. Then the attractor \mathcal{A} is a closed bounded set of $H^2(\Omega) \times H^1(\Omega)$.*

While the above regularity property is of interest in its own right, it may also serve as a tool for proving other properties of attractors -for instance- finite-dimensionality. Indeed, "squeezing" property - a fundamental tool in proving finite dimensionality [E-M-N,T,Lad,E-F-N-T]- requires a substantial amount of regularity of solutions in order to propagate the smoothness through the nonlinear dissipative term. However, it is unfortunate, that in the case of boundary damping the regularity gained (one derivative-see Theorem 2.6) is not sufficient in this respect. In order to apply the method one would need another half of the derivative and this seems a difficult if not impossible task to accomplish with the boundary damping. For this reason other methods for studying finite-dimensionality of attractors have been recently introduced [CL]. These methods, based on computations of Kolmogorov entropy, will be exploited in our studies. We shall prove that the fractal dimension of the attractor is indeed finite.

Theorem 2.7 *Under the Assumption 1 with $g'(s) > 0$, $s \in \mathbf{R}$, and $p < 1$ when $n = 3$ the dynamical system $(S(t), \mathcal{H})$ admits a compact global attractor \mathcal{A} whose fractal dimension is finite.*

Our final question deals with the issue of convergence of solutions to points of equilibria within the attractor. We already know by Theorem 2.3, that the attractor consists of the unstable manifold connecting points of equilibria. In addition, we know from Corollary 2.5 that every solution stabilizes to some equilibrium. An important issue is that of the rate of convergence of solutions to the corresponding equilibria. Once we know that every solution stabilizes to some equilibrium point as described in Corollary 2.5, we would like to know how fast that happens. An answer to this question is provided below. We shall show that under some additional hypotheses of geometric nature every solution stabilizes to some equilibrium at an exponential rate.

Theorem 2.8 *In addition to the assumptions of Theorem 2.7 we assume that the set of stationary points \mathcal{N} is finite and every equilibrium $V = (v, 0)$ is hyperbolic in the sense that the problem*

$$-\Delta w + f'(v) \cdot w = 0 \text{ in } \Omega \text{ and } \partial_\nu w + w = 0 \text{ in } \Gamma$$

has no non-trivial solutions.

Then for any $W_0 = (w_0, w_1) \in \mathcal{H}$ there exists a stationary point $V = (v, 0)$ such that

$$\|S(t)W_0 - V\|_{\mathcal{H}} \leq Ce^{-\omega t}$$

for some positive constants C and ω .

Remark 2.1 In the same way, as it was done in [B-V] for wave equations with linear internal damping, one can show that the property of finiteness and hyperbolicity of the set \mathcal{N} of equilibria is *generic*. Roughly speaking, this means that after a slight changing of nonlinearity (if it is necessary) we obtain a system with the property mentioned.

We shall make few comments about the history of the problem and the methods used. Existence of global attractors for wave equations with *internal nonlinear dissipation* has been studied in [R,F1,F2,F3]. In what follows we shall focus our discussion on the main issue addressed in the paper which is finite dimensionality and structure of attractors in the presence of nonlinear dissipation. Classical methods of proving finite dimensionality of attractors rely on one of the following main strategies: (i) proving continuous differentiability of the flow, (ii) calculating uniform Lyapunov exponents based on the linearization of the flow along trajectories within the attractor, (iii) proving additional regularity of the attractor, which, in turn, allows to establish a suitable "squeezing property" such as in [T,E-M-N,E-F-N-T,Lad]. Indeed, for (i) fractal analysis in [MP] applies (see also [B-V] and [T] for far-reaching developments of this analysis). The second approach in (ii) is very effective for problems with linear damping [T]. However in the case of nonlinear dissipation and hyperbolicity this method, if applicable, requires very strong restrictions imposed on the nonlinear terms [SZ]. For the third approach (iii) one can use either the squeezing property [T,G-T] or apply a method due to Ladyzenskaya [Lad] (see also [C]) which requires showing that co-projections of

the flow satisfy a Lipschitz condition with a constant strictly less than one. This latter property requires again, in the case of nonlinear dissipation, sufficient regularity of elements in the attractor. For the problem considered above, neither of the strategies indicated above seems applicable. The fact that the flow is hyperbolic and that the dissipation acts on the boundary and is nonlinear exclude both: continuous differentiability of the flow and sufficient regularity of the attractor. It is known that the flow $S(t)$ is not C^1 due to the hyperbolicity of equation (1.1) and nonlinearity of the damping [H-R]. Moreover, the elements on the attractor do not possess enough smoothness because of the nonlinearity of the dissipation. In fact, it is known that while an additional regularity of attractors is typical for parabolic semi-flows [H,Ha,B-V] which display some smoothing effect, it is much less expected in hyperbolic dynamics. For the *one-dimensional* wave equation with *nonlinear internal* dissipation [F1] managed to show that attractors do have finite fractal dimension. For dimension higher than one the problem is of course much more difficult. Most recently some progress has been made (see [L-R], [P] and [SZ]) with still *full interior damping*, where the dissipation is represented by a bounded operator. In the case of boundary damping the situation is even more complicated. Boundary damping is not represented by a bounded operator acting on the phase space. Thus, propagation of regularity by standard methods is out of reach. The regularity of the attractor given in Theorem 2.6 is insufficient to establish finite dimensionality of the attractor. Roughly speaking "one half extra derivative" is needed in order to apply methods based on the squeezing property as in [Lad,E-M-N,E-F-N-T]. However, the fact that the dissipation is on the boundary makes it difficult to achieve this additional regularity. The problem is simpler in the case of interior damping, as shown in [L-R], where under additional conditions imposed on the interior damping (which is required to be large) the C^∞ regularity of the attractor has been established. This particular difficulty encountered in the case of *boundary* dissipation became main motivation for searching different techniques capable to assert finite-dimensionality of attractors. As we shall see later, our method is based on a different approach which does not require additional regularity of elements on the attractor. Instead, the key ingredient of our approach consist in calculating Kolmogorov entropy and deriving from it a suitable extension of a generalized squeezing property (see (3.2) and the comments below). This property, along with a suitable string of observability/stabilizability estimates inspired by recent developments in boundary control theory of hyperbolic systems allows for an effective estimate of fractal dimension of the attractor. The additional pay-off of the method is that the obtained observability/stabilizability estimates provide also a critical ingredient for establishing the exponential decay of solutions to an equilibrium - Theorem 2.8. The main difficulty of this latter problem is the possibility of having multiple equilibria. In such cases, solutions which are in an arbitrary close neighborhood of the given equilibrium may still converge to a different equilibrium point. This makes known methods of proving uniform decay rates non applicable. In fact, the existing literature ([B-V] and the references therein) on this problem relies mostly on finite dimensional ideas where geometric arguments can be used. Instead the proof of Theorem 2.8 is analytic

and provides exact decay rates for the solutions. Thus, observability inequalities for the wave operator appear to be a common thread in characterizing long time behavior of wave dynamics with nonlinear boundary dissipation. We also refer to [CL] for a discussion of observability inequalities for general second order in time evolution equations and their applications.

Remark 2.2 One open question that is natural to ask is that of the necessity of the condition $g'(0) > 0$ in Theorem 2.7. In the absence of this condition the dissipation may be very weak. We now know that the above condition is not needed at all for uniform stability of dissipative equation [L-Ta], or for existence of global compact attractors -see Theorem 2.2, Theorem 2.3. Also, the result of Theorem 2.8 can be extended to the degenerate case $g'(0) = 0$ [CEL2]. However, when it comes to finite-dimensionality of attractors, the restriction $g'(0) > 0$ seems so far un-avoidable, even in the case of internal dissipation.

3. Proof of Theorems 2.7 and 2.8

In this manuscript we limit ourselves to brief sketches of the proofs of Theorem 2.7 and Theorem 2.8. The full length proofs with technical details and supporting lemmas are given in [CEL2]. Our main aim here is to expose conceptual ideas behind the proofs without entering too much into details of all estimates. We begin by defining a linear energy functional

$$E(w(t)) = \frac{1}{2} \int_{\Omega} |\nabla w(t)|^2 + \frac{1}{2} \int_{\Omega} |w'(t)|^2 + \frac{1}{2} \int_{\Gamma} |w(t)|^2 \equiv \frac{1}{2} \|(w(t), w'(t))\|_{\mathcal{H}}^2. \quad (3.1)$$

3.1. PROOF OF THEOREM 2.7. We first describe an abstract tool which is a generalization of squeezing properties in [Lad,E-F-N-T] and [P], and which will be used for proving finite dimensionality of an already established attractor \mathcal{A} .

We begin with a definition.

Definition 3.1 Let X be a separable Hilbert space. A seminorm $n(x)$ on X is said to be compact if $n(x_m) \rightarrow 0$ for any sequence $\{x_m\} \subset X$ such that $x_m \rightarrow 0$ weakly in X .

The following theorem is a special case of a more general result established in [CL, Sect.5].

Theorem 3.2 *Let X be a separable Hilbert space and A be a bounded closed set in X . Assume that there exists a mapping $V : A \mapsto X$ such that*

(i) $A \subseteq VA$;

(ii) V is Lipschitz on A , i.e., there exists $L > 0$ such that

$$\|Va_1 - Va_2\| \leq L\|a_1 - a_2\| \text{ for all } a_1, a_2 \in A.$$

(iii) there exist compact seminorms $n_1(x)$ and $n_2(x)$ on X such that

$$\|Va_1 - Va_2\| \leq \eta(\|a_1 - a_2\|) + K \cdot [n_1(a_1 - a_2) + n_2(Va_1 - Va_2)], \quad (3.2)$$

for all $a_1, a_2 \in A$, where $K > 0$ is a constant and $\eta : \mathbf{R}_+ \mapsto \mathbf{R}_+$ is a continuous function with the properties

$$\eta(0) = 0; \eta(s) < s, s > 0; s - \eta(s) \text{ is non-decreasing.}$$

If $\lim_{s \rightarrow 0} \{s^{-1}\eta(s)\} \equiv \eta^* < 1$, then A is a compact set in X of the finite fractal dimension. This dimension can be estimated in terms of the constants appearing in the formulation of the theorem.

Even in the case when $\eta(s) = \eta^* \cdot s$ is a linear function with $\eta^* \in [0, 1)$, this result generalizes squeezing properties given in [Lad,E-F-N-T] and most recently [P]. Indeed, Ladyzhenskaya's theorem [Lad] on finite dimension of invariant sets follows from Theorem 3.2. To see this we take $n_1 \equiv 0$ and $n_2(a) = \|Pa\|$ in relation (3.2), where P is a finite dimensional projector. Theorem 3.2 also generalizes the result by Prazak [P], which relies on the so-called "generalized squeezing property". To obtain the conclusion of Lemma 4.1 [P] on dimension we need only apply Theorem 3.2 with $n_1(a) = n_2(a) = \|Pa\|$, where P is a finite dimensional projector. One of the main advantages of our approach in comparison with results in [Lad] and [P] is that Theorem 3.2 does not contain finite-dimensional projectors in explicit form. This fact is very handy in applications to hyperbolic problems with boundary nonlinear damping. We also note that, as it is shown in [CL] by means an example, the assumption $\lim_{s \rightarrow 0} \{s^{-1}\eta(s)\} < 1$ cannot be omitted.

The key inequality in our considerations is the following stabilizability inequality.

Lemma 3.3 Stabilizability inequality. *Assume that $g'(0) > 0$. Let $u(t)$ and $v(t)$ be two solutions to (1.1)–(1.3) possessing the properties*

$$\|(u(t), u'(t))\|_{\mathcal{H}} \leq R \quad \text{and} \quad \|(v(t), v'(t))\|_{\mathcal{H}} \leq R \quad \text{for all } t \geq 0 \quad (3.3)$$

with some constant $R > 0$. Denote $z(t) \equiv u(t) - v(t)$. Then there exist positive constants C_1, C_2 and β (depending on the constants from Assumption 1 and on R and the size of Ω) such that

$$E(z(t)) \leq C_1 e^{-\beta t} E(z(0)) + C_2 \int_0^t e^{-\beta(t-s)} \|z(s)\|_{L_2(\Omega)}^2 ds \quad \text{for all } t \geq 0. \quad (3.4)$$

Since the global attractor \mathcal{A} is an invariant set we know that the solutions $(u(t), u'(t))$ and $(v(t), v'(t))$ corresponding to the initial data $(u_0, u_1), (v_0, v_1) \in \mathcal{A}$ will stay in \mathcal{A} . Moreover, relation (3.3) holds with R which is equal to the radius of the absorbing ball. Therefore (3.4) is true for this case with C_1, C_2 and β depending on the constants from Assumption 1. Thus, (3.4) implies the following corollary:

Corollary 3.4 *Under the assumptions of Lemma 3.3 we have*

$$E(z(t)) \leq C_1 e^{-\beta t} E(z(0)) + \frac{C_2}{\beta} \max_{s \in [0, t]} \|z(s)\|_{L_2(\Omega)}^2 \quad \text{for all } t \geq 0, \quad (3.5)$$

where $z(t) = u(t) - v(t)$ and the solutions $(u(t), u'(t))$ and $(v(t), v'(t))$ belong to the attractor.

To prove Lemma 3.3 we need the following observability inequality.

Lemma 3.5 Observability inequality. *Let $g'(0) > 0$ and $T > T_0 \equiv 2\left(r + \frac{1}{\sqrt{\lambda}}\right)$, where r is the radius of a minimal ball in \mathbf{R}^3 containing Ω and λ is the constant from (f-2). Assume that two solutions $u(t)$ and $v(t)$ to problem (1.1)–(1.3) possess the property*

$$\|(u(t), u'(t))\|_{\mathcal{H}} \leq R \quad \text{and} \quad \|(v(t), v'(t))\|_{\mathcal{H}} \leq R \quad \text{for all } t \in [0, T]. \quad (3.6)$$

with some constant $R > 0$. Then there exist positive constants $C_1(T)$ and $C_2(R, T)$ (depending also on the constants from Assumption 1 and the size of Ω) such that for $z(t) \equiv u(t) - v(t)$ we have the relations

$$E(z(T)) + \int_0^T E(z(t)) dt \leq C_1 \int_0^T \langle g(u'(t)) - g(v'(t)), z'(t) \rangle dt + C_2 \int_0^T \|z(t)\|_{L_2(\Omega)}^2 dt \quad (3.7)$$

and

$$E(z(T)) \leq C_1 (E(z(0)) - E(z(T))) + C_2 \int_0^T \|z(t)\|_{L_2(\Omega)}^2 dt. \quad (3.8)$$

Remark 3.1 Note that inequality (3.7) asserts reconstruction of energy, in terms of boundary observations - modulo lower order terms. It should be noted that this type of observability inequality is reminiscent of inequalities governing boundary controllability and stabilizability theory of unforced dissipative wave equation and goes back to [La1] (see also [La2, La-Li, L-Ta]). In our case this inequality needs to be established for difference of two solutions which leads to an analysis of a non-dissipative system.

Proof: In the calculations below different constants will appear. We will denote them by C_i pointing out their dependence on the parameters when it becomes important. Since the case $n = 2$ is easier to treat (because of Sobolev's embedding $H^1(\Omega) \subset L_p(\Omega)$, $1 \leq p < \infty$), we shall provide the details of the arguments for $n = 3$ only. We also note that under the condition $g'(0) > 0$ without loss of generality we can assume that

$$m_1 \leq g'(s) \leq m_2 \quad \text{for all } s \in \mathbf{R}. \quad (3.9)$$

Our starting point is the following multiplier identity (see [CEL1, formula (3.4)]):

$$\begin{aligned} \frac{1}{2} \int_{Q_T} (|z'|^2 + |\nabla z|^2) &= - \int_{\Omega} z'(h \cdot \nabla z + z)|_0^T - \int_{Q_T} (f(u) - f(v))(h \cdot \nabla z + z) \\ &\quad + \int_{\Sigma_T} \left[\partial_\nu z (h \cdot \nabla z + z) + \frac{1}{2} (h \cdot \nu) (|z'|^2 - |\nabla z|^2) \right], \end{aligned} \quad (3.10)$$

where $h(x) = x - x_0$ for some $x_0 \in \mathbf{R}^3$ and $Q_T = (0, T) \times \Omega$ and $\Sigma_T = (0, T) \times \Gamma$. Here T is a positive constant that will be determined later in the proof. Below we choose $x_0 \in \mathbf{R}^3$ such that $\sup_{x \in \Omega} |h(x)| = r$, where r is the radius of a minimal ball in \mathbf{R}^3 containing Ω .

The identity (3.10) will be transformed to an energy inequality. At first we estimate the integral containing the nonlinear function f . By using Sobolev's and Young's inequalities one obtains:

$$\|f(u(t)) - f(v(t))\|_{L_2(\Omega)} \leq C(R, \Omega, c, \delta) \cdot \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}, \quad t \in [0, T]. \quad (3.11)$$

Here and below $\delta = 1 - p$, where $p \in (0, 1)$ is the exponent from (f-1) for $n = 3$.

With $r = \sup_{x \in \Omega} |h(x)|$ and using (3.11) we have that there exists a constant C depending on $r, R, \varepsilon, \Omega, c$ and δ such that

$$- \int_{Q_T} (f(u) - f(v)) h \cdot \nabla z \leq C \int_0^T \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}^2 dt + \varepsilon \|\nabla z\|_{L_2(Q_T)}^2$$

for every $\varepsilon > 0$. It is also easy to see that

$$\left| \int_{\Omega} z'(h \cdot \nabla z + z) \right| \leq \left(r + \frac{1}{\sqrt{\lambda}} \right) E(z(t)),$$

where $r = \sup_{x \in \Omega} |h(x)|$ is the radius of a minimal ball in \mathbf{R}^3 containing Ω and λ is the constant from (f-2).

Hence, formula (3.10) and the definition of the linear energy functional (3.1) yield

$$\begin{aligned} &\int_0^T E(z(t)) dt \\ &\leq C_1(r) \left(\|z'\|_{L_2(\Sigma_T)}^2 + \|\nabla z\|_{L_2(\Sigma_T)}^2 + \|z\|_{L_2(\Sigma_T)}^2 \right) + \left(r + \frac{1}{\sqrt{\lambda}} \right) [E(z(T)) + E(z(0))] \\ &\quad + C_2(r, R, \varepsilon) \int_0^T \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}^2 dt + \varepsilon \int_0^T \|\nabla z(t)\|_{L_2(\Omega)}^2 dt. \end{aligned} \quad (3.12)$$

Multiplying the differential equation for z by z' and integration by parts results (after some calculations) in the following string of inequalities.

$$E(z(t)) \leq E(z(s)) \cdot e^{\alpha_R(t-s)} \quad \text{for all } 0 \leq s \leq t, \quad (3.13)$$

where the constant $a_R > 0$ also depends on Ω , c and δ .

$$\begin{aligned} E(z(s)) &\leq E(z(t)) + m_2 \int_s^t \int_{\Gamma} |z'(\tau)|^2 d\tau \\ &\quad + \int_s^t \left(\varepsilon \|z'(\tau)\|_{L_2(\Omega)}^2 + C(\varepsilon, R) \|z(\tau)\|_{L_{6/(1+\delta)}(\Omega)}^2 \right) d\tau \end{aligned} \quad (3.14)$$

for any $\varepsilon > 0$ and for all $0 \leq s \leq t$. In a similar way we obtain

$$\begin{aligned} &\int_s^t \langle g(u'(\tau)) - g(v'(\tau)), z'(\tau) \rangle d\tau \\ &\leq E(z(s)) - E(z(t)) + \int_s^t \left(\varepsilon \|z'(\tau)\|_{L_2(\Omega)}^2 + C(\varepsilon, R) \|z(\tau)\|_{L_{6/(1+\delta)}(\Omega)}^2 \right) d\tau \end{aligned} \quad (3.15)$$

for any $\varepsilon > 0$ and $0 \leq s \leq t$. We will use all these inequalities below.

To continue with (3.12) we estimate the tangential derivative $\nabla_{\tau} z$ on Σ_T relying on [L-T, Lemma 7.2]. This lemma states that for $0 < \alpha < T/2$ and $\eta \in (0, 1/2)$ there exists a constant $C = C(\alpha, \eta, T, \Omega)$ such that

$$\begin{aligned} &\int_{\alpha}^{T-\alpha} \int_{\Gamma} |\nabla_{\tau} z(t)|^2 dt \\ &\leq C \left(\|\partial_{\nu} z\|_{L_2(\Sigma_T)}^2 + \|z'\|_{L_2(\Sigma_T)}^2 + \|z\|_{H^{1/2+\eta}(Q_T)}^2 + \|f(u) - f(v)\|_{H^{-1/2+\eta}(Q_T)}^2 \right). \end{aligned} \quad (3.16)$$

By (3.11) the last term on the right hand side can be estimated in the following way

$$\|f(u) - f(v)\|_{H^{-1/2+\eta}(Q_T)}^2 \leq \|f(u) - f(v)\|_{L_2(Q_T)}^2 \leq C(R) \int_0^T \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}^2 dt. \quad (3.17)$$

Combining the above inequalities and accounting for the contribution of integration on $[0, \alpha] \cup [T - \alpha, T]$ gives

$$\begin{aligned} \int_0^T E(z(t)) dt &\leq C_1 \left\{ \|z'\|_{L_2(\Sigma_T)}^2 + \|\partial_{\nu} z\|_{L_2(\Sigma_T)}^2 + \|z\|_{L_2(\Sigma_T)}^2 \right\} \\ &\quad + c_0 E(z(T)) + C_2(\varepsilon, R) \left\{ \int_0^T \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}^2 dt + \|z\|_{H^{1/2+\eta}(Q_T)}^2 \right\} \\ &\quad + \varepsilon \int_0^T \left(\|z'(t)\|_{L_2(\Omega)}^2 + \|\nabla z(t)\|_{L_2(\Omega)}^2 \right) dt \end{aligned} \quad (3.18)$$

where $c_0 = 2 \left(\alpha + r + \frac{1}{\sqrt{\lambda}} \right)$. Therefore for any $T > T_0 \equiv 2 \left(r + \frac{1}{\sqrt{\lambda}} \right)$ we can choose

appropriate α and obtain an estimate of the form

$$\begin{aligned} E(z(T)) + \int_0^T E(z(t))dt & \\ & \leq C_1(T) \left\{ \|z'\|_{L_2(\Sigma_T)}^2 + \|\partial_\nu z\|_{L_2(\Sigma_T)}^2 + \|z\|_{L_2(\Sigma_T)}^2 \right\} \\ & \quad + C_2(R, T) \left\{ \int_0^T \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}^2 dt + \|z\|_{H^{1/2+\eta}(Q_T)}^2 \right\}. \end{aligned} \quad (3.19)$$

To estimate the boundary terms we use (3.9) to obtain that

$$\begin{aligned} \|z'\|_{L_2(\Sigma_T)}^2 + \|\partial_\nu z\|_{L_2(\Sigma_T)}^2 & \leq (1 + 2m_2^2)\|z'\|_{L_2(\Sigma_T)}^2 + 2\|z\|_{L_2(\Sigma_T)}^2 \\ & \leq \frac{1 + 2m_2^2}{m_1} \int_0^T \langle g(u'(t)) - g(v'(t)), z'(t) \rangle dt + 2\|z\|_{L_2(\Sigma_T)}^2. \end{aligned}$$

This last operation in connection with (3.19) yields

$$E(z(T)) + \int_0^T E(z(t))dt \leq C_1 \int_0^T \langle g(u'(t)) - g(v'(t)), z'(t) \rangle dt + C_2 \text{l.o.t.}(z), \quad (3.20)$$

where $C_1 = C_1(m_1, m_2, T)$ and $C_2 = C_2(R, T)$ and $\text{l.o.t.}(z)$ is an abbreviation for a collection of lower order terms, i.e.

$$\text{l.o.t.}(z) = \int_0^T \|z(t)\|_{L_{6/(1+\delta)}(\Omega)}^2 dt + \|z\|_{H^{1/2+\eta}(Q_T)}^2 + \|z\|_{L_2(\Sigma_T)}^2.$$

From (3.15) and (3.20) we also have

$$E(z(T)) + \int_0^T E(z(t))dt \leq C_1 (E(z(0)) - E(z(T))) + C_2 \text{l.o.t.}(z). \quad (3.21)$$

After estimating $\text{l.o.t.}(z)$ we arrive to the following inequality:

$$\text{l.o.t.}(z) \leq \varepsilon \int_0^T E(z(t))dt + C_\varepsilon \int_0^T \|z(t)\|_{L_2(\Omega)}^2 dt$$

with arbitrary $\varepsilon > 0$. Hence, (3.20) and (3.21) imply the desired relations (3.7) and (3.8). \triangle

Proof of Lemma 3.3. Under the hypotheses of Lemma 3.3 it follows from (3.8) that

$$E(z(nT)) \leq \frac{C_1}{1 + C_1} E(z((n-1)T)) + \frac{C_2}{1 + C_1} \int_{(n-1)T}^{nT} \|z(t)\|_{L_2(\Omega)}^2 dt, \quad n = 1, 2, \dots, \quad (3.22)$$

for fixed $T > T_0$. Setting $\gamma = C_1/(1 + C_1)$ one can show by induction that

$$E(z(nT)) \leq \gamma^n E(z(0)) + \frac{C_2}{1 + C_1} \sum_{k=1}^n \gamma^{n-k} \int_{(k-1)T}^{kT} \|z(t)\|_{L_2(\Omega)}^2 dt. \quad (3.23)$$

for all positive integers n . From (3.13) we have that

$$E(z(t)) \leq E(z(nT)) \cdot e^{aRt} \quad \text{for all } nT \leq t \leq (n+1)T, \quad n = 0, 1, 2, \dots \quad (3.24)$$

Let $\beta = \frac{1}{T} \ln \frac{1}{\gamma}$. Since

$$\gamma^{n-k} = \frac{1}{\gamma^2} \exp\{-\beta((n+1)T - (k-1)T)\} \leq \frac{1}{\gamma^2} \exp\{-\beta(t - \tau)\}$$

for $t \leq (n+1)T$ and $\tau \geq (k-1)T$, the desired relation (3.4) follows from (3.23) and (3.24).

3.2. COMPLETION OF THE PROOF OF THEOREM 2.7. The proof of the main theorem follows by combining the abstract result of Theorem 3.2 with the stabilizability inequality (3.5) of Corollary 3.4. As in [P] (see also [MaPr]) it is convenient to use "pieces" of trajectories for the construction of the phase space X . The details of the argument are given below.

We will apply Theorem 3.2 in the space $X = \mathcal{H} \times H^1(Q_T)$ equipped with the norm

$$\|U\|_X^2 = \|(u_0, u_1)\|_{\mathcal{H}}^2 + 2 \int_0^T E(v(t))dt, \quad \text{where } U = (u_0, u_1, v).$$

Here $T > 0$ is a constant to be determined later. On the space X we define a seminorm

$$n_T(U) := \max_{0 \leq t \leq T} \|v(t)\|_{L_2(\Omega)}$$

By the compactness of the imbedding [Si, Corollary 9]

$$H^1(Q_T) = L_2(0, T; H^1(\Omega)) \cap H^1(0, T; L_2(\Omega)) \subset C([0, T]; L_2(\Omega))$$

we obtain that $n_T(U)$ is a compact seminorm on X . Next we define the set \mathcal{A} and the map V appearing in Theorem 3.2. Consider in the space X the set

$$\mathcal{A}_T = \{U \equiv (u_0, u_1, u(t)) \text{ for } t \in [0, T] : (u_0, u_1) \in \mathcal{A}\},$$

where $u(t)$ is the solution to (1.1)-(1.3) with initial data $u(0) = u_0$, $u'(0) = u_1$ and \mathcal{A} is the attractor. The operator $V_T : \mathcal{A}_T \mapsto X$ is now defined by the formula

$$V_T : (u_0, u_1, u(t)) \mapsto (u(T), u'(T), u(T+t)) = (S(T)(u_0, u_1), u(T+t)).$$

We shall verify that all conditions of Theorem 3.2 are satisfied. For (i), this follows from the invariance property of the attractor \mathcal{A} which is equivalent to $V_T \mathcal{A}_T = \mathcal{A}_T$. As for (ii), V_T is Lipschitz continuous on \mathcal{A}_T . In order to prove this statement we will work with two solutions $u(t)$ and $v(t)$ to the original problem (1.1)-(1.3). We set $U_1 = (u_0, u_1, u(t))$, $U_2 = (v_0, v_1, v(t))$ and $z(t) = u(t) - v(t)$ and observe that

$$\begin{aligned} \frac{1}{2} \|U_1 - U_2\|_X^2 &= E(z(0)) + \int_0^T E(z(t))dt \quad \text{and} \\ \frac{1}{2} \|V_T U_1 - V_T U_2\|_X^2 &= E(z(T)) + \int_T^{2T} E(z(t))dt. \end{aligned} \quad (3.25)$$

Setting $t = T + s$ in (3.13) and integrating over the interval $[0, T]$ results in

$$\int_T^{2T} E(z(s)) ds \leq e^{a_R T} \int_0^T E(z(s)) ds.$$

When we combine this last inequality and formula (3.13) with $t = T$ and $s = 0$ we obtain the Lipschitz property of V_T with $L = e^{a_R T/2}$.

Thus, it remains to verify condition (iii) in Theorem 3.2. Integrating estimate (3.5) in t from T to $2T$ yields

$$\int_T^{2T} E(z(t)) dt \leq C_1 e^{-\beta T} E(z(0)) + C_2 T \max_{0 \leq \tau \leq 2T} \|z(\tau)\|_{L_2(\Omega)}^2, \quad (3.26)$$

where C_1 and C_2 do not depend on T . Therefore using (3.5) with $t = T$ we obtain that

$$E(z(T)) + \int_T^{2T} E(z(t)) dt \leq C_1 e^{-\beta T} E(z(0)) + C_2 \max_{0 \leq \tau \leq 2T} \|z(\tau)\|_{L_2(\Omega)}^2 \quad (3.27)$$

with C_1 independent of T . Since

$$\max_{0 \leq \tau \leq 2T} \|z(\tau)\|_{L_2(\Omega)}^2 \leq \max_{0 \leq \tau \leq T} \|z(\tau)\|_{L_2(\Omega)}^2 + \max_{0 \leq \tau \leq T} \|z(T + \tau)\|_{L_2(\Omega)}^2,$$

accounting for the definitions of V_T and the norms in X (see (3.25)), relation (3.27) can be written in the form

$$\|V_T U_1 - V_T U_2\|_X \leq \eta_T \|U_1 - U_2\|_X + K \cdot [n_T(U_1 - U_2) + n_T(V_T U_1 - V_T U_2)]$$

for all $U_1, U_2 \in \mathcal{A}_T$, where $\eta_T = C_1 e^{-\beta T}$. We can select T large enough such that $\eta_T < 1$.

Hence, all the assumptions of Theorem 3.2 are satisfied with $\eta(s) = \eta_T \cdot s$. It implies that \mathcal{A}_T is a compact set in X of finite fractal dimension.

Let $\mathcal{P} : X \rightarrow \mathcal{H}$ be the operator defined by the formula

$$\mathcal{P} : (u_0, u_1, v(t)) \rightarrow (u_0, u_1).$$

Since $\mathcal{A} = \mathcal{P}\mathcal{A}_T$ and \mathcal{P} is obviously Lipschitz continuous, we have that

$$\dim_{frac}^{\mathcal{H}} \mathcal{A} = \dim_{frac}^X \mathcal{A}_T < \infty.$$

Here \dim_{frac}^Y stands for fractal dimension of a set in the space Y .

3.3. PROOF OF THEOREM 2.8-SKETCH. We already know from Corollary 2.5 that for any $W_0 = (w_0, w_1) \in \mathcal{H}$ there exists an equilibrium point $V = (v, 0) \in \mathcal{N}$ such that

$$W(t) = S(t)W_0 \rightarrow V, \quad t \rightarrow \infty \quad (3.28)$$

where the convergence is in the strong topology of \mathcal{H} . Our goal is to show that the above convergence is at an exponential rate. Consider a new variable

$$Z(t) = (z(t), z'(t)) \equiv W(t) - V = (w(t) - v, w'(t))$$

From (3.28) we infer that for given initial condition $W_0 \in \mathcal{H}$ and any positive constant $\varepsilon > 0$ there exists $T_0 > 0$ such that for all $T > T_0$

$$\int_{T-1}^T E(w(t) - v) dt = \int_{T-1}^T E(z(t)) dt \leq \varepsilon. \quad (3.29)$$

In what follows we shall take ε sufficiently small, so the only equilibrium in the ε neighborhood is precisely V . The above is possible due to the assumption of finiteness of set of equilibria. By the definition of point of equilibrium that new variable $Z(t) = (z(t), z'(t))$ satisfies the equation

$$\begin{aligned} z'' - \Delta z + f(z + v) - f(v) &= 0 \text{ in } Q \\ \partial_\nu z + z &= -g(z') \text{ on } \Sigma \end{aligned} \quad (3.30)$$

The key for the method is the following (rather atypical) energy functional

$$\mathcal{E}(z(t)) \equiv E(z(t)) + \int_{\Omega} (F(w(t)) - F(v)) - \int_{\Omega} f(v)z(t),$$

where $F(s) = \int_0^s f(\tau) d\tau$. We have the following energy type relation.

Lemma 3.6 *Let z be any finite energy solution of (3.30). Then for any $s \leq t$ we have*

$$\mathcal{E}(z(t)) + \int_s^t \langle g(z'(\tau)), z'(\tau) \rangle d\tau = \mathcal{E}(z(s)).$$

Proof: The proof is standard but requires some calculations. We multiply both sides of equation (3.30) by z' and we integrate by parts. Computations are first performed for strong solutions and then extended to all weak solution. \triangle

Proposition 3.1 *The energy functional $\mathcal{E}(z(t))$ has the following properties:*

- $\mathcal{E}(z(t))$ is non-increasing.
- $\mathcal{E}(z(t)) \geq 0$ for all $t \geq 0$.
- If $\|Z(t)\|_{\mathcal{H}}^2 = 2E(z(t)) \leq 2R^2$ for $t \in [0, T]$, then

$$|E(z(t)) - \mathcal{E}(z(t))| \leq \varepsilon \|z(t)\|_{H^1(\Omega)}^2 + C(\varepsilon, R) \|z(t)\|_{L_2(\Omega)}^2, \quad t \in [0, T], \quad (3.31)$$

$$E(z(t)) \leq 2\mathcal{E}(z(t)) + C(R) \|z(t)\|_{L_2(\Omega)}^2, \quad t \in [0, T], \quad (3.32)$$

$$\mathcal{E}(z(t)) \leq 2E(z(t)) + C(R) \|z(t)\|_{L_2(\Omega)}^2, \quad t \in [0, T]. \quad (3.33)$$

Proof: The first assertion is a direct consequence of Lemma 3.6.

Since $Z(t) \rightarrow 0$ in \mathcal{H} by (3.28), the second assertion follows from the fact that $\mathcal{E}(z(t))$ is non-increasing.

The third assertion follows from the mean value theorem in integral form, the Cauchy-Schwarz inequality and the continuous imbedding $H^s(\Omega) \subset L_{6/(3-2s)}(\Omega)$ via the computation

$$\begin{aligned} |E(z(t)) - \mathcal{E}(z(t))| &\leq \int_{\Omega} |F(w(t)) - F(v) - F'(v)z(t)| \\ &= \int_{\Omega} \left| \int_0^1 \int_0^1 f'(v + \tau sz(t)) d\tau ds \right| |z(t)|^2 \\ &\leq C \left(1 + \|v\|_{L_{4-2\delta}(\Omega)}^2 + \|z(t)\|_{L_{4-2\delta}(\Omega)}^2 \right) \|z(t)\|_{L_4(\Omega)}^2 \leq C(R) \|z(t)\|_{H^{3/4}(\Omega)}^2. \end{aligned}$$

In order to obtain the desired estimate (3.31) one needs to use interpolation in Sobolev spaces. Finally, relations (3.32) and (3.33) are direct consequences of (3.31) and the definition of the energy $\mathcal{E}(z(t))$. \triangle The key ingredient of our proof is the following observability inequality for the equation (3.30).

Lemma 3.7 *Let z be a solution to (3.30) and such that $\sup_{t \in [0, T]} E(z(t)) \leq R^2$. Then for any $T > T_0 = 2 \left(r + \frac{1}{\sqrt{\lambda}} \right)$ (cf. Lemma 3.5) we have*

$$\mathcal{E}(T) + \int_0^T E(z(t)) dt \leq C_1(T) \int_0^T \langle g(z'(t)), z'(t) \rangle dt + C_2(T, R) \int_0^T \|z(\tau)\|_{L_2(\Omega)}^2 d\tau.$$

Proof: This inequality follows from Proposition 3.1 and relation (3.7) in Lemma 3.5 applied to solutions $u(t) = w(t)$ and $v(t) \equiv v$ of problem (1.1). \triangle We are ready to complete the proof of Theorem.

Lemma 3.8 *Let z be a solution to (3.30) and such that $\sup_{t \in [0, T]} E(z(t)) \leq R^2$. Moreover, we assume that (3.29) holds for a preassigned small ε . Then, there exists a positive constant ε_0 such that*

$$\max_{0 \leq t \leq T} \|z(t)\|_{L_2(\Omega)}^2 \leq C(R, T, \varepsilon) \int_0^T \langle g(z'(t)), z'(t) \rangle dt$$

provided $\varepsilon \leq \varepsilon_0$ and $T > T_0$, where T_0 is the same as in Lemma 3.5 and Lemma 3.7.

Proof: The proof of this Lemma is carried out via a contradiction argument. Contradiction argument along with the unique continuation property [Ru] and compactness theorems (see, e.g., [Si]) became a standard way of dispensing with lower order terms in observability estimates. However, in our context there are two new features of that argument. We must take advantage of the fact that equilibria are isolated, hence they are locally unique. The second new feature is critical use of hyperbolicity of equilibrium. For these reasons our argument is no longer

standard and can not be simply referred to the literature. The details are lengthy and provided in [CEL2]. \triangle

Now we are in position to complete the proof of Theorem 2.8.

Since the system $(S(t), \mathcal{H})$ is dissipative, we have that $\|(z(t), z'(t))\|_{\mathcal{H}} \leq R$ for all $t > 0$ and for some $R > 0$. We choose T such that (3.29) holds with $\varepsilon \leq \varepsilon_0$, where ε_0 is given in Lemma 3.8, and apply Lemma 3.7 and Lemma 3.8.

By combining the observability inequality in Lemma 3.7 with the inequality from Lemma 3.8 we obtain that

$$\mathcal{E}(z(T)) + \int_0^T E(z(t))dt \leq C \int_0^T \langle g(z'(t)), z'(t) \rangle dt. \quad (3.34)$$

Combining this with the energy identity in Lemma 3.6 yields

$$\mathcal{E}(z(T)) \leq C[\mathcal{E}(z(0)) - \mathcal{E}(z(T))].$$

Hence $\mathcal{E}(z(T)) \leq \gamma \mathcal{E}(z(0))$ for some $\gamma < 1$. Reiterating the same argument over the intervals $[mT, (m+1)T]$ gives

$$\mathcal{E}(z(mT)) \leq \gamma^m \mathcal{E}(z(0)), \quad m = 0, 1, 2, \dots$$

which implies exponential decay for the energy $\mathcal{E}(t)$. The same remains true for the linear energy functional. Indeed, from Proposition 3.1 and Lemma 3.8 we have

$$\begin{aligned} E(z(mT)) &\leq 2\mathcal{E}(z(mT)) + C(R)\|z(mT)\|_{L_2(\Omega)}^2 \\ &\leq 2\mathcal{E}(z(mT)) + C(R) \int_{mT}^{(m+1)T} \langle g(z'(t)), z'(t) \rangle dt \\ &\leq 2\mathcal{E}(z(mT)) + C(R)[\mathcal{E}(z(mT)) - \mathcal{E}(z((m+1)T))] \\ &\leq 2\mathcal{E}(z(mT)) + C(R)\mathcal{E}(z(mT)) \leq (2 + C(R))\gamma^m \mathcal{E}(z(0)) \\ &\leq C\gamma^m E(z(0)). \end{aligned}$$

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