Bol. Soc. Paran. Mat.
(3s.) **v. 22** 1 (2004): 57–65.
©SPM

Some results about positive solutions of a nonlinear equation with a weighted Laplacian

Marta García-Huidobro & Raúl Manásevich & Cecilia S. Yarur*

Contents

1. Introduction

We consider the problem of classification of bounded positive solutions to

$$(P) \qquad -\nabla\cdot(A(|x|)\nabla u) = B(|x|)|u|^{q-2}u, \quad x\in\mathbb{R}^n.$$

Here $q>2$, and $A,\ B$ are weight functions, i.e., a.e. positive measurable functions. Many authors have dealt with the non weighted case, i.e., with positive solutions to the equation

$$(E) \qquad -\Delta u = |u|^{q-2}u, \quad x\in\mathbb{R}^n,$$

where $q>2$, see for instance [4].

In this case, when $n>2$, the critical number

$$2^* = \frac{2n}{n-2}$$

appears, and it is known that

if $1<q<2^*$, *all bounded solutions have a first positive zero,*

and if $q\geq 2^*$, *then the solutions are positive in* $(0,\infty)$.

More recently, in 1993, the case of (E) with a weight in the right hand side, $B(r)=\frac{1}{1+r^\gamma}$, $\gamma>0$, that is the Matukuma equation, was studied by Ni-Yotsutani [10], Li-Ni [7], [8], [9], and Kawano-Yanagida-Yotsutani [5], where the problem

* This research was supported by FONDECYT-1030593 for the first author, Fondap Matemáticas Aplicadas for the second author and FONDECYT-1030666 for the third author.

$$-(r^{n-1}u')' = \frac{r^{n-1}}{1+r^\gamma}(u^+)^{q-1} \\ u(0) = \alpha > 0 \tag{1.1}$$

is studied. The following result is due to Kawano-Yanagida-Yotsutani, [5], 1993:
Theorem A. Let $\gamma > 0$ and $n > 2$. Then

(i) If $2 < q \le \max\{2, \frac{2(n-\gamma)}{n-2}\}$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.1) has a first positive zero in $(0, \infty)$.

(ii) If $q \ge \frac{2n}{n-2}$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.1) is positive in $(0, \infty)$ and $\lim_{r\to\infty} r^{n-2}u(r, \alpha) = \infty$.

(iii) If $\max\{p, \frac{2(n-\gamma)}{n-2}\} < q < \frac{2n}{n-2}$, then there exists a unique $\alpha^* > 0$ such that the solution $u(\cdot, \alpha)$ of (1.1) satisfies

- $u(r, \alpha) > 0$ for all $r > 0$ with $\lim_{r\to\infty} r^{n-2}u(r, \alpha) = \infty$ whenever $\alpha \in (0, \alpha^*)$.
- $u(r, \alpha^*) > 0$ for all $r > 0$ with $\lim_{r\to\infty} r^{n-2}u(r, \alpha^*) = \ell \in (0, \infty)$.
- $u(\cdot, \alpha)$ has a first zero for any $\alpha \in (\alpha^*, \infty)$.

Later, in 1995, Yanagida and Yotsutani [11] considered the case of a more general weight in the right hand side, and they studied the problem

$$-(r^{n-1}u')' = r^{n-1}K(r)(u^+)^{q-1} \\ u(0) = \alpha > 0, \tag{1.2}$$

for K satisfying

(K_1) $$K \in C^1(0, \infty),\ K > 0,\ rK(r) \in L^1(0, 1),$$

(K_2) $$\frac{rK'(r)}{K(r)} \quad \text{decreasing and nonconstant in } (0, \infty).$$

They defined the critical numbers $-\infty \le \ell < \sigma \le \infty$

$$\sigma := \lim_{r\to 0} \frac{rK'(r)}{K(r)}, \quad \ell := \lim_{r\to\infty} \frac{rK'(r)}{K(r)}, \quad \sigma > -2,\ \sigma > \ell.$$

From (K_1) $\sigma > -2$, and then they set

$$q_\sigma := \frac{2(n+\sigma)}{n-2}, \quad q_\ell := \max\{2, \frac{2(n+\ell)}{n-2}\},$$

and proved the following:
Theorem B. Let $n > 2$ and assume that the weight K satisfies (K_1) and (K_2). Then

(i) If $2 < q \leq q_\ell$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.2) has a first positive zero in $(0, \infty)$.

(ii) If $q \geq q_\sigma$, then for any $\alpha > 0$, the solution $u(\cdot, \alpha)$ of (1.2) is positive in $(0, \infty)$ and $\lim_{r\to\infty} r^{n-2}u(r, \alpha) = \infty$.

(iii) If $q_\ell < q < q_\sigma$, then there exists a unique $\alpha^* > 0$ such that the solution $u(\cdot, \alpha)$ of (1.2) satisfies

- $u(r, \alpha) > 0$ for all $r > 0$ with $\lim_{r\to\infty} r^{n-2}u(r, \alpha) = \infty$ whenever $\alpha \in (0, \alpha^*)$.
- $u(r, \alpha) > 0$ for all $r > 0$ with $\lim_{r\to\infty} r^{n-2}u(r, \alpha) = \ell \in (0, \infty)$ whenever $\alpha = \alpha^*$.
- $u(\cdot, \alpha)$ has a first zero for any $\alpha \in (\alpha^*, \infty)$.

Clearly, the result in Theorem A is a particular case of that of Theorem B, since $K(r) = \frac{1}{1+r^\gamma}$ satisfies all the assumptions with $\sigma = 0$ and $\ell = -\gamma$.

We will deal here with the case $A = B$ in (P) when the solutions are radially symmetric:

$$(P_r) \qquad \begin{cases} -(b(r)u')' = b(r)|u|^{q-2}u(r), \quad r \in (0, \infty), \\ \lim_{r\to 0} b(r)u'(r) = 0, \end{cases}$$

where $|x| = r$ and now the function $b(r) := r^{N-1}B(r)$ is a positive function satisfying some regularity and growth conditions. We will see in section 3 that under some extra assumption on the weight K in (1.2), the problem considered in [11] is a particular case of ours.

Since we are interested only in positive solutions, we will study the initial value problem

$$(IVP) \qquad \begin{cases} -(b(r)u')' = b(r)(u^+)^{q-1}, \quad r \in (0, \infty), \\ u(0) = \alpha > 0, \quad \lim_{r\to 0} b(r)u'(r) = 0. \end{cases}$$

Our note is organized as follows: in section 2 we will introduce some necessary conditions to deal with with our problem and we will state our main results which are a particular case of the work in [2]. Finally, in section 3 we compare our result with the one given in Theorem B.

2. Main results

We introduce next some necessary assumptions to deal with (IVP). We note that if u is a solution to our problem, then

$$-b(r)u'(r) = \int_0^r b(s)(u^+)^{q-1}(s)ds \ > \ 0$$

for all $r > 0$, and thus $u'(r) < 0$ for all $r > 0$. If for some positive R it happens that $u(R) = 0$, $u(r) > 0$ for $r \in (0, R)$, then for all $r \geq R$ and such that $u(r) \leq 0$, we have that

$$|u'(r)| = (b(r))^{-1} \int_0^R b(s)(u^+)^{q-1}(s)ds$$

and thus

$$u(r) = -C \int_R^r (b(\tau))^{-1} d\tau < 0 \quad \text{for some positive constant } C.$$

implying that u remains negative for all $r \geq R$. If on the contrary it holds that $u(r) > 0$ for all $r > 0$, then

$$|u'(r)| = (b(r))^{-1} \int_0^r b(s)(u^+)^{q-1}(s)ds,$$

and thus, for $r \geq s$ we have

$$|u'(r)| \geq (b(r))^{-1} \int_0^s b(\tau)(u^+)^{q-1}(\tau)d\tau,$$

implying that

$$u(s) \geq \Big(\int_0^s b(\tau)(u^+)^{q-1}(\tau)d\tau\Big) \int_s^r (b(\tau))^{-1} d\tau,$$

and we conclude that $1/b \in L^1(s, \infty)$ for all $s > 0$. Putting it in another way, if $1/b \notin L^1(1, \infty)$, then u must have a first positive zero. Therefore, keeping in mind that we are interested in the positive solutions to (P_r), there is no loss of generality in assuming that $1/b \in L^1(s, \infty)$ for all $s > 0$.

Moreover, if u is any solution to our problem, then for $r \geq s$ small enough it holds that

$$\frac{b|u'|(r) - b|u'|(s)}{(u^+)^{q-1}(r)} \geq \int_s^r b(\tau)d\tau,$$

and thus

$$b \in L^1(0, 1)$$

is a necessary condition for the existence of solutions to (IVP). Finally, it can be shown that

$$\Big(\int_0^r b(\tau)d\tau\Big)(1/b) \in L^1(0, 1)$$

is necessary and sufficient for the existence and uniqueness of solutions to (IVP). Hence, our basic assumptions on the weight b will be:

(H_1) $$b \in C^1(\mathbb{R}^+, \mathbb{R}^+), (\mathbb{R}^+ = (0, \infty))$$

(H_2) $$b \in L^1(0, 1), \qquad 1/b \in L^1(1, \infty)$$

$$\beta(r) := \int_0^r b(s)ds, \qquad h(r) = \int_r^\infty (b(s))^{-1}ds,$$

$$(H_3) \qquad (\beta/b) \in L^1(0,1).$$

By a solution to (IVP) we understand an absolutely continuous function u defined in the interval $[0,\infty)$ such that $b(r)u'$ is also absolutely continuous in the open interval $(0,\infty)$ and satisfies the equation in (IVP).

We will show that the behavior of function

$$r \mapsto B_q(r) := \beta(r)h^{q/2}(r), \tag{2.1}$$

is crucial in the study of solutions to (IVP). This function played a key role when studying the problem of existence of positive solutions to the corresponding Dirichlet problem associated to our equation, see [1]. The behavior at 0 of this function is closely related to the inclusion

$$V_0^2(b) \hookrightarrow L^q(b)$$

of weighted Sobolev spaces. For a proper definition of these spaces we refer to Kufner-Opic [6]. Now also the behavior at ∞ of this function will be crucial for our classification results. Let us define

$$\mathcal{U} := \{s \geq 2 \mid \sup_{0<r<1} B_s(r) < \infty\}, \quad \mathcal{W} := \{s \geq 2 \mid \sup_{1 \leq r < \infty} B_s(r) < \infty\},$$

and put

$$\rho_0 = \sup \mathcal{U}, \quad \rho_\infty = \inf \mathcal{W}, \tag{2.2}$$

where we set $\rho_\infty = \infty$ if $\mathcal{W} = \emptyset$. It can be proved that condition (H_3) implies that $2 \in \mathcal{U}$ and thus $\mathcal{U} \neq \emptyset$. Observe that

$$[2, \rho_0) \subseteq \mathcal{U}, \quad (\rho_\infty, \infty) \subseteq \mathcal{W}.$$

We will prove in section 2 that these critical numbers can be computed as

$$\rho_0 = \max\Big\{2, 2\liminf_{r\to 0} \frac{|\log(\beta(r))|}{|\log(h(r))|}\Big\}, \quad \rho_\infty = \max\Big\{2, 2\limsup_{r\to\infty} \frac{|\log(\beta(r))|}{|\log(h(r))|}\Big\}.$$

We will denote the unique solution to (IVP) by $u(r,\alpha)$. As it is standard in the literature, we will say that

- $u(r,\alpha)$ is a crossing solution if it has a zero in $(0,\infty)$.
- $u(r,\alpha)$ is a slowly decaying solution if $\lim_{r\to\infty} \frac{u(r)}{h(r)} = \infty$.
- $u(r,\alpha)$ is a rapidly decaying solution if $\lim_{r\to\infty} \frac{u(r)}{h(r)} = \ell \in (0,\infty)$.

In the case that u is a crossing solution, we will denote its (unique) zero by $z(\alpha)$.

Our main results consist of a classification of the solutions according to the relative position of q with respect to the critical values ρ_0 and ρ_∞. In these results, the function

$$r \mapsto c(r) := 2\frac{b^2(r)\int_r^\infty (b(s))^{-1}ds}{\beta(r)}$$

plays a fundamental role, the connection of this function with the critical values follows since

$$\liminf_{r\to 0} c(r) \le \liminf_{r\to 0} 2\frac{|\log(\beta(r))|}{|\log(h(r))|} = \rho_0,$$

and

$$\rho_\infty = 2\limsup_{r\to\infty}\frac{|\log(\beta(r))|}{|\log(h(r))|} \le \limsup_{r\to\infty} c(r).$$

Also, we note that in the non weighted case, that is, $b(r) = r^{n-1}$, $n > 2$, we have

$$\beta(r) = \frac{r^n}{n}, \quad h(r) = \frac{r^{2-n}}{n-2},$$

and thus

$$c(r) \equiv \frac{2n}{n-2}.$$

Our first classification result generalizes the non weighted case:

Theorem 2.1 *Let the weight b satisfy assumptions (H_1), (H_2) and (H_3). Let $q > 2$ be fixed and assume that $c(r) = 2\frac{b^2(r)\int_r^\infty (b(s))^{-1}ds}{\beta(r)} \equiv \rho^*$. Then $h(0) = \infty$, $\rho^* > 2$ and*

(i) If $q < \rho^*$, then $u(r,\alpha)$ a crossing solution for any $\alpha > 0$.

(ii) If $q = \rho^*$, then u is the rapidly decaying solution given by

$$u(r,\alpha) = \Big(\frac{C}{C\alpha^{1-\frac{\rho^*}{2}} + h^{1-\frac{\rho^*}{2}}}\Big)^{2/(\rho^*-2)}, \tag{2.3}$$

where C is a positive constant.

(iii) If $q > \rho^*$, then $u(r,\alpha)$ a slowly decaying solution for any $\alpha > 0$.

Finally, we generalize Theorem 2 in [11].

Theorem 2.2 *Let the weight b satisfy assumptions (H_1), (H_2) and (H_3), and assume that they also satisfy*

$$\textit{the function} \quad r \mapsto c(r) \quad \textit{is decreasing on} \quad (0,\infty).$$

If $q \le \rho^*_\infty$, then any solution of (IVP) is crossing.
If $q \ge \rho^*_0$, then any solution of (IVP) is slowly decaying.
If $\rho^*_\infty < q < \rho^*_0$, then there exists α^* such that

- $u(\cdot,\alpha)$ *is crossing for any* $\alpha > \alpha^*$.
- $u(\cdot,\alpha^*)$ *is rapidly decaying.*
- $u(\cdot,\alpha)$ *is slowly decaying for any* $\alpha < \alpha^*$.

This result, as well as some very strong generalizations will appear in [2].

3. Final remarks

In this section, we will compare our result in Theorem 2.2 with Theorem B stated in the introduction. To this end, we will show that if in addition to (K_1) and (K_2), we assume that

$$K^{1/2} \in L^1(0,1) \quad \text{and} \quad K^{1/2} \notin L^1(1,\infty), \tag{3.1}$$

then the assumptions in Theorem 2.2 are satisfied. Indeed, as in [3], we make the change of variable

$$r = r(t) := \int_0^t K^{1/2}(\tau)d\tau, \quad u(r) = v(t),$$

and the problem

$$\begin{gathered} -(t^{n-1}v')' = t^{n-1}K(t)(v^+)^{q-1}, \quad t \in (0,\infty), \quad \Big(\,' = \frac{d}{dt}\Big), \\ v(0) = \alpha > 0, \end{gathered}$$

is transformed into

$$\begin{gathered} -(b(r)u')' = b(r)(u^+)^{q-1}, \quad r \in (0,\infty), \quad \Big(\,' = \frac{d}{dr}\Big), \\ u(0) = \alpha > 0, \end{gathered}$$

where

$$b(r) = t^{n-1}K^{1/2}(t).$$

By (3.1), $r(0) = 0$ and $r(\infty) = \infty$. Next, we will see that assumptions (H_1), (H_2), and (H_3) are satisfied for this b. Clearly, we only need to check that the first in (H_2) and (H_3) are satisfied. We begin by showing that $b \in L^1(0,1)$. Indeed, by making the change of variable $r = \int_0^t K^{1/2}(\tau)d\tau$, we find that

$$\begin{aligned} \int_0^1 b(r)dr &= \int_0^{t_1} t^{n-1}K(t)dt \\ &\le t_1^{n-2}\int_0^{t_1} tK(t)dt, \end{aligned}$$

where here and in the rest of this note t_1 is defined by $1 = \int_0^{t_1} K^{1/2}(\tau)d\tau$, and thus $b \in L^1(0,1)$. Also,

$$\begin{aligned}
\int_0^1 (b(r))^{-1}\Big(\int_0^r b(\tau)d\tau\Big)dr &= \int_0^{t_1} t^{1-n}\Big(\int_0^t s^{n-1}K(s)ds\Big)dt \\
&= \frac{t^{2-n}}{2-n}\int_0^t s^{n-1}K(s)ds\,\Big|_0^{t_1} + \frac{1}{n-2}\int_0^{t_1} tK(t)dt \\
&\le \lim_{t\to 0}\frac{t^{2-n}}{n-2}\int_0^t s^{n-1}K(s)ds + \frac{1}{n-2}\int_0^{t_1} tK(t)dt \\
&\le \lim_{t\to 0}\frac{1}{n-2}\int_0^t sK(s)ds + \frac{1}{n-2}\int_0^{t_1} tK(t)dt \\
&= \frac{1}{n-2}\int_0^{t_1} tK(t)dt,
\end{aligned}$$

implying that (H_3) holds.

Finally, we will see that under (K_2), c is decreasing, and thus our theorem applies: Indeed, it can be seen that in the variable t,

$$c(r) = 2\frac{b^2(r)\int_r^\infty (b(s))^{-1}ds}{\beta(r)} = \frac{2}{n-2}\frac{t^nK(t)}{\int_0^t s^{n-1}K(s)ds},$$

and

$$\frac{tc'(t)}{c(t)} + \frac{n-2}{2}c(t) = n + \frac{tK'(t)}{K(t)},$$

hence, if (K_2) holds, it must be that

$$\frac{tc'(t)}{c(t)} + \frac{n-2}{2}c(t) \quad \text{is decreasing.}$$

Hence, if $c'(t) > 0$ for $t \in (0, t_0)$, then $\frac{tc'(t)}{c(t)}$ must decrease in $(0, t_0)$. This, together with the fact that

$$\lim_{t\to 0}\frac{tc'(t)}{c(t)} = 0,$$

implies that $c'(t) < 0$ in $(0, t_0)$, a contradiction. Hence, there are points $t > 0$ in every interval $(0, t_0)$ where $c'(t) < 0$, implying that if c is not always decreasing, it must have a minimum, which is not possible.

References

1. García-Huidobro M., Kufner A., Manásevich R., Yarur C., Radial solutions for a quasilinear equation via Hardy inequalities, *Advances in Diff. Equations*, **6** (2001), 1517-1540.
2. García-Huidobro M., Manásevich R., Yarur C., On the classification of positive solutions of a quasilinear equation containing a weighted p-laplacian, In preparation.
3. García-Huidobro M., Manásevich R., Pucci P., Serrin J., Qualitative properties of ground states for singular elliptic equations with weights. Submitted.

4. GIDAS B., SPRUCK J., Global and local behavior of positive solutions on nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34** (1981), 525-598.

5. KAWANO T., YANAGIDA E. , YOTSUTANI S., Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^N$, *Funkcial. Ekvak* **36** (1993) 121-145.

6. KUFNER A., OPIC, Hardy-Type Inequalities, Pitman Research Notes in Mathematics Series, Vol 219, Longman Scientific & Technical, Harlow 1990.

7. LI Y., NI WEI-M., On the esistence ans symmetry properties of finite total mass solutions of the Matukuma equation, the Eddington equation and their generalizations, *Arch. Rat. Mech. Anal*, **108** (1989), 175-194.

8. LI Y., NI WEI-M., On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations I, *Arch. Rat. Mech. Anal*, **118** (1992), 195-222.

9. LI Y., NI WEI-M., On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations II, *Arch. Rat. Mech. Anal*, **118** (1992), 223-243.

10. NI WEI-M., YOTSUTANI S., Semilinear elliptic equations of Matukuma-type and related topics, Japan J. Appl. Math. **5** (1988), 1-32.

11. YANAGIDA E., YOTSUTANI S., Existence of positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^N$, *Journal of Differential Equations*, **115** (1995) 477-502.

M. García-Huidobro
Departamento de Matemática,
Universidad Católica de Chile,
Casilla 306, Correo 22,
Santiago, Chile.
E-mail: mgarcia@mat.puc.cl

R. Manásevich
Departamento de Ingeniería Matemática,
FCFM, Universidad de Chile,
Casilla 170, Correo 3,
Santiago, Chile.
E-mail: manasevi@dim.uchile.cl

Cecilia S. Yarur
Departamento de Matemática y C.C.,
Universidad de Santiago de Chile,
Casilla 307, Correo 2, Santiago, Chile.
E-mail: cyarur@fermat.usach.cl