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The Stabilization Theorems For Parabolic Systems With Analytic Nonlinearity And Ljapunov Functional

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ABSTRACT: Let $u(x,t)$ denote the solution of a boundary value problem for parabolic system . We say the solution $u(x,t)$ stabilizes as t tends to plus infinity (minus infinity) if the set of all partial limits as t tends to plus infinity (menus infinity) of the solution $u(x,t)$ consists of a single stationary solution. In this communication we consider the nonlinear parabolic system with analytic dependence of $u(x,t)$ and gradient of $u(x,t)$ on the space variable and with Liapunov functional. It is shown that any solution of the problem uniformly bounded for positive t (or for negative t) stabilizes. In particular the global attractor of this kind of problem consists of stationary solution and connected orbits. The flow on global attractor is a gradient-like flow. The similar result obtained also for the Canh - Hilliard equation.

Key words: parabolic system, stabilization, Ljapunov Functional.

Contents

1. Introduction

This is a paper on recent progress in our outstanding of certain dynamical systems with Ljapunov functional. A class of parabolic systems with variational structure is a main example. As a principal topic, here, we will study the dynamical systems with particulary simple dynamic. We will have to determine just how "simple" the dynamic are. The dynamic of autonomous of ordinary differential equation $\frac{d}{dt}u = f(u), u \in R^n$ is simple in dimensions $n = 1, 2$. Let $\omega(u_0)$ denote the ω-limit set of solution $u(t)$ through $u_0 = u(0)$, i.e. the set of accumulation points of $u(t)$ as $t \to +\infty$. The α-limit set is defined analogously with $t \to -\infty$. If $u(t)$ is uniformly bounded, then we know that $\alpha(u_0), \omega(u_0)$ are nonempty, connected compact and invariant in both directors subsets of R^n. For $n \leq 2$, of course, we know much more. For $n = 2$ the Poincare-Bendixson theorem holds. For $n = 1$ both $\alpha(u_0)$ and $\omega(u_0)$ consist entirely of stationary solution. Even if $2 \leq n \leq \infty$ that statement holds for gradient systems: $f(u) = \nabla_u F(u)$, because $F(u(t))$ decreases strictly along solution, except stationary solution. But there are also substantial differences.

We say that boundary solution $u(t)$ quasi-stabilizes if the set $\omega(u_0)$ consists only stationary solution. If the set $\omega(u_0)$ consists just one stationary solution we say that solution $u(t)$ stabilizes or converges. Consequently, if $u(t)$ is bounded solution of gradient system $u(t)$ quasi-stabilizes. There are in [8], [13] examples of gradient systems with bounded nonconvergent solutions. In [8] is proved that every bounded solution stabilizes if $F(u)$ is real analytic function.

Let $u(x, t; u_0)$ denote the solution of boundary value problem for parabolic system with initial date u_0. In this paper we assume that $u_0 \in C^1$ and satisfy boundary condition. Parabolic system with boundary conditions defines a strongly continuous for $t \geq 0$ semigroup on this subspace. If $u(x, t; u_0)$ is bounded in C^1 solution the ω-limit set and the α-limit set can be defined as for ordinary differential equation above, with the same properties. In [23] (see also [4], [22], [24], [12]) T.Zelenjak proved the stabilization theorem for quasi-linear parabolic equation with one space variable and separated nonlinear boundary conditions. The proof of this theorem is based on continuous Ljapunov functional. T.Zelenjak construct this functional as a solution of a certain hyperbolic equation.

The stabilization theorem for parabolic equation with one space variable in [11], [12], [7] are proved by other method: a discrete Ljapunov functional. An extensive bibliography on discrete Ljapunov functional can be found in [7].

In this paper we consider the parabolic systems with continuous Ljapunov functional (or with variational structure) and analytic dependence of function u and of ∇u. For such type systems we proved the stabilization theorems. The analytic dependence of nonlinearity of $u, \nabla u$ is important. In [14], [15] there are examples of semi-linear boundary value problem for one parabolic equation with nonlinearity of class C^∞, with Ljapunov functional with space variable ≥ 2 and with bounded nonconvergent solution.

An important technical part of our arguments, is that investigation of nonlinear evolution problem was based on a'priori estimates of linear parabolic systems in weight Hlder classes. This estimates were obtained in [4], [5]. Use of this estimates helped us to avoid an additional restriction on nonlinear terms.

2. Preliminaries

Let Ω be a bounded domain in R^n with boundary $\partial\Omega$ of class C^3. Consider the problem:

$$\rho_j(x, u, \nabla u)u_t^j = L^j(u) \tag{1}$$

$$x \in \Omega, t > 0; j = 1, ..., m; u = (u^1, ..., u^m); \nabla u = (u^1_{x_1}, ..., u^m_{x_n});$$

$$L^j(u) = \frac{\partial F(x,u,\nabla u)}{\partial u^j} - \sum_{i=1}^{n} \frac{d}{dx_i} \frac{\partial F(x,u,\nabla u)}{\partial u^j_{x_i}}$$

with boundary condition:

$$B^j(u) = 0, \quad x \in \partial\Omega, \quad t > 0, \quad j = 1, ..., m \tag{2}$$

$$B^j(u) = \gamma^j u^j + (1-\gamma^j)(\sum_i^n \frac{\partial F(x,u,\nabla u)}{\partial u^j_{x_i}} \cos(n, x_i) - g^j(x,u))$$

with $\gamma^j = 0$ or $\gamma^j = 1$ and with initial date:

$$u^j(x,0) = u^j_0(x), \quad x \in \partial\Omega, \quad j = 1, ..., m. \tag{3}$$

In (2) n is unit normal vector on $\partial\Omega$, the functions $\rho^j(x,u,p), \quad F(x,u,p), \quad g^j(x,u)$ are smooth with respect to the space variable and real analytic with respect to $(u,p) \in R^m \times R^{mn}, \quad j = 1, ..., m;$

$$\rho^j(x,u,p) \geq \rho > 0, \quad j = 1, ..., m \tag{4}$$

$$\sum_{i,k=1}^{n} \sum_{j,l=1}^{m} \frac{\partial^2 F(x,u,p)}{\partial u^j_{x_i} \partial u^l_{x_k}} \pi^j_i \pi^l_k \geq \nu |\pi|^2 \tag{5}$$

$\rho, \nu > 0$, for every $p, \pi \in R^{mn}, \quad u \in R^m$.

3. Local and global existence theorem

Denote the E the subset of $(C^1)^m$ consisting of functions $u_0(x) = (u^1_0(x), ..., u^m_0(x))$, which satisfy the boundary conditions (2) $B^j(u_0(x)) = 0$ for $x \in \partial\Omega$.

Here we consider the classic solution to the problem (1) - (3). This means the following: the vector-function $u(x,t)$ is twice continuosly diferentiable by x and one time by t in $Q_T = \Omega \times (0,T)$ or $Q = \Omega \times (0,\infty)$. The vector function $u(x,t)$ is continuos together with partial derivatives by $x_i, \quad i = 1, ..., n$ in $\overline{Q_T}$. The vector-function $u(x,t)$ satisfaz in Q_T to the system (1), the boundary contition (2) on $\Gamma_T = \partial\Omega \times (0,T)$ (or on $\Gamma = \partial\Omega \times (0, \infty$)and the initial data (3)).

Theorem 1 *(The local in time existence and continuous dependence of initial data theorem)*

Suppose that problem (1)-(3) satisfies all the above-made assumptions above hold and $u_0(x) \in E$. Then there exists a positive T depending only the problem (1) - (3) and $||u_0||_1$ such that the problem (1)-(3) has a unique classical solution $u(x,t;u_0)$ in the cylinder Q_T.

Moreover exist $C > 0, \quad \delta > 0$ such as for all $v_0 \in E, \quad |u_0 - v_0|^{\Omega}_1 < \delta$ the solution $u(x,t;v_0)$ be classical solution to the problem (1)-(3) with initial data v_0 in the cylinder Q_T and

$$|u(x,t;u_0) - u(x,t;v_0)|^{Q_T}_{1,2+\alpha} < C|u_0 - v_0|^{\Omega}_1$$

$|.|_{1,2+\alpha}^{Q_T}, \quad \alpha \in (0,1)$ *is the norm in weight Hlder class (see [4], [5]).*

Proof.

The theorem is proved in [1], [23].

□

In such case local in time existence of classic solution of the boundary value problem (1) -(3) proved by [2], [9], [6], [1], [23].

Theorem 2 *(The global existence theorem)*

Let the conditions of theorem 1 hold. If in addition

$$|u(x,t;u_0|_1^{\Omega} \leq M_1, \quad t \in [0,T]$$

then $u(x,t;u_0) \in C^{2+\alpha}(\overline{\Omega}), \quad t \in [\epsilon, T], \quad 0 < \epsilon < T$ *and*

$$|u(x,;u_0)|_{2+\alpha}^{\Omega} \leq M_2, \quad t \in [\epsilon, T]$$

Moreover if M_1 *is bounded for* $t \to +\infty$*, then* M_2 *is also bounded for* $t \to +\infty$*.*

Proof.

The theorem is proved in [1], [23].

□

We note that for the special case of L and $n = 2$ and with Dirichlet boundary conditions A.Arkipova constructed global in time solution in [3].

4. Stabilization Theorems

Let $u(x,t;u_0)$ be the classic solution of the problem (1)-(3) and

$$|u(x,t;u_0)|_1^{\Omega} < M_1, \quad t \in [0, +\infty)$$

from theorem 2 we obtain

$$|u(x,t;u_0)|_{2+\alpha}^{\Omega} < M_2, \quad t \in [\epsilon, +\infty), \quad \epsilon > 0.$$

Therefore ω-limit set $\omega(u_0)$ of solution $u(x,t;u_0)$ or the set of partial limits for $t \to +\infty$ in the norm $C^{2+\beta}(\overline{\Omega}), \quad 0 < \beta < \alpha < 1$ of solution $u(x,t;u_0)$ is nonempty conneted set.

It is well known that the problem (1) - (3) is gradient like with respect to the Liapunov (energy) functional

$$V(u) = \int_{\Omega} F(x,u,\nabla u)dx + \int_{\partial\Omega} G(x,u), \quad G_u(x,u) = g(x,u)$$

which is to say that $V(u(x,t;u_0)$ decreasing along any non-stationary solution of the problem (1)-(3):

$$\frac{dV(u(x,t;u_0)}{dt} = \int_{\Omega} \sum_{j=1}^{m} \rho^j(x,u(x,t;u_0),\nabla u(x,t;u_0))(u_t^j(x,t;u_0))^2 dx$$

Consequently the set $\omega(u_0)$ consists of the stationary solutions $v(x)$ of the problem (1)-(3):

$$0 = L^j(v), \quad x \in \Omega, \quad 0 = B^j(v), \quad x \in \partial\Omega, \quad j = 1, ..., m.$$

We denote the set of all stationary solution of the problem (1)-(3) by S.

If the set $\omega(u_0)$ consists of just one stationary solution $v(x)$, the solution $u(x,t,;u_0)$ is stabilized (or convergent) and

$$|u(x,t;u_0) - v(x)|^{\Omega}_{2+\alpha} \to 0, \quad as \quad t \to +\infty.$$

Let the solution $u(x,t;u_0)$ to be extended for all $t < 0$ and $|u(x,t;u_0)|^{\Omega}_1 < M_1$ for all $t < 0$. The α-limit set $\alpha(u_0)$ or the set of all partial limits of solution $u(x,t;u_0)$ as $t \to +\infty$ in $C^{2+\beta}(\overline{\Omega})$ of solution $u(x,t;u_0)$ is also nonempty, conneted, and consists of stationary solutions of the problem (1)-(3). If the set $\alpha(u_0)$ consists of just one stationary solution$v(x)$, the solution $u(x,t;u_0)$ is stabilized (or convergent) in backward time.

Theorem 3 *Assume that all assumption above hold and*

$$|u(x,t;u_0)|^{\Omega}_1 < M_1, \quad t \in [0, +\infty)$$

then $u(x,t;u_0)$ *is stabilized. If*

$$|u(x,t;u_0)|^{\Omega}_1 < M_1, \quad t \in [0, -\infty)$$

then $u(x,t;u_0)$ *is stabilized in backward time.*

If

$$|u(x,t;u_0)|^{\Omega}_1 < M_1, \quad t \in [+\infty, -\infty)$$

there are $v_1, v_2 \in S, \quad v_1 \neq v_2$, *such as*

$$|u(x,t;u_0) - v_1(x)|^{\Omega}_{2+\beta} \to 0 \quad t \to +\infty$$

$$|u(x,t;u_0) - v_2(x)|^{\Omega}_{2+\beta} \to 0 \quad t \to -\infty)$$

.

Proof.

Idea of the proof of this theorem suggested by [17]. According to the result of Lojasievicz [10] concerning the real analytic functions and for any real analytic function $G(y)$ in the neighborhood of 0 with $\nabla G(0) = 0$ there exist the constants $k \in (0, 1/2), \quad l > 2, \quad K > 0$ such that

$$C|\nabla G(\gamma)| \geq d(\gamma, W)^l, \quad C|G(\gamma)| > |\nabla G(\gamma)|^{1-k}$$

$$|\gamma| < K, \quad W = \{\gamma \in R^n, \quad \nabla G(\gamma) = 0\},$$

For the proof of the main result we extend the result of Lojasievics for parabolic systems. This estention is concerning in the theorem 4 and 5 below.

Theorem 4 *There are constants $k \in (0,1/2)$, $l > 2$, $K > 0$, such that for arbitrary function $u(x) \in E$ and $v(x)$ the staedy state solution to the problem (1) - (3) $|u(x) - v(x)|_{2+\alpha} < K$ we obtain*

$$||L(u)|| > inf_{v(x)\in S}||u(x) - v(x)||^l, \quad ||L(u)|| \geq ||V(u) - V(v)||^{1-k}$$

here $||.||$ is the norm in the space $L_2(\Omega)$.

□

5. Some examples

1.Let us consider the boundary value problem for one parabolic equation with one space variable:

$$u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x), \quad x \in (0, 1), \quad t > 0$$

$$\gamma^j u(j, t) + (1 - \gamma^j)(u_x(j, t) - g^j(u(j, t)) = 0, \quad j = 0, 1, \quad t > 0.$$

$$u(x, 0) = u_0(x), \quad x \in (0, 1)$$

We assume here that all functions a, b, g^j are sufficiently smooth and $a(x, u, p) \geq a_0 > 0$

For this problem condition (B) holds if the solution $\varphi(x_0, x, y_0, y_1)$ of the Cauchy problem:

$$y'' = \frac{b(x, y, y')}{a(x, y, y'}), \quad y(x_0) = y_0, \quad y'(x_0) = y_1.$$

is determined and regular for all data $(x_0, y_0, y_1) \in [0, 1] \times R \times R$.

Theorem 5 *Let the equation above satisfy (B). Then there exists a pair of functions $\rho(x, u, p), F(x, u, p), \varphi(x, u)$ such that for each solution of problem we also have:*

$$\rho(x, u, u_x)u_t = F_u(x, u, u_x) - \frac{d}{dx}F_{u_x}(x, u, u_x), \quad x \in (0, 1), \quad t > 0$$

$$\gamma^j u(j, t)+(1-\gamma^j)(F(j, u(j, t), u_x(j, t))-\varphi^j(u(j, t)) = 0, \quad j = 0, 1, \quad t > 0, \quad j = 0, 1.$$

$$u(x, o) = u_0(x)$$

$$\rho(x, u, p) > \rho > 0, \quad F_{u_x u_x} > \nu > 0.$$

Proof.

The theorem is proved in [23], [22], [4], [24].

□

Theorem 6 *All bounded in $C^1([0,1])$ solutions of the problem are stabilized.*

Proof.
The theorem is proved in [23], [22], [4], [24].

□

2. Let us consider the parabolic equation with many space variable of the form:

$$\rho(x,u,\nabla u)u_t = F_u(x,u,\nabla u) - \sum_{i=1}^{n} \frac{d}{dx_i} F_{u_{x_i}}(x,u,\nabla u), \quad x \in \Omega, \quad t > 0$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma = \partial\Omega \times (0,\infty)$$

or

$$\sum_{i=1}^{n} F_{u_{x_i}}(x,u,\nabla u)\cos(n,x_i) - g(x,u) = 0, \quad (x,t) \in \Gamma$$

$$u(x,0) = u_0(x)$$

All bounded solutions of this problem are stabilized.
3. Let us consider the semi-linear parabolic system:

$$\rho^j(x,u,\nabla u)u_t^j = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}(a_{ik}^j(x)\frac{\partial u^j}{\partial x_k}0 + f_{u^j}, \quad (x,t) \in Q, \quad j = 1,...,m$$

$$\gamma^j u(x,t) + (1-\gamma^j)(\sum_{i,k=1}^{n} a_{ik}^j(x)\frac{\partial u^j}{\partial x_k}\cos(n,x_i) - g^j(x,u)) = 0, \quad (x,t) \in \Gamma,$$

$$j = 1,...,m, \quad \gamma^j = 1,0$$

$$u^j(x,o) = u_0^j(x), \quad x \in \Omega, \quad j = 1,...,m$$

All bounded solutions of this problem are stabilized.
In particular to this result is true for Henon - Heilis type system:

$$u_t = u_{xx} + u + n^2 - v^2, \quad v_t = v_{xx} + v - 2uv, \quad x \in (0,1), \quad t > 0$$

$$u(i,t) = v(i,t) = 0, \quad i = 0,1, \quad u(x,0) = u_0(x), \quad v(x,0) = x_0(x), \quad x \in (0,1).$$

6. Generalization of the results

1. Consider the Canh-Hilliard equation in bounded domain with smooth boundary:

$$u_t = -\gamma\triangle^2 u + div(\nabla_p a(\nabla u)) + f(x,u), \quad (x,t) \in Q\nabla_p a(p) = (a_{p_1},...,a_{p_n}), \quad \gamma > 0$$

$$-\gamma\triangle\nabla u(x,t) + (\nabla_p a(\nabla u), n) = 0, \quad u(x,t) = 0; \quad (x,t) \in \Gamma$$

$$u(x,0) = u_0(x), \quad x \in \Omega$$

If the functions a,f are analytic the stabilization theorems are true for this problem too. (See [18], [19]).

2. Let us consider the semi-linear parabolic equation with nonlocal terms:

$$\mu u_t = M(\int_\Omega (\nabla u(x,t))^2 dx) \triangle u + f(x,u). \quad (x,t) \in Q$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma$$

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

If $n \geq 2$, the functions M,f are analytic the stabilization theorems are true for this problem too. (See [16]). If $n = 1$ and the functions are sufficiently smooth the stabilization theorems are true for this problem.

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