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# A class of global weak solutions to the axisymmetric isentropic Euler equations of perfect gases in two space dimensions

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We consider the compressible isentropic Euler equations for a perfect gas ($t > 0, x \in \mathbb{R}^N$) :

$$\partial_t \rho + \sum_{1\leq k\leq N} \partial_k(\rho u_k) = 0 \tag{1}$$

(conservation of mass),

$$\partial_t(\rho u_i) + \sum_{1\leq k\leq N} \partial_k(\rho u_k u_i) + \partial_i p = 0, \tag{$2_i$}$$

$1 \leq i \leq N$ (conservation of momentum), where $\rho$ is the density, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$ the velocity, and $p(\rho)$ the pressure. We assume that $p(\rho) = a\rho^\gamma$, $a > 0$, $1 < \gamma \leq 1 + \dfrac{2}{N}$. We impose the initial conditions

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x). \tag{3}$$

One has the following results.

(I) If $\rho_0 = \bar\rho + \rho_1$, where $\bar\rho > 0$ is a constant, $\rho_1$ and $u_0 \in H^s(\mathbb{R}^N)$ with $s$ an integer $> \dfrac{N}{2} + 1$ and $\inf \rho_0 > 0$, one can find a solution to (1), (2), (3) for $t$ small (see [7]).

(II) If $\rho_0^{\frac{\gamma-1}{2}}$ and $u_0 \in H^s_{ul}(\mathbb{R}^N)$, $s$ integer $> \frac{N}{2} + 1$ and $\rho_0 > 0$, one can find a solution to (1), (2), (3) for $t$ small (Chemin [2]). Here $H^s_{ul}(\mathbb{R}^N) = \{v \in H^s_{\text{loc}}(\mathbb{R}^N), \sup_{x\in\mathbb{R}^N} ||\varphi_x v||_s < +\infty$ if $\varphi \in C_0^\infty(\mathbb{R}^N)\}$, where $\varphi_x(y) = \varphi(x-y)$ and $|| \, ||_s$ is the standard $H^s$ norm.

In general, solutions to (1), (2), (3) are not global in $t$ (Sideris [10], Rammaha [8]). In case (I), when $N = 2$ and $\rho_0, u_0$ are rotation invariant around 0 with $\rho_1 = \varepsilon\tilde\rho_1$, $u_0 = \varepsilon\tilde{u_0}$, $\tilde\rho_1, \tilde{u}_0 \in C_0^\infty(\mathbb{R}^2)$ and $|\tilde\rho_1| + |\text{div}\tilde{u_0}| \not\equiv 0$, Alinhac [1] has shown that the lifespan of solutions is $\sim \dfrac{1}{\varepsilon^2}$ ($\varepsilon$ small); see also Sideris [11].

Grassin-Serre [5] and Grassin [4] have obtained global results (see also [9]) that we are going to describe now. If $\rho$ never vanishes, it follows from (1), (2) that

$$\partial_t u_i + \sum_{1\le k\le N} u_k \partial_k u_i + \frac{\partial_i p}{\rho} = 0 \tag{$2'_i$}$$

for $1 \le i \le N$. One can symmetrize (1), ($2'_i$) ($1 \le i \le N$) by introducing $\pi = C_1^{-1}\sqrt{p'(\rho)}$, $C_1 = \dfrac{\gamma-1}{2}$. (1), ($2'_i$) ($1 \le i \le N$) become

$$\partial_t \pi + \sum_{1\le k\le N} u_k \partial_k \pi + C_1 \pi \sum_{1\le k\le N} \partial_k u_k = 0, \tag{4}$$

$$\partial_t u_i + \sum_{1\le k\le N} u_k \partial_k u_i + C_1 \pi \partial_i \pi = 0, \tag{$5_i$}$$

$1 \le i \le N$. This symmetrization has already been used by Chemin [2] for (II). Consider the initial data

$$u(x,0) = u_0(x),\ \pi(x,0) = \pi_0(x). \tag{6}$$

Grassin-Serre and Grassin have introduced the following assumptions :

$$\left.\begin{array}{l} \partial^\alpha u_0 \in L^\infty(\mathbb{R}^N) \text{ if } |\alpha| = 1,\ \partial^\alpha u_0 \in H^{s-1}(\mathbb{R}^N) \\ \text{ if } |\alpha| = 2, \inf_{x\in\mathbb{R}^N} \operatorname{dist}(sp\ du_0(x), \mathbb{R}^-) > 0, \\ \pi_0 \in H^s(\mathbb{R}^N) \text{ and } ||\pi_0||_s \text{ is small } (s \text{ integer } > \dfrac{N}{2} + 1). \end{array}\right\} \tag{7}$$

**Theorem 1 ( [5], [4]).** *If (7) is satisfied, (4), (5), (6) has a global solution when* $t > 0$, $x \in \mathbb{R}^N$.

This theorem is obtained by comparing $(\pi, u)$ with $(0, \bar{u})$, where $(0, \bar{u})$ is the solution to (4), ($5_i$),$1 \le N$, with initial data $(0, u_0)$.

The purpose of this talk is to describe a result of the same type (contained in [3]) for a class of non-smooth initial data.

We shall assume that $N = 2$ and consider initial data which are rotation invariant around 0, so $u_0(Sx) = Su_0(x)$ and $\pi_0(Sx) = \pi_0(x)$ for every rotation $S$ with center 0. It follows that

$$u_0(x) = A_0(r)\frac{x}{r} + B_0(r)\frac{x^\perp}{r} \text{ and } \pi_0(x) = \Pi_0(r) \text{ with } r = |x|,$$

$$x^\perp = (-x_2, x_1).$$

We start with $\bar{u}_0(x) = \bar{A}_0(r)\dfrac{x}{r} + \bar{B}_0(r)\dfrac{x^\perp}{r}$, satisfying (7) with $s = 3$, and consider two small perturbations of $\bar{u}_0$, namely $u_0^{(1)}, u_0^{(2)}$, rotation invariant around 0. We assume that

$$\sum_{|\alpha|\le 1} |\partial^\alpha(u_0^{(j)} - \bar{u}_0)| + \sum_{|\alpha|=2} ||\partial^\alpha(u_0^{(j)} - \bar{u}_0)||_2 \le \varepsilon.$$

Consider $\pi_0^{(j)}(x) \equiv \Pi_0^{(j)}(r) > 0$, $j = 1, 2$, such that $||\pi_0^{(j)}||_3 \leq \varepsilon$ and $\Pi_0^{(j)}(r) \geq C_0\varepsilon$ if $0 \leq (-1)^j(r-1) \leq C\varepsilon$ ($C > 0$ large enough). Put

$$(\pi_0, u_0) = \begin{cases} (\pi_0^{(1)}, u_0^{(1)}) \text{ if } r < 1, \\ (\pi_0^{(2)}, u_0^{(2)}) \text{ if } r > 1. \end{cases}$$

Write $u_0(x) = A_0(r)\dfrac{x}{r} + B_0(r)\dfrac{x^\perp}{r}$, $\pi_0(x) = \Pi_0(r)$, and assume that

$$0 < [A_0 \pm \Pi_0](1) \leq C_2\varepsilon^{2+\theta},$$

$$0 < |[B_0^2](1)| \leq C_3\varepsilon,$$

where $C_2, C_3$ are small and $0 < \theta < \dfrac{1}{2}$. Here $[F](1) = \lim_{r \xrightarrow{>} 1} F(r) - \lim_{r \xrightarrow{<} 1} F(r)$.

**Theorem 2 ( [3]).** *If $\varepsilon$ is small, there exists a weak solution to (1), (2) which is rotation invariant around 0 and global in $t > 0$, such that $\rho|_{t=0} = \tilde{C}\pi_0^{1/C_1}$, $u|_{t=0} = u_0$, where $\tilde{C} = C_1^{1/C_1}(a\gamma)^{-1/2C_1}$. This solution consists of two centered waves (in the $(r, t)$ variables) and one contact discontinuity.*

Local existence is obtained by adapting results and ideas of [6]. The global results can be proved by a continuation method.

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