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On a Transmission Problem for Dissipative Klein-Gordon-Shrödinger Equations

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ABSTRACT: In this paper we consider a transmission problem for the Cauchy problem of coupled dissipative Klein-Gordon-Shrödinger equations and we prove the existence of global solutions.

Contents

1. Introduction

Let $]0, L_3[$ be a bounded open interval of $\mathbb{R}$ such that $L_1, L_2 \in]0, L_3[$. We denote by Ω the set $]0, L_1[\cup]L_2, L_3[$.

In this work we prove the existence of strong and weak solutions of a transmission problem for the coupled Klein-Gordon-Shrödinger equations with dissipative term, given by the following system:

$$i\psi_t + \psi_{xx} + i\alpha\psi + \phi\psi = 0 \quad in \quad \Omega\times]0, \infty[\tag{1.1}$$

$$\phi_{tt} - \phi_{xx} + \phi + \beta\phi_t = |\psi|^2 \quad in \quad \Omega\times]0, \infty[\tag{1.2}$$

$$\theta_{tt} - \theta_{xx} = 0 \quad in \quad]L_1, L_2[\times]0, \infty[\tag{1.3}$$

where α and β are positive constants.

The system is subjected to the following boundary conditions.

$$\psi(0, t) = \psi(L_3, t) = \phi(0, t) = \phi(L_3, t) = 0 \tag{1.4}$$

$$\phi(L_i, t) = \theta(L_i, t) \; ; \; \phi_x(L_i, t) = \theta_x(L_i, t) \quad ; \quad i = 1, 2 \tag{1.5}$$

$$\psi_x(L_i, t) = 0 \quad ; \quad i = 1, 2 \tag{1.6}$$

and initial conditions

$$\psi(x, 0) = \psi_0(x) \quad ; \quad x \in \Omega \tag{1.7}$$

$$\phi(x, 0) = \phi_0(x) \; ; \; \phi_t(x, 0) = \phi_1(x) \quad ; \quad x \in \Omega \tag{1.8}$$

$$\theta(x, 0) = \theta_0(x) \; ; \; \theta_t(x, 0) = \theta_1(x) \quad ; \quad x \in]L_1, L_2[\tag{1.9}$$

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Controllability for transmission problems has been studied by several authors, and we mention a few works. The transmission problem for the wave equation was studied by Lions [7], where he applied the Hilbert Uniqueness Method (HUM) to show exact controllability. Latter, Lagnese [6], also applying HUM, extended this result; he showed the exact controllability for a class of hyperbolic systems which include the transmission problem for homogeneous anisotropic materials. The exact controllability for the plate equation was proved by Liu and Williams [9]. Some results about existence, uniqueness and regularity for elliptic stationary transmission problem can be found in Athanasiadis and Stratis [1] and Ladyzhenskaya and Ural'tseva [5].

Concerning stability, Liu and Williams [8] studied a transmission problem for the wave equation and showed exponential decay of the energy provided a linear feedback velocity is applied at the boundary. Marzocchi et al.[10] proved that the solution of a semi-linear transmission problem between an elastic a thermoelastic material, decays exponentially to zero.

Let us mention some works related with the Klein-Gordon -Schrödinger equations. Fukuda and Tsutsumi[4] studied the initial-boundary value problem for the coupled Klein-Gordon -Schrödinger equations in three space dimensions. In the case of one space dimension, the existence of global smooth solutions has been established by the authors [3]. Boling and Yongsheng [2] considerer the Cauchy problem of coupled dissipative proved a existence Klein-Gordon -Schrödinger equations in $\mathbb{R}^3$ and prove the existence of the maximal attractor.

The objective of this paper is to prove the existence of strong and weak solutions to problem (1.1)-(1.9). The proof of the existence is based on the Galerkin method and employed techniques in [2].

2. Notation

For brevity, we denote the space of complex-valued functions and real-valued functions and real-valued functions by the same symbols.

Let $L^p(\Omega)$ be the usual Lebesgue space of complex-valued or real-valued functions whose p-times powers are integrable with norm:

$$\begin{aligned} |u|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < +\infty \qquad (1 \leq p < +\infty). \\ |u|_\infty = ess \sup_{x \in \Omega} |u(x)| < +\infty \qquad (p = +\infty). \end{aligned} \tag{2.1}$$

In particular, $L^2(\Omega)$ is the Hilbert space with inner product and norm:

$$(u, v) = \int_\Omega u(x)\overline{v(x)} dx, \qquad |u|_2 = \|u\| = (u, u)^{1/2}. \tag{2.2}$$

$H^m(\Omega)$ (m is an interger ≥ 1) denote the complex or real Sobolev spaces whose distributional derivatives of order $\leq m$ lie in $L^2(\Omega)$ equipped with inner product and norm:

$$(u, v)_m = \sum_{j=0}^{m} \int_\Omega D^j u(x) \overline{D^j v(x)} dx, \quad \|u\|_m = (u, u)_m^{1/2}. \tag{2.3}$$

Let us define the subspace

$$H^1_L(\Omega) = \{w \in H^1(\Omega); w(0) = w(L_3) = 0\}$$

It follows that $H^1_L(\Omega)$ is a Hilbert subspace of $H^1(\Omega)$. We can prove that in $H^1_L(\Omega)$ the norm

$$\|w\|^2 = \int_\Omega |w_x(x)|^2 dx \tag{2.4}$$

and the $H^1_L(\Omega)$ norm are equivalents. Consequently, we consider $H^1_L(\Omega)$ equipped with the norm (2.4) and the scalar product

$$((v, w)) = \int_\Omega v_x(x) \cdot w_x(x) dx \tag{2.5}$$

Also let us define the subspace

$$V = \{\{u, v\} \in H^1_L(\Omega) \times H^1(]L_1, L_2[) \; ; \; u(L_i) = v(L_i) \; , \; i = 1, 2\}$$

Note that V is a closed subspace of $H^1_L(\Omega) \times H^1(]L_1, L_2[)$ which together with the norm

$$\|\{u, v\}\|^2_V = \int_\Omega |u_x(x)|^2 dx + \int_{L_1}^{L_2} |v_x(x)|^2 dx \tag{2.6}$$

is a Hilbert space.

3. Existence of solutions

In this section we establish existence and uniqueness results for problem [(1.1)−(1.9)].

First of all, we define what we will understand for strong and weak solution of the problem [(1.1) − (1.9)].

Definition 3.1 *We say that* (ψ, ϕ, θ) *is a strong solution of* [(1.1) − (1.9)] *when*

$$\begin{array}{rcl} \psi & \in & L^\infty_{loc}(0, \infty; H^2(\Omega) \cap H^1_L(\Omega)) \\ \psi_t & \in & L^\infty_{loc}(0, \infty; H^1_L(\Omega)) \\ \{\phi, \theta\} & \in & L^\infty_{loc}(0, \infty; [H^2(\Omega) \times H^2(]L_1, L_2[)] \cap V) \\ \{\phi_t, \theta_t\} & \in & L^\infty_{loc}(0, \infty; V) \\ \{\phi_{tt}, \theta_{tt}\} & \in & L^\infty_{loc}(0, \infty; L^2(\Omega) \times L^2(]L_1, L_2[)) \end{array}$$

satisfying the identities

$$\begin{array}{rcl}
i\psi_t + \psi_{xx} + i\alpha\psi + \phi\psi = 0 & in & L^\infty_{loc}(0,\infty;L^2(\Omega)) \\
\phi_{tt} - \phi_{xx} + \phi + \beta\phi_t = |\psi|^2 & in & L^\infty_{loc}(0,\infty;L^2(\Omega)) \\
\theta_{tt} - \theta_{xx} = 0 & in & L^\infty_{loc}(0,\infty;L^2(]L_1,L_2[) \\
\psi(0,t) = \psi(L_3,t) = \phi(0,t) = \phi(L_3,t) = 0 & ; & t > 0 \\
\phi(L_i,t) = \theta(L_i,t) \ ; \ \phi_x(L_i,t) = \theta_x(L_i,t) & ; & t > 0\ ,\ (i=1,2) \\
\psi_x(L_i,t) = 0 & ; & t > 0\ ,\ (i=1,2) \\
\psi(x,0) = \psi_0(x) & ; & x \in \Omega \\
\phi(x,0) = \phi_0(x) \ \ e \ \ \phi_t(x,0) = \phi_1(x) & ; & x \in \Omega \\
\theta(x,0) = \theta_0(x) \ \ e \ \ \theta_t(x,0) = \theta_1(x) & ; & x \in]L_1,L_2[
\end{array}$$

Definition 3.2 *Let $T > 0$ be real. We say that (ψ,ϕ,θ) is a weak solution of $[(1.1)-(1.9)]$ when*

$$\psi \in L^\infty(0,T;H^1_L(\Omega))$$

$$\{\phi,\theta\} \in L^\infty(0,T;V) \quad , \quad \{\phi_t,\theta_t\} \in L^\infty(0,T;L^2(\Omega)\times L^2(]L_1,L_2[))$$

satisfying the identities

$$\int_0^T \int_\Omega \left[-i\psi\bar{\Psi}_t - \psi_x\bar{\Psi}_x + i\alpha\psi\bar{\Psi} + \phi\psi\bar{\Psi}\right] dxdt = \int_\Omega i\psi_0(x)\bar{\Psi}(x,0)dx$$

$$\begin{array}{rl}
 & \int_0^T \int_\Omega \left[\phi\Phi_{tt} + \phi_x\Phi_x + \phi\Phi - \beta\phi\Phi_t - |\psi|^2\Phi\right] dxdt \\
 & + \int_0^T \int_{L_1}^{L_2} \left[\theta\Theta_{tt} + \theta_x\Theta_x\right] dxdt \\
= & \int_\Omega \phi_1(x)\Phi(x,0)dx - \int_\Omega \phi_0(x)\Phi_t(x,0)dx + \beta\int_\Omega \phi_0(x)\Phi(x,0)dx \\
 & + \int_{L_1}^{L_2} \theta_1(x)\Theta(x,0)dx + \int_{L_1}^{L_2} \theta_0(x)\Theta_t(x,0)dx
\end{array}$$

for all $\Psi \in C^1([0,T];H^1_L(\Omega))$, $\{\Phi,\Theta\} \in C^2([0,T];V)$ and a.e $t \in [0,T]$ such that

$$\Psi(T) = \Phi(T) = \Phi_t(T) = \Theta(T) = \Theta_t(T) = 0$$

The existence of strong solution to system $[(1.1)-(1.9)]$ is given in the following theorem:

Theorem 1 *Given*

$$\begin{array}{rcl}
\psi_0 & \in & H^2(\Omega)\cap H^1_L(\Omega) \\
\{\phi_0,\theta_0\} & \in & [H^2(\Omega)\times H^2(]L_1,L_2[)]\cap V \\
\{\phi_1,\theta_1\} & \in & V
\end{array}$$

with

$$\begin{array}{rcl} \psi_{0x}(L_i) & = & 0 \ ; \ (i=1,2) \\ \phi_{0x}(Li) & = & \theta_{0x}(L_i) \ ; \ (i=1,2) \end{array}$$

there exists only a strong solution of $[(1.1)-(1.9)]$.

Proof. We follow a standard Faedo-Galerkin method and we divide the proof in four steps.

Step 1 *(Approximate System).* Let us denote by $\{u_i; i \in \mathbb{N}\}$ a basis of $H^2(\Omega) \cap H^1_L(\Omega)$ and by $\{\{v_i, w_i\}; i \in \mathbb{N}\}$ a basis of $[H^2(\Omega) \times H^2(]L_1, L_2[)] \cap V$. We denote by

$$H_\nu = span\{u_1, u_2, \cdots, u_\nu\}$$

$$V_\nu = span\{\{v_1, w_1\}, \{v_2.w_2\}, \cdots, \{v_\nu, w_\nu\}\}$$

Let

$$\psi^\nu(x,t) = \sum_{i=1}^{\nu} a_{i\nu}(t) u_i \qquad (a_{i\nu}(t) : Complex-valued)$$

and

$$\{\phi^\nu(x,t), \theta^\nu(x,t)\} = \sum_{i=1}^{\nu} b_{i\nu}(t)\{v_i, w_i\} \qquad (b_{i\nu}(t) : Real-valued)$$

be solutions of the system $(j = 1, 2, \cdots, \nu)$ of ordinary differential equations

$$\int_\Omega [i\psi^\nu_t \bar{u}_j - \psi^\nu_x \bar{u}_{j,x} + i\alpha\psi^\nu \bar{u}_j + \phi^\nu \psi^\nu \bar{u}_j] \, dx = 0 \tag{3.1}$$

$$\begin{aligned} \int_\Omega \left[\phi^\nu_{tt} v_j + \phi^\nu_x v_{j,x} + \phi^\nu v_j + \beta \phi^\nu_t v_j - |\psi^\nu|^2 v_j\right] dx & \\ + \int_{L_1}^{L_2} [\theta^\nu_{tt} w_j + \theta^\nu_x w_{j,x}] \, dx & = 0 \end{aligned} \tag{3.2}$$

which satisfy the initial data

$$\psi^\nu(0) = \psi_0 \ , \ \{\phi^\nu(0), \theta^\nu(0)\} = \{\phi_0, \theta_0\} \ , \ \{\phi^\nu_t(0), \theta^\nu_t(0)\} = \{\phi_1, \theta_1\}$$

Standard theorems in the theory of ordinary differential equations ensure that this system has the solutions $\{\psi^m, \phi^m, \psi^m\}$ $(m = 1, 2, 3, \cdots)$ locally in time which are uniquely determined by initial data, for each m.

Step 2 *(Estimate I).* Multiplying (3.1) by $\overline{a_{j\nu}(t)}$, summing over j and taking imaginary parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\psi^\nu(t)\|^2 + \alpha\|\psi^\nu(t)\|^2 = 0$$

It follows that

$$\|\psi^\nu(t)\|^2 + \alpha \int_0^t \|\psi^\nu(s)\|^2 ds = \|\psi_0\|^2 \tag{3.3}$$

From (3.3) it follows that:

$$\psi^\nu \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)) \tag{3.4}$$

Step 3 *(Estimate II)*. Multiplying (3.1) by $-\overline{a_j'(t)}$ and summing over j, we have

$$\begin{aligned} -i\|\psi_t^\nu(t)\|^2 + (\psi_x^\nu(t), \psi_{xt}^\nu(t)) \\ -i\alpha(\psi^\nu(t), \psi_t^\nu(t)) - \int_\Omega \phi^\nu\psi^\nu\overline{\psi_t^\nu}dx &= 0 \end{aligned} \tag{3.5}$$

Multiplying (3.1) by $-\alpha\overline{a_j(t)}$ and summing over j, we have

$$\begin{aligned} -i\alpha(\psi_t^\nu(t), \psi^\nu(t)) + \alpha\|\psi_x^\nu(t)\|^2 \\ +i\alpha^2\|\psi^\nu(t)\|^2 - \alpha\int_\Omega \phi^\nu|\psi^\nu|^2dx &= 0 \end{aligned} \tag{3.6}$$

Taking real parts in [(3.5) − (3.6)], we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\psi_x^\nu(t)\|^2 - Re(i\alpha(\psi^\nu(t), \psi_t^\nu(t))) \\ -Re\left(\int_\Omega \phi^\nu\psi^\nu\overline{\psi_t^\nu}dx\right) &= 0 \end{aligned} \tag{3.7}$$

$$-Re(i\alpha(\psi_t^\nu, \psi^\nu(t))) + \alpha\|\psi_x^\nu(t)\|^2 - \alpha\int_\Omega \phi^\nu|\psi^\nu|^2dx = 0 \tag{3.8}$$

Summing (3.7) and (3.8), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\psi_x^\nu(t)\|^2 + \alpha\|\psi_x^\nu(t)\|^2 \\ -Re(\phi^\nu\psi^\nu, \psi_t^\nu) - \alpha(\phi^\nu\psi^\nu, \psi^\nu) &= 0 \end{aligned} \tag{3.9}$$

Noticing that

$$-Re(\phi^\nu\psi^\nu, \psi_t^\nu) = -\frac{1}{2}\frac{d}{dt}(\phi^\nu, |\psi^\nu|^2) + \frac{1}{2}(\phi_t^\nu, |\psi^\nu|^2) \tag{3.10}$$

We infer from (3.9) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\psi_x^\nu(t)\|^2 - \int_\Omega \phi^\nu|\psi^\nu|^2dx\right) + \alpha\|\psi_x^\nu\|^2 \\ +\frac{1}{2}\int_\Omega \phi_t^\nu|\psi^\nu|^2dx - \alpha\int_\Omega \phi^\nu|\psi^\nu|^2dx &= 0 \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} \frac{d}{dt}\left(2\|\psi_x^\nu(t)\|^2 - 2\int_\Omega \phi^\nu|\psi^\nu|^2dx\right) + 4\alpha\|\psi_x^\nu\|^2 \\ +2\int_\Omega \phi_t^\nu|\psi^\nu|^2dx - 4\alpha\int_\Omega \phi^\nu|\psi^\nu|^2dx &= 0 \end{aligned} \tag{3.12}$$

We introduce the transformations

$$\eta^\nu(t) = \phi_t^\nu(t) + \delta\phi^\nu(t)$$

and

$$\gamma^\nu(t) = \theta_t^\nu(t) + \delta\theta^\nu(t)$$

where $\delta = \min\{\frac{\beta}{2}, \frac{1}{2\beta}\}$. Them (3.2) is equivalent to.

$$\begin{aligned} &(\eta_t^\nu(t), v_j) + (\beta-\delta)(\eta^\nu(t), v_j) + (1-\delta(\beta-\delta))(\phi^\nu(t), v_j) \\ &+(\phi_x^\nu(t), v_{j,x}) + \int_{L_1}^{L_2} \gamma_t^\nu w_j dx + \delta^2 \int_{L_1}^{L_2} \theta^\nu w_j dx + \int_{L_1}^{L_2} \theta_x^\nu w_j dx = \\ &+ \int_\Omega |\psi^\nu|^2 v_j dx + \delta \int_{L_1}^{L_2} \gamma^\nu w_j dx \end{aligned} \tag{3.13}$$

Multiplying (3.13) by $b_j'(t) + \delta b_j(t)$ and summing over j, we have.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\eta^\nu(t)\|^2 + (1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 + \|\phi_x^\nu(t)\|^2\right] \\ &+\frac{1}{2}\frac{d}{dt}\left[\int_{L_1}^{L_2} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1}^{L_2} |\theta^\nu|^2 dx + \int_{L_1}^{L_2} |\theta_x^\nu|^2 dx\right] \\ &+(\beta-\delta)\|\eta^\nu(t)\|^2 + \delta(1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 \\ &+\delta\|\phi_x^\nu(t)\|^2 + \delta^3 \int_{L_1}^{L_2} |\theta^\nu|^2 dx + \delta \int_{L_1}^{L_2} |\theta_x^\nu|^2 dx = \\ &+ \int_\Omega \eta^\nu |\psi^\nu|^2 dx + \delta \int_{L_1}^{L_2} |\gamma^\nu|^2 dx = \\ &+ \int_\Omega \phi_t^\nu |\psi^\nu|^2 dx + \delta \int_\Omega \phi |\psi^\nu|^2 dx + \delta \int_{L_1}^{L_2} |\gamma^\nu|^2 dx \end{aligned}$$

or

$$\begin{aligned} &\frac{d}{dt}\left[\|\eta^\nu(t)\|^2 + (1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 + \|\phi_x^\nu(t)\|^2\right] \\ &+\frac{d}{dt}\left[\int_{L_1}^{L_2} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1}^{L_2} |\theta^\nu|^2 dx + \int_{L_1}^{L_2} |\theta_x^\nu|^2 dx\right] \\ &+2(\beta-\delta)\|\eta^\nu(t)\|^2 + 2\delta(1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 \\ &+2\delta\|\phi_x^\nu(t)\|^2 + 2\delta^3 \int_{L_1}^{L_2} |\theta^\nu|^2 dx + 2\delta \int_{L_1}^{L_2} |\theta_x^\nu|^2 dx = \\ &+2 \int_\Omega \eta^\nu |\psi^\nu|^2 dx + 2\delta \int_{L_1}^{L_2} |\gamma^\nu|^2 dx = \\ &+2 \int_\Omega \phi_t^\nu |\psi^\nu|^2 dx + 2\delta \int_\Omega \phi |\psi^\nu|^2 dx + 2\delta \int_{L_1}^{L_2} |\gamma^\nu|^2 dx \end{aligned} \tag{3.14}$$

then (3.12) + (3.14) implies that

$$\frac{d}{dt} H^\nu(t) + I^\nu(t) = 0 \tag{3.15}$$

where

$$\begin{aligned} H^\nu(t) &= 2\|\psi_x^\nu(t)\|^2 - 2\int_\Omega \phi^\nu|\psi^\nu|^2dx + \|\eta^\nu(t)\|^2 \\ &\quad +(1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 + \|\phi_x^\nu(t)\|^2 \\ &\quad + \int_{L_1}^{L_2}|\gamma^\nu|^2dx + \delta^2\int_{L_1}^{L_2}|\theta^\nu|^2dx + \int_{L_1}^{L_2}|\theta_x^\nu|^2dx, \end{aligned} \tag{3.16}$$

$$\begin{aligned} I^\nu(t) &= +4\alpha\|\psi_x^\nu\|^2 - 2(2\alpha+\delta)\int_\Omega \phi^\nu|\psi^\nu|^2dx + 2(\beta-\delta)\|\eta^\nu(t)\|^2 \\ &\quad +2\delta(1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 + 2\delta\|\phi_x^\nu(t)\|^2 \\ &\quad +2\delta^3\int_{L_1}^{L_2}|\theta^\nu|^2dx + 2\delta\int_{L_1}^{L_2}|\theta_x^\nu|^2dx - 2\delta\int_{L_1}^{L_2}|\gamma^\nu|^2dx \end{aligned} \tag{3.17}$$

For arbitrary $\epsilon_1, \epsilon_2 > 0$,

$$\left|\int_\Omega \phi^\nu|\psi^\nu|^2dx\right| \le \epsilon_1\|\psi_x^\nu(t)\|^2 + \epsilon_2\|\phi_x^\nu(t)\|^2 + c(\epsilon_1,\epsilon_2)\|\psi^\nu(t)\|^6 \tag{3.18}$$

Taking $\epsilon_1 = \frac{1}{2}$, $\epsilon_2 = \frac{1}{4}$ in (3.18), we deduce that

$$\begin{aligned} H^\nu(t) &\ge \|\psi_x^\nu(t)\|^2 + \|\eta^\nu(t)\|^2 + (1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 + \frac{1}{2}\|\phi_x^\nu(t)\|^2 \\ &\quad + \int_{L_1}^{L_2}|\gamma^\nu|^2dx + \delta^2\int_{L_1}^{L_2}|\theta^\nu|^2dx + \int_{L_1}^{L_2}|\theta_x^\nu|^2dx - c\|\psi^\nu(t)\|^6, \end{aligned} \tag{3.19}$$

$$\begin{aligned} H^\nu(t) &\le 3\|\psi_x^\nu(t)\|^2 + \|\eta^\nu(t)\|^2 + (1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 + \frac{3}{2}\|\phi_x^\nu(t)\|^2 \\ &\quad +c\|\psi^\nu(t)\|^6 + \int_{L_1}^{L_2}|\gamma^\nu|^2dx + \delta^2\int_{L_1}^{L_2}|\theta^\nu|^2dx + \int_{L_1}^{L_2}|\theta_x^\nu|^2dx, \end{aligned} \tag{3.20}$$

Taking $\epsilon_1 = \frac{\alpha}{2\alpha+\delta}$, $\epsilon_2 = \frac{\delta}{2(2\alpha+\delta)}$ in (3.18), we see that

$$\begin{aligned} I^\nu(t) &\ge +2\alpha\|\psi_x^\nu\|^2 + 2(\beta-\alpha)\|\eta^\nu(t)\|^2 + 2\delta(1-\delta(\beta-\delta))\|\phi^\nu(t)\|^2 \\ &\quad +\delta\|\phi_x^\nu(t)\|^2 - c\|\psi^\nu(t)\|^6 + 2\delta^3\int_{L_1}^{L_2}|\theta^\nu|^2dx \\ &\quad +2\delta\int_{L_1}^{L_2}|\theta_x^\nu|^2dx - 2\delta\int_{L_1}^{L_2}|\gamma^\nu|^2dx \end{aligned} \tag{3.21}$$

Thus from (3.20) and (3.21) we find a $\beta_1 > 0$ such that

$$\beta_1 H^\nu(t) \le I^\nu(t) + C\|\psi^\nu(t)\|^6 + C\int_{L_1}^{L_2}|\gamma^\nu|^2dx. \tag{3.22}$$

Therefore we derive from (3.15) and (3.22) that

$$\frac{d}{dt}H^\nu(t)+\beta_1 H^\nu(t)\le C\|\psi^\nu(t)\|^6+C\int_{L_1}^{L_2}|\gamma^\nu|^2dx. \tag{3.23}$$

From (3.4) and (3.23) we obtain

$$\frac{d}{dt}H^\nu(t)+\beta_1 H^\nu(t)\le C+C\int_{L_1}^{L_2}|\gamma^\nu|^2dx. \tag{3.24}$$

It follows that

$$H^\nu(t)\le C|H^\nu(0)|+C\int_{L_1}^{L_2}|\gamma^\nu|^2dx. \tag{3.25}$$

From (3.25) and observing that $|H_\nu(0)|$ is bounded, we have

$$H^\nu(t)\le C+C\int_{L_1}^{L_2}|\gamma^\nu|^2dx. \tag{3.26}$$

From (3.19), (3.26) and using Gronwall inequality we obtain

$$\begin{aligned}&\|\psi_x^\nu(t)\|^2+\|\eta^\nu(t)\|^2+\|\phi^\nu(t)\|^2+\|\phi_x^\nu(t)\|^2\\ &+\int_{L_1}^{L_2}|\gamma^\nu|^2dx+\int_{L_1}^{L_2}|\theta^\nu|^2dx+\int_{L_1}^{L_2}|\theta_x^\nu|^2dx \quad\le\quad C(T).\end{aligned} \tag{3.27}$$

From (3.27) it follows that:

$$\psi^\nu \ \text{ is bounded in } \ L^\infty(0,T;H_L^1(\Omega)) \tag{3.28}$$

$$(\phi^\nu,\theta^\nu) \ \text{ is bounded in } \ L^\infty(0,T;V) \tag{3.29}$$

$$(\phi_t^\nu,\theta_t^\nu) \ \text{ is bounded in } \ L^\infty(0,T;L^2(\Omega)\times L^2(\,]L_1,L_2[\,)) \tag{3.30}$$

Step 4 *(Estimate III)* First, we are going to estimate $\|\psi_t^\nu(0)\|$, $\|\phi_{tt}^\nu(0)\|$ and $\|\theta_{tt}^\nu(0)\|$. Indeed, from (16)-(17) and observing that

$$\psi_{0x}(L_i)=0 \ \ ; \ \ (i=1,2)$$

$$\phi_{0x}(Li)=\theta_{0x}(L_i) \ \ ; \ \ (i=1,2)$$

we have

$$\begin{aligned}\|\psi_t^\nu(0)\|^2+\|\phi_{tt}^\nu(0)\|^2+\|\theta_{tt}^\nu(0)\|^2 \quad = \quad & i(\psi_{0xx}^\nu,\psi_t^\nu(0))-\alpha(\phi_0,\psi_t^\nu(0))\\ &+i(\phi_0\psi_0,\psi_t^\nu(0))+(\phi_{0xx},\phi_{tt}^\nu(0))\\ &-(\psi_0,\phi_{tt}^\nu(0))-\beta(\phi_1,\phi_{tt}^\nu(0))\\ &+(|\psi_0|^2,\phi_{tt}^\nu(0))+(\theta_{0xx},\theta_{tt}^\nu(0))\end{aligned} \tag{3.31}$$

If follows that

$$\|\psi_t^\nu(0)\| + \|\phi_{tt}^\nu(0)\| + \|\theta_{tt}^\nu(0)\| \quad \leq \quad C \;\; ; \;\; \forall \;\; \nu \in \mathbb{N} \tag{3.32}$$

Now, taking the derivate of (3.1) and (3.2) with respect to t and, using arguments of step 3, we get that

$$\begin{array}{rl} & \frac{1}{2}\frac{d}{dt}\left[\|\psi_t^\nu(t)\|^2 + \|\phi_{tt}^\nu(t)\|^2 + \|\phi_{xt}^\nu(t)\|^2 + \|\phi_t^\nu(t)\|^2\right] \\ & +\frac{1}{2}\frac{d}{dt}\left[\int_{L_1}^{L_2}|\theta_{tt}^\nu|^2dx + \int_{L_1}^{L_2}|\theta_{xt}^\nu|^2dx\right] + \alpha\|\psi_t^\nu\|^2 + \beta\|\phi_{tt}^\nu\|^2 \\ = & -Im\int_\Omega(\phi_t^\nu\psi^\nu\overline{\psi_t^\nu})dx + 2\int_\Omega\psi^\nu\overline{\psi_t^\nu}\phi^\nu dx \\ \leq & C[\|\psi_t^\nu(t)\|^2 + \|\phi_{xt}^\nu(t)\|^2 + \|\phi_{xt}^\nu(t)\|^2] \end{array} \tag{3.33}$$

Integration (3.33) from zero to t, for $0 \leq t \leq T$, $T > 0$ any real number and observing the estimate (3.32), we have

$$\begin{array}{rl} & \|\psi_t^\nu(t)\|^2 + \|\phi_{tt}^\nu(t)\|^2 + \|\phi_{xt}^\nu(t)\|^2 + \|\phi_t^\nu(t)\|^2 \\ & +\int_{L_1}^{L_2}|\theta_{tt}^\nu|^2dx + \int_{L_1}^{L_2}|\theta_{xt}^\nu|^2dx \\ \leq & C + C\int_0^T[\|\psi_t^\nu(s)\|^2 + \|\phi_{xt}^\nu(s)\|^2 + \|\phi_{tt}^\nu(s)\|^2]ds \end{array} \tag{3.34}$$

Applying Gronwall inequality to (3.34), we obtain:

$$\begin{array}{rl} & \|\psi_t^\nu(t)\|^2 + \|\phi_{tt}^\nu(t)\|^2 + \|\phi_{xt}^\nu(t)\|^2 \\ & +\|\phi_t^\nu(t)\|^2 + \int_{L_1}^{L_2}|\theta_{tt}^\nu|^2dx + \int_{L_1}^{L_2}|\theta_{xt}^\nu|^2dx \quad \leq \quad C \end{array} \tag{3.35}$$

independent of ν, for all t in $[0, T]$.

From (3.35) it follows that:

$$\psi_t^\nu \;\text{ is bounded in }\; L^\infty(0, T; L^2(\Omega)) \tag{3.36}$$

$$(\phi_t^\nu, \theta_t^\nu) \;\text{ is bounded in }\; L^\infty(0, T; V) \tag{3.37}$$

$$(\phi_{tt}^\nu, \theta_{tt}^\nu) \;\text{ is bounded in }\; L^\infty(0, T; L^2(\Omega) \times L^2(\,]L_1, L_2[\,)) \tag{3.38}$$

The rest of the proof of the existence of strong solution is a matter routine.

The existence of weak solution to system $[(1.1) - (1.9)]$ is given in the following theorem:

Theorem 2 *Given*

$$\psi_0 \in H_L^1(\Omega) \;\;,\quad \{\phi_0, \theta_0\} \in V \;\; \textit{and} \;\; \{\phi_1, \theta_1\} \in L^2(\Omega) \times L^2(]L_1, L_2[)$$

there exists only a weak solution of $[(1.1) - (1.9)]$.

Proof. Given $\psi_0 \in H^1_L(\Omega)$, $\{\phi_0, \theta_0\} \in V$ and $\{\phi_1, \theta_1\} \in L^2(\Omega) \times L^2(]L_1, L_2[)$, there exists $\psi_0^\nu \in H^2(\Omega) \cap H^1_L(\Omega)$, $\{\phi_0^\nu, \theta_0^\nu\} \in [H^2(\Omega) \times H^2(]L_1, L_2[)] \cap V$ and $\{\phi_1, \theta_1\} \in V$ such that

$$\begin{array}{rclll} \psi_0^\nu & \longrightarrow & \psi_0 & \text{strongly} & in\ \ H^1_L(\Omega) \\ \{\phi_0^\nu, \theta_0^\nu\} & \longrightarrow & \{\phi_0, \theta_0\} & \text{strongly} & in\ \ V \\ \{\phi_1^\nu, \theta_1^\nu\} & \longrightarrow & \{\phi_1, \theta_1\} & \text{strongly} & in\ \ L^2(\Omega) \times L^2(]L_1, L_2[) \end{array} \tag{3.39}$$

and

$$\begin{array}{rcl} \psi_{0x}(L_i) = 0 & ; & (i = 1, 2) \\ \phi_{0x}(Li) = \theta_{0x}(L_i) & ; & (i = 1, 2) \end{array}$$

With ψ_0^ν, $\{\phi_0^\nu, \theta_0^\nu\}$ and $\{\phi_1^\nu, \theta_1^\nu\}$, above defined, we determine an unique strong solution $\{\psi, \phi^\nu, \theta^\nu\}$ satisfying all conditions of **Theorem** (1).

Using similar arguments of step 3 of **Theorem** (1). we have

$$\begin{array}{rlll} \psi^\nu & is & bounded\ \ in & L^\infty(0, T, H^l_L(\Omega)) \\ \{\phi^\nu, \theta^\nu\} & is & bounded\ \ in & L^\infty(0, T, V) \\ \{\phi_t^\nu, \theta_t^\nu\} & is & bounded\ \ in & L^\infty(0, T, L^2(\Omega) \times L^2(]L_1, L_2[)) \end{array}$$

If follows that

$$\begin{array}{rcccl} \psi^\nu & \overset{*}{\rightharpoonup} & \psi & in & L^\infty(0, T, H^1_L(\Omega)) \\ \{\phi^\nu, \theta^\nu\} & \overset{*}{\rightharpoonup} & \{\phi, \theta\} & in & L^\infty(0, T, V) \\ \{\phi_t^\nu, \theta_t^\nu\} & \overset{*}{\rightharpoonup} & \{\phi_t, \theta_t\} & in & L^\infty(0, T, L^2(\Omega) \times L^2(]L_1, L_2[)) \end{array}$$

We suppose that $\{\psi^\nu, \phi^\nu, \theta^\nu\}$ and $\{\psi^\sigma, \phi^\sigma, \theta^\sigma\}$ are two strong solutions of $[(1.1) - (1.9)]$ with initial data

$$\{\psi_0^\nu, \phi_0^\nu, \theta_0^\nu\} \quad and \quad \{\psi^\sigma, \phi^\sigma, \theta^\sigma\}$$

After direct calculations, we have

$$\begin{array}{rl} & \frac{1}{2}\frac{d}{dt}E^{\nu\sigma}(t) + \alpha\|\psi^\nu - \psi^\sigma\|^2 + \beta\|\phi_t^\nu - \phi_t^\sigma\|^2 \\ \leq & C\left[\|\psi^\nu(t) - \psi^\sigma(t)\|^2 + \|\phi_t^\nu(t) - \phi_t^\sigma(t)\|^2 + \|\phi^\nu(t) - \phi^\sigma(t)\|^2\right] \end{array} \tag{3.40}$$

where

$$\begin{array}{rcl} E^{\nu\sigma}(t) & = & \|\psi^\nu(t) - \psi^\sigma(t)\|^2 + \|\phi_t^\nu(t) - \phi_t^\sigma(t)\|^2 + \|\phi_x^\nu(t) - \phi_x^\sigma(t)\|^2 \\ & & +\|\phi^\nu(t) - \phi^\sigma(t)\|^2 + \displaystyle\int_{L_1}^{L_2} |\theta_t^\nu - \theta_t^\sigma|^2 dx + \int_{L_1}^{L_2} |\theta_x^\nu - \theta_x^\sigma|^2 dx \end{array} \tag{3.41}$$

By Gronwall inequality, we have

$$E^{\nu\sigma}(t) \quad \leq \quad C(T)E^{\nu\sigma}(t) \tag{3.42}$$

From (3.39) and (3.42), we obtain

$$\begin{array}{rcll} \psi^\nu & \longrightarrow & \psi & in \quad C([0,T];L^2(\Omega)) \\ \{\phi^\nu,\theta^\nu\} & \longrightarrow & \{\phi,\theta\} & in \quad C([0,T];V) \\ \{\phi_t^\nu,\theta_t^\nu\} & \longrightarrow & \{\phi_t,\theta_t\} & in \quad C([0,T];L^2(\Omega)\times L^2(]L_1,L_2[)) \end{array} \tag{3.43}$$

The rest of the proof of the existence of weak solution is a matter routine.

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