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The nonlinear transmission problem with memory

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ABSTRACT: In this work we study a nonlinear transmission problem for the wave equation with boundary dissipation of memory type. The material is constituted by two different elastic components. We have a transmission problem with damping boundary condition of memory type. We prove the global existence and uniformly decay of the solution to zero as time goes to infinity.

Key words: Wave Equation, Asymptotic Behavior, memory.

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1. Introduction

In this work we model the oscillation of a solid composed by two different elastic materials, and we suppose that its external boundary is inside a viscoelastic fluid producing a dissipative mechanism of memory type while its internal boundary is clamped. The corresponding mathematical equations which model this situation is called a transmission problem with boundary dissipation.

Boundary dissipation was studied for several authors, for example, [19,20,21,22, 24,25,26,30] and the references therein, all of them dealing with frictional damping. Models with memory dissipation are physically and mathematically more interesting, physically because our model follows the constitutive equations for materials with memory and Mathematically because the estimates we need to show the exponential decay are more delicate and depends on the relaxation function, see for example [2] and the references therein.

Memory dissipation is produced by the interaction of materials with memory. Such types of dissipation are subtle and their analysis are more delicate than the frictional damping, because introduce another type of technical difficulties. So, we have only a few works in this direction.

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In this work we show the existence of solutions of a nonlinear transmission problem with boundary dissipation of memory type. Moreover we will prove that under suitable conditions on the relaxation functions the solution will decay uniformly as time goes to infinity. The transmission problem we consider here is the following

$$\rho_1 u_{tt} - \gamma_1 \Delta u + f(u) = 0, \quad \text{in} \quad \Omega_1 \times]0,T[, \tag{1.1}$$

$$\rho_2 v_{tt} - \gamma_2 \Delta v + g(v) = 0, \quad \text{in} \quad \Omega_2 \times]0,T[, \tag{1.2}$$

with boundary condition

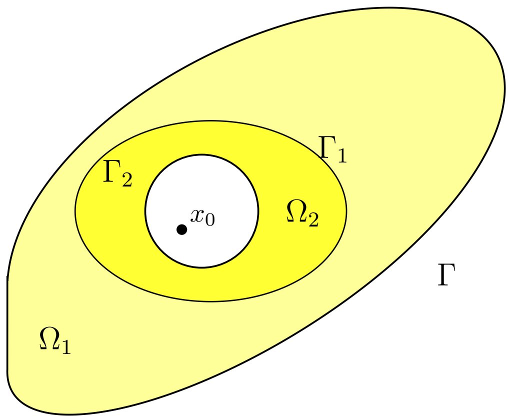
$$u(x,t) + \int_0^t k(t-\tau)\frac{\partial u}{\partial \nu} d\tau = 0 \qquad \text{on} \qquad \Gamma \tag{1.3}$$

and satisfying the transmission condition

$$u = v, \qquad \text{and} \qquad \gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial v}{\partial \nu} \qquad \text{on} \qquad \Gamma_1. \tag{1.4}$$

Additionally we assume that v satisfies Dirichlet boundary condition over Γ_2, that is

$$v(x,t) = 0, \quad \text{on} \quad \Gamma_2 \times]0,T[, \tag{1.5}$$



and verifies the following initial conditions

$$u(x,0) = u_0(x), \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega_1$$
$$v(x,0) = v_0(x), \quad \text{and} \quad v_t(x,0) = v_1(x) \quad \text{in } \Omega_2.$$

2. Existence of solutions

Lemma 2.1 *For any function $\alpha \in C^1$ and for any $\varphi \in W^{1,2}(0,T)$ we have that*

$$\int_0^t \alpha(t-\tau)\varphi(\tau)d\tau\varphi_t = -\frac{1}{2}\alpha(t)|\varphi(t)|^2 + \frac{1}{2}\alpha'\Box\varphi - \frac{1}{2}\frac{d}{dt}\left\{\alpha\Box\varphi - \left(\int_0^t \alpha\right)|\varphi|^2\right\}. \tag{2.1}$$

Let us denote by a a function satisfying

$$k(0)a + k' * a = -\frac{k'}{k(0)}. \tag{2.2}$$

By $*$ we are denoting the convolution product, that is $k * g(\cdot,t) = \int_0^t k(t-\tau)g(\cdot,\tau)\,d\tau$. The function a is called the resolvent kernel of k. Using the Volterra's resolvent, we have

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)}u_t - a * u_t$$

after performing an integration by parts, the above identity is equivalent to

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)}u_t - a(0)u - a' * u + a(t)u_0. \tag{2.3}$$

The hypotheses we use on a are the following

$$a(t) > 0, \quad a'(t) < 0, \quad a''(t) > 0, \ \forall t \geq 0 \tag{2.4}$$

$$-c_0 a'(t) \leq a''(t) \leq -c_1 a'(t), \ \forall t \geq 0, \tag{2.5}$$

where c_i are positive constants. To facilitate our calculation we introduce the following notations

$$(\alpha \Box f)(t) = \int_0^t \alpha(t-\tau)\,|f(t) - f(\tau)|^2\,d\tau, \tag{2.6}$$

$$(\alpha \Diamond f)(t) = \int_0^t g(t-\tau)\,[f(t) - f(\tau)]\,d\tau. \tag{2.7}$$

We easily see that

$$(\alpha * f)(t) = \left(\int_0^t \alpha(s)ds\right) f(t) - (\alpha \Diamond f)(t). \tag{2.8}$$

About the hypothesis (2.2) we known that the behavior of a is similar as the behavior of k. We can find the following lemma in [33]. If b and α satisfy

$$b + \alpha = -b * \alpha,$$

then

Lemma 2.2 *(i) Let us suppose that*

$$|\alpha(t)| \leq c_\alpha \mathrm{e}^{-\gamma t}, \ \forall t > 0$$

for some $\gamma > 0$ *and* $c_\alpha > 0$*, then for any* $0 < \varepsilon < \gamma$ *and* $c_\alpha < \gamma - \varepsilon$ *we have*

$$|b(t)| \leq \frac{c_\alpha(\gamma - \varepsilon)}{\gamma - \varepsilon - c_\alpha}\mathrm{e}^{-\varepsilon t}, \ \forall t > 0.$$

(ii) If α satisfies

$$|\alpha(t)| \leq c_\alpha (1+t)^{-p},$$

for some $p > 1$, $c_\alpha > 0$ and

$$\frac{1}{c_\alpha} > c_p := \sup_{0 \leq t < \infty} \int_0^t (1+t)^p (1+t-\tau)^{-p} (1+\tau)^{-p} d\tau,$$

then we have

$$|b(t)| \leq \frac{c_\alpha}{1 - c_\alpha c_p} (1+t)^{-p}, \forall\, t > 0.$$

Let us introduce the following vector spaces

$$W = \{ w \in H^1(\Omega_2); \quad w(x) = 0 \text{ on } \Gamma_2 \}$$

and

$$V = \{(u,v) \in H^1(\Omega_1) \times W;\ u = v \ \text{ on } \ \Gamma_1\}.$$

let us consider $f, g \in C^1(\mathbb{R})$ satisfying

$$|f(s)| \leq C_1 |s|^\rho + C_2 \quad \text{and} \quad |g(s)| \leq C_1 |s|^\rho + C_2, \tag{2.9}$$

$$|f'(s)| \leq C_1 |s|^{\rho-1} + C_2 \quad \text{and} \quad |g'(s)| \leq C_1 |s|^{\rho-1} + C_2, \tag{2.10}$$

where C_1 and C_2 are positive constants and $1 \leq \rho < \infty$ when the space dimension $n \leq 2$ and we take $1 \leq \rho \leq \frac{n}{n-2}$ when $n \geq 3$. We also assume that for any $s \in \mathbb{R}$,

$$F(s) = \int_0^s f(\sigma)\, d\sigma \geq 0 \quad \text{and} \quad G(s) = \int_0^s g(\sigma)\, d\sigma \geq 0. \tag{2.11}$$

Let us introduce the definition of weak solution to system (1.1)–(1.5).

Definition 2.3 *We say that the couple (u, v) is a weak solution of (1.1)–(1.5) when*

$$(u,v) \in L^\infty(0,T;V) \quad \textit{and} \quad (u_t, v_t) \in L^\infty(0,T; L^2(\Omega_1) \times L^2(\Omega_2)),$$

and satisfies the following identity

$$\begin{aligned}
&\int_0^T \int_{\Omega_1} [\rho_1 u \phi_{tt} + \gamma_1 \nabla u \nabla \phi + f(u)\phi]\, dxdt \\
&\qquad + \int_0^T \int_{\Omega_2} [\rho_2 v \psi_{tt} + \gamma_2 \nabla v \nabla \psi + g(v)\psi]\, dxdt \\
&= \int_{\Omega_1} u_1 \phi(0) dx - \int_{\Omega_1} u_0 \phi_t(0) dx + \int_{\Omega_2} v_1 \psi(0) dx - \int_{\Omega_2} v_0 \psi_t(0) dx \\
&\qquad - \int_\Gamma \left(\frac{1}{k(0)} u_t + a(0) u + a' * u - a(t) u_0 \right) \phi\, d\Gamma,
\end{aligned}$$

for any $(\phi, \psi) \in C^2(0,T;V)$ such that

$$\phi(T) = \phi_t(T) = \psi(T) = \psi_t(T) = 0.$$

In order to show the existence of strong solutions we need a regularity result for the elliptic system associated to the problem (1.1)–(1.5). For the reader's convenience we recall the following result whose proof can be found in the book by O. A. Ladyzhenskaya and N. N. Ural'tseva ([34], Theorem 16.2).

Lemma 2.4 *For any given functions $F \in L^2(\Omega_1)$ and $G \in L^2(\Omega_2)$ and $g \in H^{\frac{1}{2}}(\Gamma)$ and $\gamma_1, \gamma_2 \in \mathbb{R}^+$, there exists only one solution (u, v) of*

$$\begin{aligned} -\gamma_1 \Delta u &= F \quad in \quad \Omega_1, \\ -\gamma_2 \Delta v &= G \quad in \quad \Omega_2, \\ v(x) &= 0 \quad on \quad \Gamma_2 \\ \frac{\partial u}{\partial \nu} &= g, \ \ on\ \Gamma, \end{aligned}$$

$$u(x) = v(x) \quad on \quad \Gamma_1 \quad and \quad \gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial v}{\partial \nu} \quad on \quad \Gamma_1,$$

satisfying

$$u \in H^2(\Omega_1) \quad and \quad v \in H^2(\Omega_2).$$

The existence result is summarized in the following theorem.

Theorem 2.5 *Let us suppose that f and g are C^1-functions verifying conditions (2.9)–(2.10) and let us take initial data such that*

$$(u_0, v_0) \in V\, and \quad (u_1, v_1) \in L^2(\Omega_1) \times L^2(\Omega_2),\ u_0 = 0 \ \ on\ \ \Gamma.$$

Then, there exists a solution (u, v) of system (1.1)–(1.5) satisfying

$$(u, v) \in C(0, T; V) \cap C^1(0, T; L^2(\Omega_1) \times L^2(\Omega_2)).$$

In addition, if

$$(u_0, v_0) \in H^2(\Omega_1) \times H^2(\Omega_2) \quad and \quad (u_1, v_1) \in V,$$

satisfying the compatibility conditions

$$\frac{\partial u_0}{\partial \nu} = -\frac{1}{k(0)} u_1 - a u_0 \ \ on\, \Gamma$$

$u_0 = v_0$ *and* $\gamma_1 \frac{\partial u}{\partial \nu} = \gamma_2 \frac{\partial v}{\partial \nu}$, *on* Γ_1 *then*

$$(u, v) \in C(0, T; H^2(\Omega_1) \times H^2(\Omega_2)) \cap C^1(0, T; V) \cap C^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2)).$$

Proof: To show the existence of solutions we use the Galerkin methods. □

3. Asymptotic behavior

In this section we prove that the solution decay exponentially as time go to infinity. First of all we need some preliminaries results.

Lemma 3.1 *Under above notation we have that*

$$\frac{d}{dt}E(t) = -\frac{1}{k(0)}\int_\Gamma |u_t|^2\, d\Gamma + \frac{a'(t)}{2}\int_\Gamma |u|^2\, d\Gamma - \frac{1}{2}\int_\Gamma a''\Box u\, d\Gamma,$$

where

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega_1} \rho_1|u_t|^2 + \gamma_1|\nabla u|^2 + 2F(u)dx + \gamma_1\int_\Gamma a(t)|u|^2 - a'\Box u\, d\Gamma \\ &\quad + \frac{1}{2}\int_{\Omega_2} \rho_2|v_t|^2 + \gamma_2|\nabla v|^2 + 2G(v)dx. \end{aligned} \tag{3.1}$$

Proof: Multiply by u_t equation (1.1) and by v_t equation (1.2), summing up and using identity (2.3) and Lemma 2.1 we get the result. □ Let us take f and g such that

$$0 \le F(s) := \int_0^s f(t)dt \le \frac{1}{m+1} sf(s), \tag{3.2}$$

$$0 \le G(s) := \int_0^s g(t)dt \le \frac{1}{l+1} sg(s), \tag{3.3}$$

$$F(s) \le G(s) \tag{3.4}$$

where $l, m > 1$, and let us consider

$$\delta < \min\left\{\frac{l-1}{l+1}n, \frac{m-1}{m+1}n, 1\right\}. \tag{3.5}$$

Let us denote by

$$J_0(t) = \int_{\Omega_1} \rho_1 u_t q\cdot\nabla u\, dx + \int_{\Omega_2} \rho_2 v_t q\cdot\nabla v\, dx.$$

Lemma 3.2 *Let us consider $q(x) = x - x_0 \in C^1(\overline{\Omega})$, $\gamma_1 > \gamma_2$ and $\rho_1 > \rho_2$. Then any strong solution of (1.1)–(1.5) satisfies:*

$$\begin{aligned} \frac{d}{dt}J_0(t) &\le \gamma_1\int_\Gamma \frac{\partial u}{\partial \nu} q\cdot\nabla u\, dx - \frac{\gamma_1}{2}\int_\Gamma q\cdot\nu|\nabla u|^2\, dx + \frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2\, d\Gamma \\ &\quad -\frac{n}{2}\int_{\Omega_1} \rho_1|u_t|^2 - \gamma_1|u|^2 dx + n\int_{\Omega_1} F(u)dx - \gamma_1\int_{\Omega_1} |\nabla u|^2\, dx \\ &\quad -\frac{n}{2}\int_{\Omega_2} \rho_2|v_t|^2 - \gamma_2|\nabla v|^2\, dx + n\int_{\Omega_2} G(v)\, dx - \gamma_2\int_{\Omega_2} |\nabla v|^2 dx. \end{aligned}$$

Lemma 3.3 *Under the above relations we have that*

$$\begin{aligned}\frac{d}{dt}\left\{\int_{\Omega_1}\rho_1 u u_t dx + \int_{\Omega_2}\rho_2 v_t v dx\right\} &= \int_{\Omega_1}\rho_1 |u_t|^2 - \gamma_1|\nabla u|^2 dx + \gamma_1\int_\Gamma \frac{\partial u}{\partial\nu} u d\Gamma - \int_{\Omega_1} f(u) u dx\\ &+ \int_{\Omega_2}\rho_2 |v_t|^2 - \gamma_2|\nabla v|^2 dx - \int_{\Omega_2} g(v) v dx.\end{aligned}$$

Proof: Multiply (1.1) by u and (1.2) by v and summing up the product the our result follows. □

Let us define the functional

$$\Phi(t) = J_0(t) + \left(\frac{n-\delta}{2}\right)\left[\int_{\Omega_1}\rho_1 u u_t dx + \int_{\Omega_2}\rho_2 v_t v dx\right]$$

where we consider $q(x) = x - x_0$ as before.

Lemma 3.4 *Under the above hypothesis of Lemma 3.2 we have that there exist positive constant δ_0 such that*

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\le C\int_\Gamma \left|\frac{\partial u}{\partial\nu}\right|^2 d\Gamma + \left(\frac{n-\delta}{2}\right)\gamma_1\int_\Gamma u\frac{\partial u}{\partial\nu} d\Gamma - \delta_0 E_0(t)\\ &+\frac{\rho_1}{2}\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma,\end{aligned}$$

where

$$E_0(t) = \frac{1}{2}\int_{\Omega_1}\rho_1|u_t|^2 + \gamma_1|\nabla u|^2 + F(u)dx + \frac{1}{2}\int_{\Omega_2}\rho_2|v_t|^2 + \gamma_2|\nabla v|^2 + G(v)dx.$$

Finally we have,

Theorem 3.5 *With the same hypotheses as Lemma 3.4 we have that there exists a positive constants such that*

$$E(t) \le CE(0)\exp(-\delta_1 t).$$

Proof: Note that from (1.3) and (2.8) we have

$$\frac{\partial u}{\partial\nu} = -\frac{1}{k(0)}u_t - a(t)u - a'\diamondsuit u$$

from where it follows

$$|\frac{\partial u}{\partial\nu}|^2 \le 2\left\{\frac{1}{k^2(0)}|u_t|^2 + a^2(t)|u|^2 + |a'\diamondsuit u|^2\right\}.$$

Since

$$|a'\diamondsuit u|^2 = \left|\int_0^t a'(t-s)\{u(s)-u(t)\}ds\right|^2 \le \left(\int_0^t |a'(t-s)|ds\right)|a'|\Box u.$$

From where and (2.5) it follows that

$$\left|\frac{\partial u}{\partial \nu}\right|^2 \leq k_0\{|u_t|^2 + a(t)|u|^2 + a'\Box u\}. \tag{3.6}$$

On the other hand,

$$\begin{aligned}
\left|\int_\Gamma u\frac{\partial u}{\partial \nu}d\Gamma\right| &\leq \left(\int_\Gamma |u|^2\, d\Gamma\right)^{\frac{1}{2}}\left(\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma\right)^{\frac{1}{2}} \\
&\leq \delta_1 \int_\Gamma |u|^2\, d\Gamma + C_{\delta_1}\int_\Gamma \left\{|u_t|^2 + a(t)|u|^2 + a'\Box u\right\} d\Gamma \\
&\leq \delta_1 \int_\Gamma |u|^2 d\Gamma + C\int_\Gamma \left\{|u_t|^2 + |u|^2 + a'\Box u\right\} d\Gamma. \qquad (3.7)
\end{aligned}$$

Since

$$\int_\Gamma |u|^2\, d\Gamma \leq C\int_\Omega |\nabla u|^2 + |\nabla v|^2\, dx.$$

we have that

$$\mathcal{L}(t) = NE(t) + \Phi(t)$$

satisfies

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq & -\frac{N\gamma_1}{k(0)}\int_\Gamma |u_t|^2 d\Gamma + \frac{N\gamma_1 a'(t)}{2}\int_\Gamma |u|^2 d\Gamma - \frac{N\gamma_1}{2}\int_\Gamma a''\Box u d\Gamma \\
& + C\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma + \left(\frac{n-\delta}{2}\right)\gamma_1\int_\Gamma u\frac{\partial u}{\partial \nu}d\Gamma \\
& - \frac{\delta_0}{2}E_0(t) + \rho_1\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma.
\end{aligned}$$

Using (4.2) and (4.3) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\left(\frac{N\gamma_1}{k(0)} - C_2\right)\int_\Gamma |u_t|^2 d\Gamma - \left(\frac{N\gamma_1}{2} - C_2\right)\int_\Gamma a''\Box u d\Gamma - \frac{\delta_0}{2}E_0(t).$$

Then we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\delta_0}{2}E(t) \leq -c\mathcal{L}(t), \tag{3.8}$$

from where our conclusion follows. □

4. Polynomial rate of decay

Here our attention will be focused on the uniform rate of decay when k decays polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Let us consider the following hypothesis,

$$\begin{aligned}
0 &< a(t) \leq b_0(1+t)^{-p}, \\
-b_1 a^{1+\frac{1}{p}}(t) &\leq a'(t) \leq -b_2 a^{1+\frac{1}{p}}(t), \\
b_3[-a'(t)]^{1+\frac{1}{p+1}} &\leq a''(t) \leq b_4[-a'(t)]^{1+\frac{1}{p+1}},
\end{aligned} \tag{4.1}$$

where $p > 1$ and $b_i > 0$ for $i = 0, \ldots, 4$. The following lemmas will play an important role in the sequel.

Lemma 4.1 *Let m and h be integrable functions, $0 \leq r < 1$ and $q > 0$. Then, for $t \geq 0$*

$$\int_0^t |m(t-s)h(s)|ds \leq \left(\int_0^t |m(t-s)|^{1+\frac{1-r}{q}}|h(s)|ds\right)^{\frac{q}{q+1}} \left(\int_0^t |m(t-s)|^r|h(s)|ds\right)^{\frac{1}{q+1}}.$$

Proof: In fact, let us take

$$v(s) := |m(t-s)|^{1-\frac{r}{q+1}}|h(s)|^{\frac{q}{q+1}}, \quad w(s) := |m(t-s)|^{\frac{r}{q+1}}|h(s)|^{\frac{1}{q+1}}.$$

Applying Hölder's inequality to $|m(s)h(s)| = v(s)w(s)$ with exponents $\delta = \frac{q}{q+1}$ for v and $\delta^* = q+1$ for w our conclusion follows. □

Lemma 4.2 *Let us denote by $\phi \in L^\infty(0,T;L^2(\Gamma))$. Then, for $p > 1$, $0 \leq r < 1$ and $t \geq 0$, we have*

$$\left(\int_\Gamma |a'|\Box\phi d\Gamma\right)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \leq 2\left(\int_0^t |a'(s)|^r ds||\phi||^2_{L^\infty(0,t;L^2(\Gamma))}\right)^{\frac{1}{(1-r)(p+1)}} \int_\Gamma |a'|^{1+\frac{1}{p+1}}\Box\phi d\Gamma,$$

while for $r = 0$ we get

$$\left(\int_{\Gamma 1} |a'|\Box\phi d\Gamma\right)^{\frac{p+2}{p+1}} \leq 2\left(\int_0^t ||\phi(s,.)||^2_{L^2(\Gamma)}ds + t||\phi(s,.)||^2_{L^2(\Gamma)}\right)^{p+1} \int_\Gamma |a'|^{1+\frac{1}{p+1}}\Box\phi d\Gamma.$$

Proof: The above inequalities are a immediate consequence of Lemma 4.1 taking

$$m(s) := |a'(s)|, \quad h(s) := \int_\Gamma |\phi(t,x) - \phi_(s,x)|^2 d\Gamma, \quad q := (1-r)(p+1).$$

This concludes our assertion. □

Theorem 4.3 *Let us suppose that the initial data $(u_0, u_1) \in H^2(\Omega) \times V$. If the resolvent kernel $a(t)$ satisfies condition (4.1), then there is a positive constant c such that*

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0).$$

Proof: Note that from (1.3) and (2.8) we have

$$\frac{\partial u}{\partial \nu} = -\frac{1}{k(0)}u_t - a(t)u - a'\Diamond u$$

from where it follows

$$|\frac{\partial u}{\partial \nu}|^2 \leq 2\left\{\frac{1}{a^2(0)}|u_t|^2 + a^2(t)|u|^2 + |a'\Diamond u|^2\right\}.$$

Since

$$|a'\Diamond u|^2 = \left|\int_0^t a'(t-s)\{u(s)-u(t)\}ds\right|^2 \leq \left(\int_0^t |a'(t-s)|^{1-\frac{1}{p}}ds\right)[-a']^{1+\frac{1}{p}}\Box u.$$

From where and (2.5) it follows that

$$\left|\frac{\partial u}{\partial \nu}\right|^2 \leq k_0\{|u_t|^2 + [-a']^{1+\frac{1}{p}}(t)|u|^2 + [-a']^{1+\frac{1}{p}}\Box u\}. \tag{4.2}$$

On the other hand,

$$\begin{aligned}
\left|\int_\Gamma u\frac{\partial u}{\partial \nu}d\Gamma\right| &\leq \left(\int_\Gamma |u|^2\,d\Gamma\right)^{\frac{1}{2}}\left(\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma\right)^{\frac{1}{2}}\\
&\leq \delta_1\int_\Gamma |u|^2\,d\Gamma + \delta_1\int_\Gamma \left\{|u_t|^2 + [-a']^{1+\frac{1}{p}}(t)|u|^2 + [-a']^{1+\frac{1}{p}}\Box u\right\}d\Gamma\\
&\leq C\int_\Gamma \left\{|u_t|^2 + |u|^2 + [-a']^{1+\frac{1}{p}}\Box u\right\}d\Gamma. \qquad (4.3)
\end{aligned}$$

Since

$$\int_\Gamma |u|^2\,d\Gamma \leq C\int_\Omega |\nabla u|^2 + |\nabla v|^2\,dx.$$

we have that

$$\mathcal{L}(t) = NE(t) + \Phi(t)$$

satisfies

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq\; & -\frac{N\gamma_1}{k(0)}\int_\Gamma |u_t|^2 d\Gamma + \frac{N\gamma_1 a'(t)}{2}\int_\Gamma |u|^2 d\Gamma - \frac{N\gamma_1}{2}\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma\\
& + C\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma + C\int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma + \left(\frac{n-\delta}{2}\right)\gamma_1\int_\Gamma u\frac{\partial u}{\partial \nu}d\Gamma\\
& - \frac{\delta_0}{2}E_0(t) + \rho_1\int_\Gamma q\cdot\nu|u_t|^2 d\Gamma.
\end{aligned}$$

Using (4.2) and (4.3) we conclude that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq\; & -\left(\frac{N\gamma_1}{k(0)} - C_2\right)\int_\Gamma |u_t|^2 d\Gamma - \left(\frac{N\gamma_1}{2} - C_2\right)\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma\\
& - \frac{\delta_0}{2}E_0(t), \qquad (4.4)
\end{aligned}$$

from where we have that for N large enough we get

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{N\gamma_1}{2k(0)}\int_\Gamma |u_t|^2 d\Gamma - \frac{N\gamma_1}{4}\int_\Gamma [-a']^{1+\frac{1}{p}}\Box u d\Gamma - \frac{\delta_0}{2}E_0(t). \tag{4.5}$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. In this condition from hypothesis (4.1) we have

$$\int_0^\infty [-a']^r \leq c \int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for} \quad i = 1, 2, 3, 4.$$

Using this estimate in Lemma 4.2 we get

$$\int_\Gamma [-a']^{1+\frac{1}{p+1}} \Box u d\Gamma \quad \geq \quad cE(0)^{-\frac{1}{(1-r)(p+1)}} \left(\int_\Gamma [-a'] \Box u d\Gamma \right)^{1+\frac{1}{(1-r)(p+1)}}, \tag{4.6}$$

On the other hand, since the energy is bounded we have

$$E(t)^{1+\frac{1}{(1-r)(p+1)}} \leq cE(0)^{\frac{1}{(1-r)(p+1)}} E(t). \tag{4.7}$$

Substitution of (4.6)-(4.7) into (4.5) we arrive to

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \leq & -cE(0)^{-\frac{1}{(1-r)(p+1)}} E(t)^{1+\frac{1}{(1-r)(p+1)}} \\ & - cE(0)^{-\frac{1}{(1-r)(p+1)}} \left(\int_\Gamma [-a'] \Box u d\Gamma \right)^{1+\frac{1}{(1-r)(p+1)}}. \end{aligned}$$

Since there exists positive constants satisfying

$$c_0 E(t) \leq \mathcal{L}(t) \leq c_1 E(t) \tag{4.8}$$

We get

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}} \mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}}. \tag{4.9}$$

Therefore, using a Gronwall's type argument we conclude that

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}} \mathcal{L}(0). \tag{4.10}$$

Since $(1-r)(p+1) > 1$ we get, for $t \geq 0$, the following bounds

$$\begin{aligned} t\|u(t,.)\|^2_{L^2(\Gamma)} \leq ct\mathcal{L}(t) \leq \infty, \\ \int_0^t \|u(s,.)\|^2_{L^2(\Gamma)} \leq c\int_0^\infty \mathcal{L}(t) \leq \infty. \end{aligned}$$

Using the above estimates in Lemma 4.2 with $r = 0$, we get

$$\int_\Gamma [-a']^{1+\frac{1}{p+1}} \Box u d\Gamma \geq \frac{c}{E(0)^{\frac{1}{p+1}}} \left(\int_\Gamma [-a'] \Box u d\Gamma \right)^{1+\frac{1}{p+1}}.$$

Using these inequalities and the same arguments as in the derivation of (4.9), we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}}.$$

So we obtain

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}}\mathcal{L}(0),$$

using inequality (4.8) we conclude that

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0),$$

which completes the proof. □

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