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Splitting 3-plane sub-bundles over the product of two real projective spaces

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ABSTRACT: Let α be a real vector bundle of fiber dimension three over the product $\mathbb{R}P(m) \times \mathbb{R}P(n)$ which splits as a Whitney sum of line bundles. We show that the necessary and sufficient conditions for α to embed as a sub-bundle of a certain family of vector bundles β of fiber dimension $m+n$ is the vanishing of the last three Stiefel-Whitney classes of the virtual bundle0 $\beta - \alpha$. Among the target bundles β we consider the tangent bundle.[1]

Contents

The problem of deciding if a vector bundle α can be realized as a sub-bundle of another vector bundle β over a manifold M has been considered by several authors. Immersion problems and also the existence of a k-field frame on a manifold M are among the applications of this question. The most used techniques to approach such problems are Postnikov decomposition ([5], [6]) and the singularity method developed by Ulrich Koschorke [2].

This question can also be formulated as the existence of a monomorphism of vector bundles from α into β. In this paper the manifold is the product of two real projective spaces $\mathbb{R}P(m) \times \mathbb{R}P(n)$, α is a vector bundle of fiber dimension 3 and β has the same fiber dimension $m+n$ as the dimension of the manifold and they are listed below.

α	β
1) ε^3	1) ε^{m+n}
2) $\gamma \oplus \varepsilon^2$	2) $TP(m) \oplus \varepsilon^n$
3) $\gamma \oplus \gamma \oplus \varepsilon^1$	3) $\gamma^\perp \oplus \varepsilon^n$
4) $\gamma \oplus \gamma \oplus \gamma$	4) $TP(m) \oplus TP(n)$
5) $\varepsilon^2 \oplus \xi$	5) $\gamma^\perp \oplus TP(n)$
6) $\varepsilon^1 \oplus \xi \oplus \xi$	6) $\gamma^\perp \oplus \xi^\perp$
7) $\xi \oplus \xi \oplus \xi$	7) $\varepsilon^m \oplus TP(n)$
8) $\gamma \oplus \xi \oplus \varepsilon^1$	8) $\varepsilon^m \oplus \xi^\perp$
9) $\gamma \oplus \gamma \oplus \xi$	9) $TP(m) \oplus \xi^\perp$
10) $\gamma \oplus \xi \oplus \xi$	

Here ε^n always represents the trivial vector bundle of dimension n, γ and ξ are the canonical line bundles over the projective spaces $\mathbb{R}P(m)$ and $\mathbb{R}P(n)$, respectively. The bundles $TP(m)$ and $TP(n)$ are their tangent bundles. We denote by

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$\gamma^\perp$ and $\xi^\perp$ the orthogonal complement of γ and ξ, respectively. We recall that $\gamma \oplus \gamma^\perp \cong \varepsilon^{m+1}$ and $\gamma \otimes \gamma^\perp \cong TP(m)$ over $\mathbb{R}P(m)$ while $\xi \oplus \xi^\perp \cong \varepsilon^{n+1}$ and $\xi \otimes \xi^\perp \cong TP(n)$ over $\mathbb{R}P(n)$. Let p be the projection of $\mathbb{R}P(m) \times \mathbb{R}P(n)$ over any of the factors. We denote the pullback of any vector bundle under p and the vector bundle itself by the same notation. We assume m and n to be greater or equal than 3.

A motivation for considering this list of vector bundles α comes from the following facts:

1. Any vector bundle of fiber dimension two over $\mathbb{R}P(m)$ is isomorphic to either ε^2, $\varepsilon^1 \oplus \gamma$ or $\gamma \oplus \gamma$.

2. Any vector bundle of fiber dimension three over $\mathbb{R}P(m)$ that is a restriction of a vector bundle over $\mathbb{R}P(\infty)$ is decomposable as a Whitney sum of line bundles.

Fact 1 can be verified by noticing that oriented vector bundles of fiber dimension 2 over $\mathbb{R}P(m)$ are classified by $H^2(\mathbb{R}P(m), \mathbb{Z})$ which is isomorphic to $\mathbb{Z}_2$. On the other hand, nonorientable vector bundles of fiber dimension 2 are classified by $H^2(\mathbb{R}P(m), \mathbb{Z}_w)$, the cohomology group with coefficients twisted by $w = w_1(\gamma)$ and for $m \geq 3$ this group is trivial [4].

Fact 2 follows from the fact that there is a bijection between $[\mathbb{R}P(\infty), BO(3)]$, the set of homotopy classes of maps from $\mathbb{R}P(\infty)$ to $BO(3)$ and $Rep(\mathbb{Z}_2, O(3))$, the set of equivalence classes of representation $\mathbb{Z}_2$ in $O(3)$. This follows from a result of Dwyer and Zabrodsky ([1] or [3]). Since $Rep(\mathbb{Z}_2, O(3))$ is equal to $Hom(\mathbb{Z}_2, O(3)) \ / \ Inn(O(3))$ there are four classes, corresponding to the following four non isomorphic vector bundles: ε^3, $\varepsilon^2 \oplus \gamma$, $\varepsilon^1 \oplus \gamma \oplus \gamma$ and $\gamma \oplus \gamma \oplus \gamma$.

Since $H^1(\mathbb{R}P(m) \times \mathbb{R}P(n), \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the line bundles over $\mathbb{R}P(m) \times \mathbb{R}P(n)$ are isomorphic to one of the following line bundles: ε^1, γ, ξ and $\gamma \otimes \xi$.

In this work we did not consider the line bundle $\gamma \otimes \xi$ as a splitting component of α because the very first obstructions to the problem will already break into many cases.

The first evidence one can get for the existence of a monomorphism from α to β comes from the Stiefel-Whitney classes. That is, if there is a monomorphism from α into β, then there is a vector bundle, say ζ, such that $\beta \cong \alpha \oplus \zeta$ and then

$$w_{r-i}(\zeta) = w_{r-i}(\beta - \alpha) = 0,$$

for $i = 0, 1, \ldots, \dim(\alpha) - 1$, where $r = \dim(\beta)$. Then we are facing the task of computing the three last Stiefel-Whitney classes $w_i(\alpha - \beta)$, $i = m+n, m+n-1, m+n-2$, for the ninety possibilities of our original setting. This can be done rather smoothly because of the algebraic simplicity of the cohomology of the product $\mathbb{R}P(m) \times \mathbb{R}P(n)$.

We prove then, in a constructive way in most of the cases, the following theorem:

Theorem 1 *If $\alpha = r\gamma \oplus s\xi \oplus \varepsilon^t$, with $r, s, t \geq 0$ and $r + s + t = 3$ and $\beta = \beta_1 \oplus \beta_2$, with $\beta_1 = \varepsilon^m, TP(m)$ or $\gamma^\perp$, $\beta_2 = \varepsilon^n, TP(n)$ or $\xi^\perp$ over the product $\mathbb{R}P(m) \times \mathbb{R}P(n)$, where $m, n \geq 3$, then there is a monomorphism from α into β if, and only if, $w_i(\beta - \alpha) = 0$ for $i = m + n - 2, m + n - 1$ and $m + n$.*

The cases when $\beta = \varepsilon^m \oplus TP(n)$, $\varepsilon^m \oplus \xi^\perp$ and $TP(m) \oplus \xi^\perp$ are, in a sense, dual to the cases $\beta = TP(m) \oplus \varepsilon^n$, $\gamma^\perp \oplus \varepsilon^m$ and $\gamma^\perp \oplus TP(n)$, and so we only consider the first six bundles in the list on the right side.

First we compute the Stiefel-Whitney classes in order to prove the theorem.

Let u and v represent the generators of $H^1(I\!RP(m); \mathbb{Z}_2)$ and $H^1(I\!RP(n); \mathbb{Z}_2)$, respectively. Then, $H^k(I\!RP(m) \times I\!RP(n); \mathbb{Z}_2)$ is generated by all possible products $u^i v^j$ such that $i + j = k$. In particular, for $k = m + n - 2, m + n - 1$ and $m + n$ we can choose the following ordered basis:

$\{u^m v^{n-2}, u^{m-1} v^{n-1}, u^{m-2} v^n\}$ for $H^{m+n-2}(I\!RP(m) \times I\!RP(n); \mathbb{Z}_2)$,

$\{u^m v^{n-1}, u^{m-1} v^n\}$ for $H^{m+n-1}(I\!RP(m) \times I\!RP(n); \mathbb{Z}_2)$,

$\{u^m v^n\}$ for $H^{m+n}(I\!RP(m) \times I\!RP(n); \mathbb{Z}_2)$.

To avoid similar calculations that occurs in more dual cases (as when $\beta = \varepsilon^m \oplus \xi^\perp$ and $\alpha = \gamma \oplus \gamma \oplus \xi$ or $\gamma \oplus \xi \oplus \xi$) we consider the total Stiefel-Whitney classes given below. When $\alpha = \varepsilon^3$ and $\beta = \varepsilon^{m+n}, TP(m) \oplus \varepsilon^n$ and $\gamma^\perp \oplus \varepsilon^n$, the solution is clear.

1) $w(\varepsilon^{m+n} - \gamma \oplus \varepsilon^2) = (1 + u)^{-1}$
2) $w(\varepsilon^{m+n} - \gamma \oplus \gamma \oplus \varepsilon^1) = (1 + u)^{-2}$
3) $w(\varepsilon^{m+n} - \gamma \oplus \gamma \oplus \gamma) = (1 + u)^{-3}$
4) $w(\varepsilon^{m+n} - \gamma \oplus \xi \oplus \varepsilon^1) = (1 + u)^{-1}(1 + v)^{-1}$
5) $w(\varepsilon^{m+n} - \gamma \oplus \gamma \oplus \xi) = (1 + u)^{-2}(1 + v)^{-1}$
6) $w(TP(m) \oplus \varepsilon^n - \gamma \oplus \varepsilon^2) = (1 + u)^m$
7) $w(TP(m) \oplus \varepsilon^n - \gamma \oplus \gamma \oplus \varepsilon^1) = (1 + u)^{m-1}$
8) $w(TP(m) \oplus \varepsilon^n - \gamma \oplus \gamma \oplus \gamma) = (1 + u)^{m-2}$
9) $w(TP(m) \oplus \varepsilon^n - \varepsilon^2 \oplus \xi) = (1 + u)^{m+1}(1 + v)^{-1}$
10) $w(TP(m) \oplus \varepsilon^n - \varepsilon^1 \oplus \xi \oplus \xi) = (1 + u)^{m+1}(1 + v)^{-2}$
11) $w(TP(m) \oplus \varepsilon^n - \xi \oplus \xi \oplus \xi) = (1 + u)^{m+1}(1 + v)^{-3}$
12) $w(TP(m) \oplus \varepsilon^n - \gamma \oplus \xi \oplus \varepsilon^1) = (1 + u)^m(1 + v)^{-1}$
13) $w(TP(m) \oplus \varepsilon^n - \gamma \oplus \gamma \oplus \xi) = (1 + u)^{m-1}(1 + v)^{-1}$
14) $w(TP(m) \oplus \varepsilon^n - \gamma \oplus \xi \oplus \xi) = (1 + u)^m(1 + v)^{-2}$
15) $w(\gamma^\perp \oplus \varepsilon^n - \gamma \oplus \varepsilon^2) = (1 + u)^{-2}$
16) $w(\gamma^\perp \oplus \varepsilon^n - \gamma \oplus \gamma \oplus \varepsilon^1) = (1 + u)^{-3}$
17) $w(\gamma^\perp \oplus \varepsilon^n - \gamma \oplus \gamma \oplus \gamma) = (1 + u)^{-4}$
18) $w(\gamma^\perp \oplus \varepsilon^n - \varepsilon^2 \oplus \xi) = (1 + u)^{-1}(1 + v)^{-1}$
19) $w(\gamma^\perp \oplus \varepsilon^n - \varepsilon^1 \oplus \xi \oplus \xi) = (1 + u)^{-1}(1 + v)^{-2}$
20) $w(\gamma^\perp \oplus \varepsilon^n - \xi \oplus \xi \oplus \xi) = (1 + u)^{-1}(1 + v)^{-3}$
21) $w(\gamma^\perp \oplus \varepsilon^n - \gamma \oplus \xi \oplus \varepsilon^1) = (1 + u)^{-2}(1 + v)^{-1}$
22) $w(\gamma^\perp \oplus \varepsilon^n - \gamma \oplus \gamma \oplus \xi) = (1 + u)^{-3}(1 + v)^{-1}$
23) $w(\gamma^\perp \oplus \varepsilon^n - \gamma \oplus \xi \oplus \xi) = (1 + u)^{-2}(1 + v)^{-2}$
24) $w(TP(m) \oplus TP(n) - \varepsilon^3) = (1 + u)^{m+1}(1 + v)^{n+1}$
25) $w(TP(m) \oplus TP(n) - \gamma \oplus \varepsilon^2) = (1 + u)^m(1 + v)^{n+1}$

26) $w(TP(m) \oplus TP(n) - \gamma \oplus \gamma \oplus \varepsilon^1) = (1+u)^{m-1}(1+v)^{n+1}$
27) $w(TP(m) \oplus TP(n) - \gamma \oplus \gamma \oplus \gamma) = (1+u)^{m-2}(1+v)^{n+1}$
28) $w(TP(m) \oplus TP(n) - \gamma \oplus \xi \oplus \varepsilon^1) = (1+u)^{m}(1+v)^{n}$
29) $w(TP(m) \oplus TP(n) - \gamma \oplus \gamma \oplus \xi) = (1+u)^{m-1}(1+v)^{n}$
30) $w(\gamma^\perp \oplus TP(n) - \varepsilon^3) = (1+u)^{-1}(1+v)^{n+1}$
31) $w(\gamma^\perp \oplus TP(n) - \gamma \oplus \varepsilon^2) = (1+u)^{-2}(1+v)^{n+1}$
32) $w(\gamma^\perp \oplus TP(n) - \gamma \oplus \gamma \oplus \varepsilon^1) = (1+u)^{-3}(1+v)^{n+1}$
33) $w(\gamma^\perp \oplus TP(n) - \gamma \oplus \gamma \oplus \gamma) = (1+u)^{-4}(1+v)^{n+1}$
34) $w(\gamma^\perp \oplus TP(n) - \varepsilon^2 \oplus \xi) = (1+u)^{-1}(1+v)^{n}$
35) $w(\gamma^\perp \oplus TP(n) - \varepsilon^1 \oplus \xi \oplus \xi) = (1+u)^{-1}(1+v)^{n-1}$
36) $w(\gamma^\perp \oplus TP(n) - \xi \oplus \xi \oplus \xi) = (1+u)^{-1}(1+v)^{n-2}$
37) $w(\gamma^\perp \oplus TP(n) - \gamma \oplus \xi \oplus \varepsilon^1) = (1+u)^{-2}(1+v)^{n}$
38) $w(\gamma^\perp \oplus TP(n) - \gamma \oplus \gamma \oplus \xi) = (1+u)^{-3}(1+v)^{n}$
39) $w(\gamma^\perp \oplus TP(n) - \gamma \oplus \xi \oplus \xi) = (1+u)^{-2}(1+v)^{n-1}$
40) $w(\gamma^\perp \oplus \xi^\perp - \varepsilon^3) = (1+u)^{-1}(1+v)^{-1}$
41) $w(\gamma^\perp \oplus \xi^\perp - \gamma \oplus \varepsilon^2) = (1+u)^{-2}(1+v)^{-1}$
42) $w(\gamma^\perp \oplus \xi^\perp - \gamma \oplus \gamma \oplus \varepsilon^1) = (1+u)^{-3}(1+v)^{-1}$
43) $w(\gamma^\perp \oplus \xi^\perp - \gamma \oplus \gamma \oplus \gamma) = (1+u)^{-4}(1+v)^{-1}$
44) $w(\gamma^\perp \oplus \xi^\perp - \gamma \oplus \xi \oplus \varepsilon^1) = (1+u)^{-2}(1+v)^{-2}$
45) $w(\gamma^\perp \oplus \xi^\perp - \gamma \oplus \gamma \oplus \xi) = (1+u)^{-3}(1+v)^{-2}$

Since we want to compute the last three Stiefel-Whitney classes, we only have to know the three last terms of each factor of $(1+u)^i$ where $i = -1, -2, -3, -4, m+1, m, m-1$ and $m-2$, where $m, n \geq 3$. These are given by the following table:

$$(1+u)^{-1} = 1 + u + u^2 + \cdots + u^{m-2} + u^{m-1} + u^m, \quad \forall m,$$

$$(1+u)^{-2} = \begin{cases} 1 + u^2 + u^4 + \cdots + u^{m-2} + 0 + u^m, & m \equiv 0(2) \\ 1 + u^2 + u^4 + \cdots + 0 + u^{m-1} + 0, & m \equiv 1(2), \end{cases}$$

$$(1+u)^{-3} = \begin{cases} 1 + u + u^4 + u^5 + \cdots + 0 + 0 + u^m, & m \equiv 0(4) \\ 1 + u + u^4 + u^5 + \cdots + 0 + u^{m-1} + u^m, & m \equiv 1(4) \\ 1 + u + u^4 + u^5 + \cdots + u^{m-2} + u^{m-1} + 0, & m \equiv 2(4) \\ 1 + u + u^4 + u^5 + \cdots + u^{m-2} + 0 + 0, & m \equiv 3(4), \end{cases}$$

$$(1+u)^{-4} = \begin{cases} 1 + u^4 + u^8 + \cdots + 0 + 0 + u^m, & m \equiv 0(4) \\ 1 + u^4 + u^8 + \cdots + 0 + u^{m-1} + 0, & m \equiv 1(4) \\ 1 + u^4 + u^8 + \cdots + u^{m-2} + 0 + 0, & m \equiv 2(4) \\ 1 + u^4 + u^8 + \cdots + 0 + +0 + 0, & m \equiv 3(4), \end{cases}$$

$$(1+u)^{m+1} = \begin{cases} 1+\cdots+0+0+u^m, & m \equiv 0(4) \\ 1+\cdots+0+u^{m-1}+0, & m \equiv 1(4) \\ 1+\cdots+u^{m-2}+u^{m-1}+u^m, & m \equiv 2(4) \\ 1+\cdots+0++0+0, & m \equiv 3(4), \end{cases}$$

$$(1+u)^{m} = \begin{cases} 1+\cdots+0+0+u^m, & m \equiv 0(4) \\ 1+\cdots+0+u^{m-1}+u^m, & m \equiv 1(4) \\ 1+\cdots+u^{m-2}+0+u^m, & m \equiv 2(4) \\ 1+\cdots+u^{m-2}+u^{m-1}+u^m, & m \equiv 3(4), \end{cases}$$

$$(1+u)^{m-1} = \begin{cases} 1+\cdots+u^{m-2}+u^{m-1}+0, & m \equiv 0(2) \\ 1+\cdots+0+u^{m-1}+0, & m \equiv 1(2), \end{cases}$$

$$(1+u)^{m-2} = 1+\cdots+u^{m-2}+0+0, \quad \forall m.$$

We denote: $w_k(\zeta_i) = w_k(\beta-\alpha)$ where $i = 1, 2, \ldots, 45$, and we use the ordered basis choosen before. The cases where the last three Stiefel-Whitney classes vanish are:

Cases $1, 2, 3, 6, 7, 8, 15, 16, 17$, for any n, m.
If $i = 9, 10, 43$, for $m \equiv 3(4)$ and any n.
If $i = 11$, for $m \equiv 1(4)$ and $n \equiv 3(4)$ or $m \equiv 3(4)$ and any n.
If $i = 24$, for $m \equiv 3(4)$ or $n \equiv 3(4)$.
If $i = 25, 26, 30, 31$, for any m and $n \equiv 3(4)$.
If $i = 27$, for any m and $n \equiv 1(2)$.
If $i = 32$, for any m and $n \equiv 3(4)$ or $m \equiv 3(4)$ and $n \equiv 1(4)$.
If $i = 33$, for $m \equiv 2(4)$ and $n \equiv 1(4)$ or $m \equiv 3(4)$ or $n \equiv 3(4)$.
If $i = 45$, for $m \equiv 3(4)$ and $n \equiv 1(2)$. Otherwise at least one of the three last Stiefel-Whitney classes is not zero. Therefore there is no monomorphism. We use some basic results:

Lemma 1 If $m \equiv 1(2)$ then $TP(m) \cong \varepsilon^1 \oplus \theta^{m-1}$.

Proof This follows from the Poincaré-Hopf Theorem.

Lemma 2 If $m \equiv 3(4)$ then $TP(m) \cong \varepsilon^3 \oplus \zeta^{m-3}$.

Proof If $m \equiv 3(4)$ then $\binom{m+1}{2} \equiv 0(2)$ and so $\mathbb{R}P(m)$ is a spin manifold. Then we can use the following fact due to Emery Thomas: If M is a spin manifold with dim $M \equiv 3(4)$, then span$(M) \geq 3$. See [5], corollary 1.2.

Lemma 3 If α and β are smooth vector bundle of dimensions a and b, respectively, over a closed connected n-dimensional manifold M. If $n + a \leq b$, then there exists a monomorphism $\alpha \hookrightarrow \beta$.

Proof This can be obtained by singularity approach due to Ulrich Koschorke. See [2], exercise 1.13.

Recall that $TP(m) \oplus \varepsilon^1 \cong \gamma \oplus \gamma \oplus \cdots \oplus \gamma \;\; ((m+1) - \text{times})$.

Cases 1-3 ($\alpha = \gamma \oplus \varepsilon^2, \gamma \oplus \gamma \oplus \varepsilon^1$ and $\gamma \oplus \gamma \oplus \gamma$, $\beta = \varepsilon^{m+n}$) For any 3-plane bundle α there is a monomorphism $\alpha \hookrightarrow \varepsilon^{m+3}$ over $\mathbb{R}P(m)$ (Lemma 3). In particular for $\alpha = \gamma \oplus \varepsilon^2, \gamma \oplus \gamma \oplus \varepsilon^1$ and $\alpha = \gamma \oplus \gamma \oplus \gamma$. We can pull these monomorphisms back over the product $\mathbb{R}P(m) \times \mathbb{R}P(n)$ in order to get $\gamma \oplus \varepsilon^2, \gamma \oplus \gamma \oplus \varepsilon^1, \gamma \oplus \gamma \oplus \gamma \hookrightarrow \varepsilon^{m+3} \oplus \varepsilon^{n-3} \cong \varepsilon^{m+n}$.

Cases 6-8 ($\alpha = \gamma \oplus \varepsilon^2$, $\gamma \oplus \gamma \oplus \varepsilon^1$ and $\gamma \oplus \gamma \oplus \gamma$, $\beta = TP^m \oplus \varepsilon^n$) Since $TP(m) \oplus \varepsilon^1 \cong \gamma \oplus \cdots \oplus \gamma$ ($(m+1)$-times), then $\gamma \oplus \varepsilon^2$, $\gamma \oplus \gamma \oplus \varepsilon^1$, $\gamma \oplus \gamma \oplus \gamma \hookrightarrow TP(m) \oplus \varepsilon^n \cong (\gamma \oplus \cdots \oplus \gamma) \oplus \varepsilon^{n-1}$.

Cases 9-11 ($\alpha = \varepsilon^2 \oplus \xi$, $\varepsilon^1 \oplus \xi \oplus \xi$ and $\xi \oplus \xi \oplus \xi$, $\beta = TP^m \oplus \varepsilon^n$) If $m \equiv 3(4)$, then $TP(m) \cong \varepsilon^3 \oplus \zeta^{m-3}$ (Lemma 2). Then, $TP(m) \oplus \varepsilon^n \cong \zeta^{m-3} \oplus \varepsilon^{n+3}$. Over the factor $\mathbb{R}P(n)$, $\alpha \hookrightarrow \varepsilon^{n+3}$, for any 3-plane α. In particular, $\varepsilon^2 \oplus \xi$, $\varepsilon^1 \oplus \xi \oplus \xi$ and $\xi \oplus \xi \oplus \xi \hookrightarrow \varepsilon^{n+3}$. We can pull these monomorphisms back over the product $\mathbb{R}P(m) \times \mathbb{R}P(n)$ to get the desired monomorphisms $\varepsilon^2 \oplus \xi$, $\varepsilon^1 \oplus \xi \oplus \xi$ and $\xi \oplus \xi \oplus \xi \hookrightarrow TP(m) \oplus \varepsilon^n$. For case 9 alone we can use: Over the factor $\mathbb{R}P(n)$, $\varepsilon^{n+1} \cong \xi \oplus \xi^\perp$. Taking the pullback of this decomposition we can write $\varepsilon^2 \oplus \xi \hookrightarrow TP(m) \oplus \varepsilon^n \cong (\zeta^{m-3} \oplus \varepsilon^3) \oplus \varepsilon^n \cong \zeta^{m-3} \oplus \varepsilon^2 \oplus \varepsilon^{n+1} \cong \zeta^{m-3} \oplus \varepsilon^2 \oplus \xi \oplus \xi^\perp$.

We still have to consider, in case 11 ($\alpha = \xi \oplus \xi \oplus \xi$ and $\beta = TP(m) \oplus \varepsilon^n$), the situation $m \equiv 1(4)$ and $n \equiv 3(4)$. Since $m \equiv 1(4)$, $TP(m) \oplus \varepsilon^n \cong \theta^{m-1} \oplus \varepsilon^{n+1} \cong \theta^{m-1} \oplus \xi \oplus \xi^\perp$. Tensorizing with ξ we get $\xi \otimes (TP(m) \oplus \varepsilon^n) \cong (\xi \otimes \theta^{m-1}) \oplus \varepsilon^1 \oplus TP(n) \cong (\xi \otimes \theta^{m-1}) \oplus \varepsilon^1 \oplus \zeta^{n-3} \oplus \varepsilon^3$ because $n \equiv 3(4)$ (Lemma 2). Tensorizing once more with ξ we get $TP(m) \oplus \varepsilon^n \cong \theta^{m-1} \oplus \xi \oplus \xi \oplus \xi \oplus \xi \oplus (\xi \otimes \zeta^{n-3})$. This shows we can get the desired monomorphism.

Cases 15-17 ($\alpha = \gamma \oplus \varepsilon^2$, $\gamma \oplus \gamma \oplus \varepsilon^1$ and $\gamma \oplus \gamma \oplus \gamma$, $\beta = \gamma^\perp \oplus \varepsilon^n$) Same argument as in cases 1-3 proves that $\gamma \oplus \varepsilon^2$, $\gamma \oplus \gamma \oplus \varepsilon^1$, $\gamma \oplus \gamma \oplus \gamma \hookrightarrow \gamma^\perp \oplus \varepsilon^n$.

Case 24 ($\alpha = \varepsilon^3$, $\beta = TP^m \oplus TP^n$) If $m \equiv 3(4)$ or $n \equiv 3(4)$, then $\varepsilon^3 \hookrightarrow TP(m) \oplus TP(n)$.

Cases 25, 26 ($\alpha = \gamma \oplus \varepsilon^2$ and $\gamma \oplus \gamma \oplus \varepsilon^1$, $\beta = TP^m \oplus TP^n$) If $n \equiv 3(4)$, $TP(m) \oplus TP(n) \cong TP(m) \oplus (\varepsilon^3 \oplus \eta^{n-3}) \cong (TP(m) \oplus \varepsilon^1) \oplus (\varepsilon^2 \oplus \eta^{n-3}) \cong (\gamma \oplus \cdots \oplus \gamma) \oplus \varepsilon^2 \oplus \eta^{n-3}, ((m+1)$-copies). So $\gamma \oplus \varepsilon^2$, $\gamma \oplus \gamma \oplus \varepsilon^1 \hookrightarrow (\gamma \oplus \cdots \oplus \gamma) \oplus \varepsilon^2 \oplus \eta^{n-3} \cong TP(m) \oplus TP(n)$.

Case 27 ($\alpha = \gamma \oplus \gamma \oplus \gamma$, $\beta = TP^m \oplus TP^n$) If $n \equiv 1(2)$, $TP(m) \oplus TP(n) \cong TP(m) \oplus \varepsilon^1 \oplus \theta^{n-1} \cong \gamma \oplus \cdots \oplus \gamma \oplus \theta^{n-1}, ((m+1)$-copies). Then $\gamma \oplus \gamma \oplus \gamma \hookrightarrow TP(m) \oplus TP(n)$.

Case 30 ($\alpha = \varepsilon^3$, $\beta = \gamma^\perp \oplus TP(n)$) If $n \equiv 3(4)$ then $TP(n) \cong \varepsilon^3 \oplus \eta^{n-3}$ and then $\varepsilon^3 \hookrightarrow \gamma^\perp \oplus TP(n)$ (Lemma 2).

Case 31 ($\alpha = \gamma \oplus \varepsilon^2$, $\beta = \gamma^\perp \oplus TP(n)$) For any 3-plane bundle α, there is a monomorphism $\alpha \hookrightarrow \gamma^\perp \oplus \varepsilon^3$ over $\mathbb{R}P(m)$. If $n \equiv 3(4)$, then we can pullback over the product $\mathbb{R}P(m) \times \mathbb{R}P(n)$ the existent monomorphism $\gamma \oplus \varepsilon^2 \hookrightarrow \gamma^\perp \oplus \varepsilon^3$ to get $\gamma \oplus \varepsilon^2 \hookrightarrow \gamma^\perp \oplus \varepsilon^3 \oplus \eta^{n-3} \cong \gamma^\perp \oplus TP(n)$.

Case 32 ($\alpha = \gamma \oplus \gamma \oplus \varepsilon^1$, $\beta = \gamma^\perp \oplus TP(n)$) If $n \equiv 3(4)$, the same argument as in case 31 gives a monomorphism $\gamma \oplus \gamma \oplus \varepsilon^1 \hookrightarrow \gamma^\perp \oplus TP(n)$. If $n \equiv 1(4)$ and $m \equiv 3(4)$ we can do the following: $\gamma^\perp \oplus TP(n) \cong \gamma^\perp \oplus \varepsilon^1 \oplus \theta^{n-1}$. Tensorizing with γ we get $(\gamma \otimes \gamma^\perp) \oplus \gamma \oplus (\gamma \otimes \theta^{n-1}) \cong TP(m) \oplus \gamma \oplus (\gamma \otimes \theta^{n-1}) \cong (\varepsilon^3 \oplus \zeta^{m-3}) \oplus \gamma \oplus (\gamma \otimes \theta^{n-1})$. Tensorizing with γ once more we get $\gamma^\perp \oplus TP(n) \cong \gamma \oplus \gamma \oplus \gamma \oplus (\gamma \otimes \zeta^{n-3}) \oplus \varepsilon^1 \oplus \theta^{n-1}$, and then there is a monomorphism $\gamma \oplus \gamma \oplus \varepsilon^1 \hookrightarrow \gamma^\perp \oplus TP(n)$.

Case 33 ($\alpha = \gamma\oplus\gamma\oplus\gamma$, $\beta = \gamma^\perp\oplus TP(n)$) If $n \equiv 3(4)$, then the argument used in case 31 shows that there is a monomorphism $\gamma\oplus\gamma\oplus\gamma \hookrightarrow \gamma^\perp\oplus TP(n)$. If $m \equiv 3(4)$, the double tensorization argument given in case 32 shows that $\gamma^\perp \oplus TP(n) \cong \gamma\oplus\gamma\oplus\gamma\oplus(\gamma\otimes\zeta^{n-3})\oplus TP(n)$. Then $\gamma\oplus\gamma\oplus\gamma \hookrightarrow \gamma^\perp\oplus TP(n)$.

Suppose $m \equiv 2(4)$ and $n \equiv 1(4)$. Then $TP(n) \cong \varepsilon^1 \oplus \theta^{n-1}$. It suffices to prove that $\gamma\oplus\gamma\oplus\gamma \hookrightarrow \gamma^\perp\oplus\varepsilon^1$ over the factor $\mathbb{R}P(m)$. There exists a bundle monomorphism $\varepsilon^3 \hookrightarrow TP(m+1) \cong \gamma\otimes\gamma^\perp$ over $\mathbb{R}P(m+1)$ by Lemma 2. Tensor product with γ yields $\gamma\oplus\gamma\oplus\gamma \hookrightarrow \gamma^\perp$ over $\mathbb{R}P(m+1)$. Restriction of this bundle monomorphism under the inclusion $i : \mathbb{R}P(m) \to \mathbb{R}P(m+1)$ gives $\gamma\oplus\gamma\oplus\gamma \hookrightarrow i^*\gamma^\perp \cong \gamma^\perp\oplus\varepsilon^1$ on $\mathbb{R}P(m)$.

Case 43 ($\alpha = \gamma\oplus\gamma\oplus\gamma$, $\beta = \gamma^\perp\oplus\xi^\perp$) If $m \equiv 3(4)$ the double tensorizing argument shows that there is a monomorphism from $\gamma\oplus\gamma\oplus\gamma$ into $\gamma^\perp\oplus\xi^\perp$.

Case 45 ($\alpha = \gamma\oplus\gamma\oplus\xi$, $\beta = \gamma^\perp\oplus\xi^\perp$) If $m \equiv 3(4)$ and $n \equiv 1(2)$ then $\gamma\otimes(\gamma^\perp\oplus\xi^\perp) \cong (\gamma\otimes\gamma^\perp)\oplus(\gamma\otimes\xi^\perp) \cong TP(m)\oplus(\gamma\otimes\xi^\perp) \cong \varepsilon^3\oplus\zeta^{m-3}\oplus(\gamma\otimes\xi^\perp)$. Tensorizing with γ once more gives $\gamma^\perp\oplus\xi^\perp \cong \gamma\oplus\gamma\oplus\gamma\oplus(\gamma\otimes\zeta^{m-3})\oplus\xi^\perp$. Now, tensorizing twice with ξ gives $\gamma^\perp\oplus\xi^\perp \cong \gamma\oplus\gamma\oplus\gamma\oplus(\gamma\otimes\zeta^{m-3})\oplus\xi\oplus(\xi\otimes\theta^{n-1})$. Then there is a monomorphism from $\gamma\oplus\gamma\oplus\xi$ into $\gamma^\perp\oplus\xi^\perp$.

Remark 1 In same cases, the geometric arguments show that we can embed more copies of γ (or ξ) than the ones we claimed. Also, some proofs work for smaller m or n, as long as $m+n \geq 3$.

References

1. W. Dwyer and A. Zabrodsky, Maps between classifying spaces, Lecture Notes in Math., 1298(1986), 106-119.
2. U. Koschorke,Vector Fields and Other Vector Bundle Morphisms - A Singularity Approach,Lecture Notes in Math.,Springer Verlag,847(1980)
3. J. R. Martino,Classifying spaces and their maps., Contemporary Math., 188(1995), 153-190
4. M. H. de P. L. Mello, Sobre a existência de sub-fibrados de planos em fibrados não orientáveis, Tese de Doutorado,Pontifícia Universidade Católica,(1985)
5. E. Thomas, Real and Complex Vector Fields on Manifolds, Journal of Mathematics and Mechanics,16(1967)
6. E. Thomas, Fields of tangent 2-planes on even-dimensional manifolds, Ann. of Math.,2-1986(1967)

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