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Mittag-Leffler stability for a one-dimensional fractional elastic-porous system: nonstandard frictional damping and nonstandard Kelvin-Voigt damping

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ABSTRACT: In this paper, we investigate the asymptotic behavior of solutions for a one-dimensional fractional elastic-porous system. We dissipate the system by two damping devices. The elastic equation is dissipated by a nonstandard frictional damping (frictional damping of fractional order) and the porous equation by a nonstandard Kelvin-Voigt damping (Kelvin-Voigt damping of fractional order). We prove that the system is Mittag-Leffler stable under certain conditions on the coefficients of the system and without imposing the equal wave speeds condition $\frac{\mu}{\rho} = \frac{\delta}{J}$. The result is new and opens the door for more research areas on porous-elastic systems and other problems.

Key Words: Elastic-porous systems, Mittag-Leffler, Energy method, Fractional calculus.

Contents

1. Introduction

An elastic-porous system typically refers to a material or structure that combines elasticity and porosity. Elasticity refers to the property of a material to deform when subjected to stress and return to its original shape when the stress is removed. Materials like rubber bands and springs exhibit elasticity. Porosity refers to the presence of void spaces or pores within a material. These voids can be interconnected or isolated. Porous materials often have applications in filtration, insulation, and absorption due to their ability to hold fluids or gases within their structure.

Understanding the stability of elastic porous systems is crucial for ensuring their structural integrity. This knowledge helps in designing materials and structures that can withstand various loads and environmental conditions without failure. For examples, in many engineering applications, such as in civil engineering when building foundations or in biomedical engineering while dealing with implants, safety is paramount importance. Analyzing the stability of elastic porous systems helps in predicting potential failure modes and designing systems that meet safety standards. Moreover, elastic porous systems are often used in environmental applications, such as soil stabilization and groundwater remediation. Hence, understanding their stability helps in assessing their long-term effectiveness and potential environmental impact.

Motivated by the aforementioned issues, many researchers have studied the well-posedness, stability

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and instability of elastic-porous systems. We start with the work of Quintanilla [1], where he showed that the following elastic-porous system:

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta \phi_x, & x \in (0,\pi),\ t > 0, \\ \rho_0 \kappa \phi_{tt} = \alpha \phi_{xx} - \beta u_x - \tau \phi_t - \xi \phi, & x \in (0,\pi),\ t > 0, \end{cases} \tag{1.1}$$

has a slow decay only and one damping in the porous equation $(-\tau\phi_t)$ is not strong enough to ensure an exponential decay. Apalara [2] proved that replacing the porous dissipation $(-\tau\phi_t)$ by a memory damping leads to general decay rate under the assumption of equal-speed wave propagations. Feng and Yin [3] extended the result in [2] to the case of non-equal wave speeds. Some other researchers such as Casas and Quintanilla [4], Magana and Quintanilla [5], Pamplona et al. [6], Messaoudi and Fareh [7], Han and Xu [8], Djellali et al. [9] proved the stability of such systems by employing other different damping terms.

Recently, numerous papers focusing on replacing the standard derivatives by fractional derivatives due to the wide use of their applications in various scientific and engineering fields, including viscoelasticity [10] and image processing [11,12]. In the context of elasticity, fractional derivatives can be employed to model materials with non-local or memory-dependent behaviors. For example: Fractional derivatives can be incorporated to model damping and energy dissipation in elastic structures. The inclusion of fractional derivatives allows for a more accurate representation of the dissipation process, especially in situations where traditional models may not capture the observed behavior [13].

The standard diffusion models in continuous media arise from fundamental concepts: momentum and energy conservation. The classical swelling equations are used to describe how materials' volumes change as liquids or solvents are absorbed. However, some of the standard diffusion models can not describe adequately the behavior of many phenomena such as ballistic motions across membranes, and material dynamics between a pure viscous fluid state and a pure elastic solid state. Using fractional models is a practical technique to get around these problems and constraints and capture the complicated diffusion that takes place in random, fractal, and heterogeneous media, such as porous media [14]. Equations for fractional diffusion or fractional wave propagation that describe anomalous diffusion processes and phenomena are becoming more and more popular these days. For examples, Othmani and Tatar [15] studied the well-posedness and stability of the fractional telegraph problem. Al-Homidan and Tatar [16] and [17] investigated the stability of two fractional problems; the first one represents an interpolation between the heat and the wave equations and the second is for a fractional Timoshenko system. For more results on fractional diffusion models, we refer to [18,19,20,21,22,23,24,25,26,27] and the references therein.

Motivated by the importance and efficiency of fractional derivatives, **we consider** the following fractional elastic-porous system:

$$\begin{cases} \rho\ {}^C D^\alpha \left({}^C D^\alpha u\right) - \mu u_{xx} - b z_x + \gamma_1\ {}^C D^\alpha u(t) = 0, & \text{in } (0,1)\times(0,\infty), \\ J\ {}^C D^\alpha \left({}^C D^\alpha z\right) - \delta z_{xx} + b u_x + \xi z + \gamma_2\ {}^C D^\alpha z_{xx}(t) = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{1.2}$$

where z is the volume fraction, and u is the displacement of a solid elastic material. The strictly positive constants J and ρ represent the product of the mass density by the equilibrated inertia and the mass density, respectively. The parameters $\mu, \delta, \xi > 0$, b is a real number, and the damping coefficients parameters $\gamma_1, \gamma_2 > 0$. In addition, we assume that $\mu\xi - b^2 > 0$.

The term ${}^C D^\alpha u(t)$ is the nonstandard frictional damping (standard when $\alpha = 1$) and the term ${}^C D^\alpha z_{xx}$ is the nonstandard Kelvin-Voigt damping (standard when $\alpha = 1$). The Kelvin-Voigt damping model is effectively applied in elastic porous systems to account for the complex interactions between the solid matrix and the fluid within the pores. These systems are often encountered in various engineering and scientific fields.

We consider the system (1.2) subject to the following boundary and initial conditions:

$$\begin{cases} z(x,0) = z_0(x),\ \ z_t(x,0) = z_1(x),\ \ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x), z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0, \end{cases} \tag{1.3}$$

where the initial conditions z_0, z_1, u_0, u_1, are fixed data.

Our aim in this work is to study the asymptotic behavior of the solutions of this fractional system. In our result, we established a Mittag-Leffler stability in which the exponential stability is special case. Unlike the previous studies, we obtain this exponential stability without the equal wave speeds.

Our result will significantly extend many earlier results in the literature, in particular, the ones in [1], [2] and [3].

2. Existence and Uniqueness Result

In this section, we sketch briefly how to prove the existence and uniqueness for our problem (1.2). To this end, we let $U = \left(u, \widetilde{u}, z, \widetilde{z}\right)^T$, where $\widetilde{z} = {}^C D^\alpha z$, $\widetilde{u} = {}^C D^\alpha u$ and $U_0 = \left(u_0, \widetilde{u_0}, z_0, \widetilde{z}_0\right)^T$. Therefore, the system (1.2) can be written in the abstract form as follows:

$$\begin{cases} {}^C D^\alpha U = \mathcal{A}U,\ 0 < \alpha < 1, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\mu}{\rho}\partial_{xx} & -\frac{\gamma_1}{\rho} & \frac{b}{\rho}\partial_x & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{b}{J}\partial_x & 0 & \frac{\delta}{J}\partial_{xx} - \frac{\xi}{J} & -\frac{\gamma_2}{J}\partial_{xx} \end{pmatrix}.$$

We define

$$\mathcal{H} := H_0^1(0,1) \times L^2\left(0,1\right) \times H_0^1(0,1) \times L^2\left(0,1\right),$$

and the domain

$$D(\mathcal{A}) := \left\{ U = \left(u, \widetilde{u}, z, \widetilde{z}\right)^T \in \mathcal{H} : u, z \in H^2(0,1) \cap H_0^1(0,1), \widetilde{u}, \widetilde{z} \in H_0^1(0,1) \right\}.$$

Assuming that $U_0 \in D(\mathcal{A})$, the solution is classical

$$U \in C\left((0,\infty), D(\mathcal{A})\right) \cap C^1\left((0,\infty), \mathcal{H})\right), \tag{2.2}$$

and fulfils $t^{-\alpha} * z_t \in C^1\left((0,\infty), L^2\left(0,1\right)\right)$ and $t^{-\alpha} * u_t \in C^1\left((0,\infty), L^2\left(0,1\right)\right)$.
This can be shown by utilizing the results in [28] where the author proved the existence of mild and classical solutions for the following abstract non homogeneous fractional integro-differential problem:

$$\begin{cases} {}^C D^\gamma U(t,x) = \mathcal{P}U(t,x) + f\left(t, U(t)\right) - \int_0^t \mathcal{B}(t-s)U(s,x)ds,\ 0 < \gamma < 1, \\ U(0,x) = U_0(x) \in X, \end{cases} \tag{2.3}$$

where X is a Banach space, $\mathcal{P}$ and $\mathcal{B}$ are defined on a common domain $D(\mathcal{P})$. He showed that the above system possesses a unique solution

$$U\left(t\right) := R_\gamma\left(t\right)U_0 \in C\left([0,\infty); D(\mathcal{P})\right) \cap C^\gamma\left((0,\infty); X\right),$$

where R_γ is the $\gamma-$resolvent of the system, provided that $U_0 \in D(\mathcal{P})$. His proof relies on the notion of $\gamma-$resolvent and some spectral theory arguments.

Remark 2.1 *Notice that replacing* ${}^CD^\alpha\left({}^CD^\alpha u\right)$ *and* ${}^CD^\alpha\left({}^CD^\alpha z\right)$ *in our system* (1.2) *by* ${}^CD^{2\alpha}u$ *and* ${}^CD^{2\alpha}z$*, respectively leads to the following abstract problem:*

$$\begin{cases} {}^CD^{2\alpha}U(x,t) = \mathcal{A}U(x,t)\ 0 < 2\alpha < 2, \\ U(0,x) = 0, \quad U(1,x) = 0, \end{cases} \tag{2.4}$$

where the operator $\mathcal{A}$ *is positive definite and given by*

$$\mathcal{A} = \begin{pmatrix} \frac{\mu}{\rho}\partial_{xx} - \frac{\gamma_1}{\rho}\ {}^CD^\alpha & \frac{b}{\rho}\partial_x \\ \\ -\frac{b}{J}\partial_x & \frac{\delta}{J}\partial_{xx} - \frac{\xi}{J} - \frac{\gamma_2}{J}\ {}^CD^\alpha\partial_{xx} \end{pmatrix}. \tag{2.5}$$

The proof of the well-posedness of the abstract system (2.4) *can be found in [29] under the condition* $w'(0,x) = w_1(x) = 0$ *for* $w = u$ *and* $w = z$*. As a matter of fact, this condition annihilates the term which is equal to the difference between* ${}^CD^{2\alpha}w$ *and* ${}^CD^\alpha\left({}^CD^\alpha w\right)$*.*

3. Preliminaries

In this section, we present the necessary background for our results.

Definition 3.1 *[30] : The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* χ *is defined by*

$$I^\alpha\chi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi(s)ds,\ \alpha > 0,$$

where $\Gamma(\alpha)$ *is the gamma function. This definition holds for any measurable function* χ *provided that the right-hand side exists.*

Definition 3.2 *[30] : The Caputo fractional derivative of order* α*,* $n-1 < \alpha < n$*,* $n \in \mathbb{Z}^+$*, is defined by*

$${}^CD^\alpha\chi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\chi^{(n)}(s)ds.$$

In particular, for $0 < \alpha < 1$*,*

$${}^CD^\alpha\chi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\chi'(s)ds,\ 0 < \alpha < 1.$$

The Riemann-Liouville fractional derivative of order α *is defined by*

$${}^{RL}D^\alpha\chi(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}\chi(s)ds,\ 0 < \alpha < 1,$$

provided that the integral exists. The relationship between the two derivatives is given by

$${}^{RL}D^\alpha\chi(t) = \frac{\chi(0)t^{-\alpha}}{\Gamma(1-\alpha)} + {}^CD^\alpha\chi(t),\ 0 < \alpha < 1,\ t > 0.$$

This relationship will be crucial in applying the following lemma on differentiation under the integral sign.

We recall the two-parametric and one-parametric Mittag-Leffler functions

$$E_{\alpha,\beta}(z) := \sum\nolimits_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,$$

and

$$E_\alpha(z) := \sum\nolimits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)},\ \mathrm{Re}(\alpha)>0,$$

resp. It is useful to notice that $E_{\alpha,1}(z) \equiv E_\alpha(z)$.

Proposition 3.1 *[31]: Let $\chi(t)$ be a differentiable function on $[0,\infty)$. If there exists a positive constant γ and a real number α in the interval $(0,1)$ such that*

$${}^{C}D^{\alpha}\chi(t) \leq -\gamma\chi(t),$$

for all $t \geq 0$, where ${}^{C}D^{\alpha}\chi(t)$ denotes the Caputo fractional derivative of order α of $\chi(t)$, then

$$\chi(t) \leq \chi(0)E_\alpha(-\gamma t^\alpha),$$

for all $t \geq 0$. Here, $E_\alpha(z)$ is the one-parameter Mittag-Leffler function. Moreover, if the fractional derivative is of Riemann-Liouville type, denoted by ${}^{RL}D^{\alpha}\chi(t)$, and the inequality holds, then the decay rate is given by

$$\chi(t) \leq \chi(0)t^{\alpha-1}E_{\alpha,\alpha}(-\gamma t^\alpha),$$

for all $t \geq 0$, where $E_{\alpha,\alpha}(z)$ is the two-parameter Mittag-Leffler function.

Proposition 3.2 *[32]: For $\alpha,\beta>0$, we have*

$$\lambda z^\alpha E_{\alpha,\alpha+\beta}(\lambda z^\alpha) = E_{\alpha,\beta}(\lambda z^\alpha) - \frac{1}{\Gamma(\beta)}.$$

Proposition 3.3 *[32]: For $\mu,\alpha,\beta>0$, we have*

$$\frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1}E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}dt = z^{\mu+\beta-1}E_{\alpha,\mu+\beta}(\lambda z^\alpha),\ z>0.$$

The following result concerns the Riemann-Liouville fractional derivative and the integral sign.

Proposition 3.4 *[30]: The following holds provided that $\psi(t)$ is a continuous function and $I^{1-\alpha}\chi(t) \in C^1([0,\infty))$, $0<\alpha<1$:*

$${}^{RL}D^{\alpha}\int_0^t \chi(t-s)\psi(s)ds = \int_0^t \psi(s)^{RL}D^{\alpha}\chi(t-s)ds + \psi(t)\lim_{t\to 0^+} I^{1-\alpha}\chi(t),\ t>0.$$

We conclude this section with the following proposition regarding the Caputo fractional derivative of the product of two functions.

Proposition 3.5 *[33]: Let $f(t)$ and $h(t)$ be absolutely continuous functions on $[0,T]$. Then, for $0<\alpha<1$, we have*

$$\begin{aligned} & f(t)^{C}D^{\alpha}h(t) + h(t)^{C}D^{\alpha}f(t) \\ = \quad & {}^{C}D^{\alpha}(fh)(t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^\xi \frac{f'(\eta)d\eta}{(t-\eta)^\alpha}\int_0^\xi \frac{h'(s)ds}{(t-s)^\alpha},\ t\in[0,T]. \end{aligned} \tag{3.1}$$

In particular

$$\begin{aligned} {}^{C}D^{\alpha}(f^2(t)) &= 2f(t)^{C}D^{\alpha}f(t) - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi \frac{f'(\eta)d\eta}{(t-\eta)^\alpha}\right)^2 \\ &\leq 2f(t)^{C}D^{\alpha}f(t). \end{aligned} \tag{3.2}$$

4. The energy

By multiplying the first and the second equations in (1.2) by $D^{\alpha}u$ and $D^{\alpha}z$ respectively, and integrating over $(0,1)$, we obtain

$$\begin{aligned} & \rho\int_0^1 D^{\alpha}uD^{\alpha}\left(D^{\alpha}u\right)dx + J\int_0^1 D^{\alpha}zD^{\alpha}\left(D^{\alpha}z\right)dx \\ = \; & \mu\int_0^1 D^{\alpha}uu_{xx}dx + b\int_0^1 D^{\alpha}uz_xdx + \delta\int_0^1 D^{\alpha}zz_{xx}dx \\ & -b\int_0^1 D^{\alpha}zu_xdx - \xi\int_0^1 zD^{\alpha}zdx - \gamma_1\int_0^1 \left(D^{\alpha}u\right)^2dx - \gamma_2\int_0^1 \left(D^{\alpha}z_x\right)^2dx. \end{aligned} \tag{4.1}$$

Using integration by parts, and the product rule, we find that

$$\begin{aligned} & \rho\int_0^1 D^{\alpha}uD^{\alpha}\left(D^{\alpha}u\right)dx + J\int_0^1 D^{\alpha}zD^{\alpha}\left(D^{\alpha}z\right)dx \\ = \; & -\frac{\mu}{2}D^{\alpha}||u_x||^2 - \frac{\mu\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2dx \\ & -\frac{\delta}{2}D^{\alpha}||z_x||^2 - \frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2dx \\ & -bD^{\alpha}\left(zu_x\right) - \frac{b\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{u_x'(s)ds}{(t-s)^{\alpha}}dx \\ & -\frac{\xi}{2}D^{\alpha}||z||^2 - \frac{\xi\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2dx \\ & -\gamma_1\int_0^1\left(D^{\alpha}u\right)^2dx - \gamma_2\int_0^1\left(D^{\alpha}z_x\right)^2dx. \end{aligned} \tag{4.2}$$

The above equation motivates introducing the functional

$$E(t) = \frac{1}{2}\int_0^1 \{\rho||D^{\alpha}u||^2 + J||D^{\alpha}z||^2 + \mu u_x^2 + \delta z_x^2 + \xi z^2 + 2bzu_x\}dx. \tag{4.3}$$

Consequently, The above considerations lead to the following lemma.

Lemma 4.1 *The functional $E(t)$ satisfies along solutions of* (1.2)

$$\begin{aligned}
D^{\alpha}E(t) \quad = \quad & -\frac{\alpha\rho}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}u(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\alpha J}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}z(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\mu\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u'_x(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z'_x(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{b\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{u'_x(s)ds}{(t-s)^{\alpha}}dx \\
& -\frac{\xi\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\gamma_1\int_0^1 (D^{\alpha}u)^2\,dx-\gamma_2\int_0^1 (D^{\alpha}z_x)^2\,dx. \qquad (4.4)
\end{aligned}$$

Proof: Clearly, a direct fractional differentiation of $E(t)$, using the product rules, yields

$$\begin{aligned}
D^{\alpha}E(t) \quad = \quad & J\int_0^1 D^{\alpha}zD^{\alpha}\left(D^{\alpha}z\right)dx+\rho\int_0^1 D^{\alpha}uD^{\alpha}\left(D^{\alpha}u\right)dx \\
& -\frac{\alpha\rho}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}u(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\alpha J}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}z(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& +bD^{\alpha}(zu_x)+\frac{\mu}{2}D^{\alpha}||u_x||^2+\frac{\delta}{2}D^{\alpha}||z_x||^2+\frac{\xi}{2}D^{\alpha}||z||^2. \qquad (4.5)
\end{aligned}$$

Using (4.2), equation (4.5) becomes

$$
\begin{aligned}
D^{\alpha}E(t) \;=\;& -\frac{\alpha\rho}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}u(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx\\
& -\frac{\alpha J}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}z(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx\\
& -\frac{\mu\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx\\
& -\frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx\\
& -\frac{b\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{u_x'(s)ds}{(t-s)^{\alpha}}dx\\
& -\frac{\xi\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx\\
& -\gamma_1\int_0^1 (D^{\alpha}u)^2\,dx-\gamma_2\int_0^1 (D^{\alpha}z_x)^2\,dx. \qquad (4.6)
\end{aligned}
$$

Hence, the proof of (4.4) is finished.

Remark 4.1 • *The assumption $\mu\xi - b^2 \geq 0$ together with the fact that*

$$\mu u_x^2 + \xi z^2 + 2bzu_x = \left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\sqrt{\xi}z + \frac{b}{\sqrt{\xi}}u_x\right)^2$$

prove the non-negativity of the proposed energy functional (4.3).

- *The above identity allows us to write*

$$
\begin{aligned}
D^{\alpha}E(t) \leq & -\frac{\mu\alpha}{2\Gamma(1-\alpha)}\left(\mu-\frac{b^2}{\xi}\right)\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2\\
& -\frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2\\
& -\gamma_1\int_0^1 (D^{\alpha}u)^2\,dx-\gamma_2\int_0^1 (D^{\alpha}z_x)^2\,dx \leq 0. \qquad (4.7)
\end{aligned}
$$

- *Noticing that (4.4), after canceling the negative terms, also can be written in the form*

$$
\begin{aligned}
D^{\alpha}E(t) \quad \leq \quad & -\frac{\alpha\rho}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}u(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\alpha J}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}z(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{b\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{u_x'(s)ds}{(t-s)^{\alpha}}dx \\
& -\frac{\mu\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\xi\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx. \qquad (4.8)
\end{aligned}
$$

The use of Young's inequality, the above relation becomes

$$
\begin{aligned}
D^{\alpha}E(t) \quad \leq \quad & -\frac{\alpha\rho}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}u(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\alpha J}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}z(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& +\frac{(\frac{b^2}{4\varepsilon}-\frac{\xi}{2})\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& +\frac{(\varepsilon-\frac{\mu}{2})\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx. \qquad (4.9)
\end{aligned}
$$

Choosing $\varepsilon=\frac{b^2}{2\xi}$ gives $\frac{b^2}{4\varepsilon}-\frac{\xi}{2}=0$ and $\varepsilon-\frac{\mu}{2}\leq 0$ (thanks to the assumption $\mu\xi-b^2\geq 0$). Using these, we get

$$
\begin{aligned}
D^{\alpha}E(t) \quad \leq \quad & -\frac{\alpha\rho}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}u(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\alpha J}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{(D^{\alpha}z(\eta))'d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& +\frac{(\varepsilon-\frac{\mu}{2})\alpha}{\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \\
& -\frac{\delta\alpha}{2\Gamma(1-\alpha)}\int_0^1\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{z_x'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx. \qquad (4.10)
\end{aligned}
$$

We will omit the superscript "C" from ${}^{C}D^{\alpha}$ for convenience. Also, we shall use the notation C to denote a generic positive constant.

5. Technical lemmas

Lemma 5.1 *The functional*

$$\chi_1(t) = \rho \int_0^1 u D^\alpha u dx + \frac{\gamma_1}{2} \int_0^1 u^2 dx, \tag{5.1}$$

satisfies, for any $t > 0$,

$$\begin{aligned} D^\alpha \chi_1(t) &\leq \rho \|D^\alpha u\|^2 - \mu \int_0^1 u_x^2 dx + b \int_0^1 u z_x dx \\ &+ \frac{\rho\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha u(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\ &+ \frac{\rho\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{u_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 dx. \end{aligned} \tag{5.2}$$

Proof: Taking the Caputo derivative of order α of the functional χ_1 along solutions of (1.2) shows that

$$\begin{aligned} D^\alpha \chi_1(t) &= \rho \|D^\alpha u\|^2 - \mu \int_0^1 u_x^2 \, dx + b \int_0^1 u z_x dx \\ &- \frac{\rho\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{u'(\eta) d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{[D^\alpha u(s)]' ds}{(t-s)^\alpha} \\ &- \frac{\gamma_1 \alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{u'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 dx. \end{aligned}$$

Integration by parts and the application of Young's and Poincare's inequalities lead to (5.2). □

Lemma 5.2 *The functional*

$$\chi_2(t) = J \int_0^1 z D^\alpha z(t) dx - \frac{\gamma_2}{2} \int_0^1 z_x^2 dx, \tag{5.3}$$

satisfies, for any $t > 0$,

$$\begin{aligned} D^\alpha \chi_2(t) &\leq J ||D^\alpha z||^2 - \delta ||z_x||^2 + b \int_0^1 u z_x dx - \xi ||z||^2 \\ &+ \frac{\alpha J}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha z(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\ &+ \frac{(J+\gamma_2)\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{z_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 dx. \end{aligned} \tag{5.4}$$

Proof: Taking the Caputo derivative of order α of the functional χ_2 along solutions of (1.2), integrating by parts, and using the fractional product rule yields

$$\begin{aligned} D^\alpha \chi_2(t) &= J ||D^\alpha z||^2 - \delta ||z_x||^2 + b \int_0^1 u z_x dx - \xi ||z||^2 \\ &- \frac{J\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{z'(\eta) d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{[D^\alpha z(s)]' ds}{(t-s)^\alpha} \\ &+ \frac{\gamma_2 \alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{z_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 dx. \end{aligned} \tag{5.5}$$

Hence, application of the Young's and Poincare's inequalities, leads to (5.4). □

6. Stability result

In this section, we state and prove our main result.

Lemma 6.1 *The functional $\mathcal{L}$ defined by*

$$\mathcal{L}(t) = NE(t) + \chi_1(t) + \chi_2(t), \tag{6.1}$$

satisfies, for a positive constant N (large enough) and for all $t \geq t_0$,

$$D^\alpha \mathcal{L}(t) \leq -mE(t), \tag{6.2}$$

for some $m > 0$, and satisfies, for some constants $a, b > 0$, the equivalence

$$aE(t) \leq \mathcal{L}(t) \leq bE(t), \quad t \geq 0. \tag{6.3}$$

Proof: Differentiating the functional $\mathcal{L}$ and utilizing all the above estimates in (5.2) and (5.4), recalling (4.4), and applying Poincare's inequality, it follows that

$$\begin{aligned}
D^\alpha \mathcal{L}(t) \leq & -(\gamma_1 N - \rho)\,||D^\alpha u||^2 - (\gamma_2 N - J)\,||D^\alpha z||^2 \\
& - \int_0^1 \left(\mu u_x^2 + \xi z + 2buz_x\right) dx - \delta \int_0^1 z_x^2 dx \\
& - (\delta N - (J + \gamma_2)) \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{z_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 \\
& - (\mu N - \rho) \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{u_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 \\
& - (N-1) \frac{\rho\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha u(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
& - (N-1) \frac{\alpha J}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha z(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx.
\end{aligned}$$

Using Remark 4.1, the above relation becomes:

$$\begin{aligned}
D^\alpha \mathcal{L}(t) \leq & -(\gamma_1 N - \rho)\,||D^\alpha u||^2 - (\gamma_2 N - J)\,||D^\alpha z||^2 \\
& - \left(\mu - \frac{b^2}{\xi}\right) \int_0^1 u_x^2 dx - \delta \int_0^1 z_x^2 dx \\
& - (\mu N - (J + \gamma_2)) \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{z_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 \\
& - (\mu N - \rho) \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{u_x'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 \\
& - (N-1) \frac{\rho\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha u(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
& - (N-1) \frac{\alpha J}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha z(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx.
\end{aligned}$$

Recalling the condition $\left(\mu - \frac{b^2}{\xi}\right) > 0$ and selecting $N > 1$ large enough so that the relation (6.3) remains valid and

$$\gamma_1 N - \rho > 1, \quad \gamma_2 N - J > 1, \quad \delta N - (J + \gamma_2) > 1, \quad \mu N - \rho > 1.$$

Therefore, we have for some positive constant c_1

$$D^\alpha \mathcal{L}(t) \leq -c_1 \left[\rho||D^\alpha u||^2 + J||D^\alpha z||^2 + ||u_x||^2 + ||z_x||^2\right].$$

Applying Young's inequality on the terms in the energy functional (4.3), we have for some positive constant c_2

$$E(t) \leq c_2 \left[\rho||D^\alpha u||^2 + J||D^\alpha z||^2 + ||u_x||^2 + ||z_x||^2\right].$$

These two relations imply

$$D^\alpha \mathcal{L}(t) \leq -cE(t).$$

This ends the proof of (6.2). For the proof of (6.3), it is easy to prove that

$$|\chi_i(t)| \leq m_i E(t) \quad i = 1, 2,$$

for some positive constants m_i. Then, we obtain

$$|\mathcal{L} - NE| \leq mE,$$

where $m = \max\{m_i\}, i = 1, 2$. Now, the above relation gives

$$(N - m)E(t) \leq \mathcal{L}(t) \leq (m + N)E(t).$$

Select $N > m$ ends the proof of (6.3). □

6.1. The main result

In this section, we state our main result.

Theorem 6.1 *If $\mu\xi - b^2 > 0$, then, there exist positive constants β_1 and β_2 such that the energy of (1.2) is Mittag-Leffler stable, that is; for all $t > 0$,*

$$E(t) \leq \beta_1 E_\alpha \left(-\beta_2 t^\alpha\right). \tag{6.4}$$

Proof: The proof of this theorem follows directly by combing (6.2), (6.3), and Proposition 3.1. □

7. Conclusion and remarks

In this work, we studied the stability of a one-dimensional elastic-porous system with fractional derivatives and Dirichlet boundary conditions. We dissipated the elastic equation by a nonstandard frictional damping (frictional damping with fractional derivative) and the porous equation by nonstandard Kelvin-Voigt damping (Kelvin-Voigt damping with fractional damping). We proved that the system is Mittag-Leffler stable under certain conditions on the coefficients of the system imposed by the physics of the system even in the integer order case and without imposing the wave speeds condition $\frac{\mu}{\rho} = \frac{\delta}{J}$. The result is new and opens a research area for porous-elastic system.

Availability of data and materials

No data were used to support this study.

Competing interests

The author declares no competing interests.

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